CHAPTER TWO

1. Since $AB = BC$; since the two angles at B are equal; and since the angles at A and C are both right angles, it follows by the angle-side-angle theorem that $\triangle EBC$ is congruent to $\triangle SBA$ and therefore that $SA = EC$.

2. Because both angles at E are right angles; because AE is common to the two triangles; and because the two angles CAE are equal to one another, it follows by the angle-side-angle theorem that $\triangle AET$ is congruent to $\triangle AES$. Therefore $SE = ET$.

3. $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Therefore the oblong number $n(n+1)$ is double the triangular number T_n.

4. $n^2 = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$, and the summands are the triangular numbers T_{n-1} and T_n.

5. $\frac{8n(n+1)}{2} + 1 = 4n^2 + 4n + 1 = (2n+1)^2$.

6. Examples using the first formula are (3,4,5), (5,12,13), (7,24,25), (9,40,41), (11,60,61). Examples using the second formula are (8,15,17), (12,35,37), (16,63,65), (20,99,101), (24,143,145).

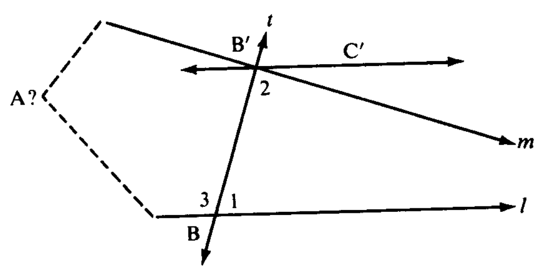
7. Consider the right triangle ABC where AB has unit length and the hypotenuse BC has length 2. Then the square on the leg AC is three times the square on the leg AB. Assume the legs AB and AC are commensurable, so that each is represented by the number of times it is measured by their greatest common measure, and assume further that these numbers are relatively prime, for otherwise there would be a larger common measure. Thus the squares on AC and AB are represented by square numbers, where the former is three times the latter. It follows that leg AC is divisible by three and therefore its square is divisible by nine. Since the square on AB is one third that on AC, it is divisible by three, and hence the side AB itself is divisible by three, contradicting the assumption that the numbers measuring the two legs are relatively prime.

9. Let ABC be the given triangle. Extend BC to D and draw CE parallel to AB. By I–29, angles BAC and ACE are equal, as are angles ABC and ECD. Therefore angle ACD equals the sum of the angles ABC and BAC. If we add angle ACB to each of these, we get that the sum of the three interior angles of the triangle is equal to the straight angle BCD. Because this latter angle equals two right angles, the theorem is proved.

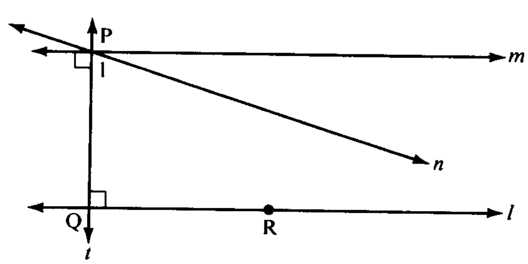
10. Place the given rectangle $BEFG$ so that BE is in a straight line with AB. Extend FG to H so that AH is parallel to BG. Connect HB and extend it until it meets the extension of FE at D. Through D draw DL parallel to FH and extend GB and HA so they meet DL in M and L respectively. Then HD is the diagonal of the rectangle $FDLH$ and

so divides it into two equal triangles HFD and HLD. Because triangle BED is equal to triangle BMD and also triangle BGH is equal to triangle BAH, it follows that the remainders, namely rectangles $BEFG$ and $ABML$ are equal. Thus $ABML$ has been applied to AB and is equal to the given rectangle $BEFG$.

11. In this proof, we shall refer to certain propositions in Euclid's Book I, all of which are proved before Euclid first uses Postulate 5. (That occurs in proposition 29.) First, assume Playfair's axiom. Suppose line t crosses lines m and l and that the sum of the two interior angles (angles 1 and 2 in the diagram) is less than two right angles. We know that the sum of angles 1 and 3 is equal to two right angles. Therefore $\angle 2 < \angle 3$. Now on line BB' and point B' construct line $B'C'$ such that $\angle C'B'B = \angle 3$ (Proposition 23). Therefore, line $B'C'$ is parallel to line l (Proposition 27). Therefore, by Playfair's axiom, line m is not parallel to line l. It therefore meets l. We must show that the two lines meet on the same side as C'. If the meeting point A is on the opposite side, then $\angle 2$ is an exterior angle to triangle ABB', yet it is smaller than $\angle 3$, one of the interior angles, contradicting proposition 16. We have therefore derived Euclid's postulate 5.



Second, assume Euclid's postulate 5. Let l be a given line and P a point outside the line. Construct the line t perpendicular to l through P (Proposition 12). Next, construct the line m perpendicular to line t at P (Proposition 11). Since the alternate interior angles formed by line t crossing lines m and l are both right and therefore are equal, it follows from Proposition 27 that m is parallel to l. Now suppose n is any other line through P. We will show that n meets l and is therefore not parallel to l. Let $\angle 1$ be the acute angle that n makes with t. Then the sum of angle 1 and angle PQR is less than two right angles. By postulate 5, the lines meet.



Note that in this proof, we have actually proved the equivalence of Euclid's Postulate 5 to the statement that given a line l and a point P not on l, there is at most one line through P which is parallel to l. The other part of Playfair's Axiom was proved (in the second part above) without use of postulate 5 and was not used at all in the first part.

12. One possibility: If the line has length a and is cut at a point with coordinate x, then $4ax + (a - x)^2 = (a + x)^2$. This is a valid identity.

13. In the circle ABC, let the angle BEC be an angle at the center and the angle BAC be an angle at the circumference which cuts off the same arc BC. Connect AE and continue the line to F. Since $EA = EB$, $\angle EAB = \angle EBA$. Since $\angle BEF$ equals the sum of those two angles, $\angle BEF$ is double $\angle EAB$. Similarly, $\angle FEC$ is double $\angle EAC$. Therefore the entire $\angle BEC$ is double the entire $\angle BAC$. Note that this argument holds as long as line EF is within $\angle BEC$. If it is not, an analogous argument by subtraction holds.

14. Let $\angle BAC$ be an angle cutting off the diameter BC of the circle. Connect A to the center E of the circle. Since $EB = EA$, it follows that $\angle EBA = \angle EAB$. Similarly, $\angle ECA = \angle EAC$. Therefore the sum of $\angle EBA$ and $\angle ECA$ is equal to $\angle BAC$. But the sum of all three angles equals two right angles. Therefore, twice $\angle BAC$ is equal to two right angles, and angle BAC is itself a right angle.

15. In the circle, inscribe a side AC of an equilateral triangle and a side AB of an equilateral pentagon. Then arc BC is the difference between one-third and one-fifth of the circumference of the circle. That is, arc $BC = \frac{2}{15}$ of the circumference. Thus, if we bisect that arc at E, then lines BE and EC will each be a side of a regular 15-gon.

16. Let the triangle be ABC and draw DE parallel to BC cutting the side AB at D and the side AC at E. Connect BE and CD. Then triangles BDE and CDE are equal in area, having the same base and in the same parallels. Therefore, triangle BDE is to triangle ADE and triangle CDE is to triangle ADE. But triangles withe the same altitude are to one another as their bases. Thus triangle BDE is to triangle ADE as BD is to AD, and triangle CDE is to triangle ADE and CE is to AE. It follows that BD is to AD as CE is to AE, as desired.

17. Let ABC be the triangle, and let the angle at A be bisected by AD, where D lies on the side BC. Now draw CE parallel to AD, meeting BA extended at E. Now angle CAD is equal to angle BAD by hypothesis. But also angle CAD equals angle ACE and angle BAD equals angle AEC, since in both cases we have a transversal falling across parallel lines. It follows that angle AEC equals angle ACE, and therefore that $AC = AE$. By proposition VI-2, we know that BD is to DC as BA is to AE. Therefore BD is to DC as BA is to AC, as claimed.

18. Let $a = s_1 b + r_1$, $b = s_2 r_1 + r_2$, $\ldots$, $r_{k-1} = s_{k+1} r_k$. Then r_k divides r_{k-1} and therefore also $r_{k-2}, \ldots, b, a$. If there were a greater common divisor of a and b, it would divide r_1, r_2, $\ldots$, r_k. Since it is impossible for a greater number to divide a smaller, we have shown that r_k is in fact the greatest common divisor of a and b.

19.

$$\begin{aligned}963 &= 1\cdot 657+306\\ 657 &= 2\cdot 306+45\\ 306 &= 6\cdot 45+36\\ 45 &= 1\cdot 36+9\\ 36 &= 4\cdot 9+0\end{aligned}$$

Therefore, the greatest common divisor of 963 and 657 is 9.

20. Since $1-x=x^2$, we have

$$1 = 1\cdot x + (1-x) = 1\cdot x + x^2$$

$$x = 1\cdot x^2 + (x-x^2) = 1\cdot x^2 + x(1-x) = 1\cdot x^2 + x^3$$
$$x^2 = 1\cdot x^3 + (x^2-x^3) = 1\cdot x^3 + x^2(1-x) = 1\cdot x^3 + x^4$$
$$\cdots$$

Thus $1:x$ can be expressed in the form $(1,1,1,\ldots)$.

21.

$$\begin{array}{rcccc} 46 &=& 7\cdot 6 &+& 4\\ 6 &=& 1\cdot 4 &+& 2\\ 4 &=& 2\cdot 2 && \end{array} \qquad \begin{array}{rcccc} 23 &=& 7\cdot 3 &+& 2\\ 3 &=& 1\cdot 2 &+& 1\\ 2 &=& 2\cdot 1 && \end{array}$$

Note that the multiples 7, 1, 2 in the first example equal the multiples 7, 1, 2 in the second.

22. In Figure 2.16 (left), let $AB=7$ and the area of the given figure be 10. The construction described on p. 72 then determines x to be BS. This value is $\frac{7}{2}-\sqrt{\frac{49}{4}-10}=\frac{7}{2}-\sqrt{\frac{9}{4}}=\frac{7}{2}-\frac{3}{2}=2$. The second solution is $BE+ES=AE+ES=AS$. This value is $\frac{7}{2}+\sqrt{\frac{49}{4}-10}=\frac{7}{2}+\sqrt{\frac{9}{4}}=\frac{7}{2}+\frac{3}{2}=5$.

23. In Figure 2.16 (right), let $AB=10$ and the area of the given figure be 39. The construction described on p. 72 then determines x to be BS. This value is $\sqrt{5^2+39}-5=\sqrt{64}-5=8-5=3$.

24. Suppose m factors two different ways as a product of primes: $m=pqr\cdots s=p'q'r'\cdots s'$. Since p divides $pqr\cdots s$, it must also divide $p'q'r'\cdots s'$. By VII–30, p must divide one of the prime factors, say p'. But since both p and p' are prime, we must have $p=p'$. After canceling these two factors from their respective products, we can then repeat the argument to show that each prime factor on the left is equal to a prime factor on the right and conversely.

25. One standard modern proof is as follows. Assume there are only finitely many prime numbers p_1, p_2, p_3, …, p_n. Let $N = p_1p_2p_3 \cdots p_n + 1$. There are then two possibilities. Either N is prime or N is divisible by a prime other than the given ones, since division by any of those leaves remainder 1. Both cases contradict the original hypothesis, which therefore cannot be true.

26. We begin with a square inscribed in a circle of radius 1. If we divide the square into four isosceles triangles, each with vertex angle a right angle, then the base of each triangle has length $b_1 = \sqrt{2}$ and height $h_1 = \frac{\sqrt{2}}{2}$. Then the area A_1 of the square is equal to $4 \cdot \frac{1}{2}b_1h_1 = 2b_1h_1 = 2$. If we next construct an octagon by bisecting the vertex angles of each of these triangles and connecting the points on the circumference, the octagon is formed of eight isosceles triangles. The base of each triangle has length

$$b_2 = \sqrt{\left(\frac{b_1}{2}\right)^2 + (1-h_1)^2} = \sqrt{\left(\frac{b_1}{2}\right)^2 + h_1^2 - 2h_1 + 1} = \sqrt{2-2h_1} = \sqrt{2-\sqrt{2}}$$

and height

$$h_2 = \sqrt{1 - \left(\frac{b_2}{2}\right)^2} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Thus the octagon has area $A_2 = 8 \cdot \frac{1}{2}b_2h_2 = 4b_2h_2 = 2\sqrt{2} = 2.828427$. If we continue in this way by always bisecting the vertex angles of the triangles to construct a new polygon, we get that the area A_n of the nth polygon is given by the formula $A_n = 2^{n+1} \cdot \frac{1}{2}b_nh_n = 2^nb_nh_n$, where

$$b_n = \sqrt{\left(\frac{b_{n-1}}{2}\right)^2 + (1-h_{n-1})^2} = \sqrt{\left(\frac{b_{n-1}}{2}\right)^2 + h_{n-1}^2 - 2h_{n-1} + 1} = \sqrt{2-2h_{n-1}}$$

and

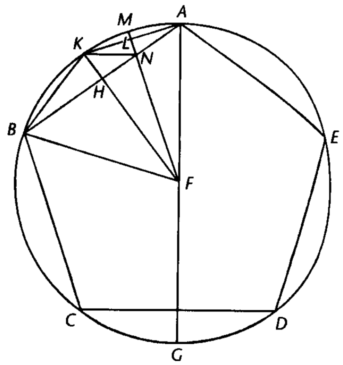
$$h_n = \sqrt{1 - \left(\frac{b_n}{2}\right)^2}.$$

The next two results using this formula are $A_3 = 3.061467$ and $A_4 = 3.121445$.

27. Since BC is the side of a decagon, triangle EBC is a 36-72-72 triangle. Thus $\angle ECD = 108°$. Since CD, the side of a hexagon, is equal to the radius CE, it follows that triangle ECD is an isosceles triangle with base angles equal to 36°. Thus triangle EBD is a 36-72-72 triangle and is similar to triangle EBC. Therefore $BD : EC = EC : BC$ or $BD : CD = CD : BC$ and the point C divides the line segment BD in extreme and mean ratio.

28. Let $ABCDE$ be the pentagon inscribed in the circle with center F. Connect AF and extend it to meet the circle at G. Draw FH perpendicular to AB and extend it to

meet the circle at K. Connect AK. Then AK is a side of the decagon inscribed in the circle, while $BF = AF$ is the side of the hexagon inscribed in the circle. Draw FL perpendicular to AK; let N be its intersection with AB and M be its intersection with the circle. Connect KN. Now triangles ANK and AKB are isosceles triangles with a common base angle at A. Therefore, the triangles are similar. So $BA : AK = AK : AN$, or $AK^2 = BA \cdot AN$. Further, note that arc BKM has measure 54°, while arc BCG has measure 108°. It follows that $\angle BFN = \angle BAF$. Since triangles BFN and BAF also have angle FBA in common, the triangles are similar. Therefore, $BA : BF = BF : BN$, or $BF^2 = BA \cdot BN$. We therefore have $AK^2 + BF^2 = BA \cdot AN + BA \cdot BN = BA \cdot (AN + BN) = BA^2$. That is, the sum of the squares on the side of the decagon and the side of the hexagon is equal to the square on the side of the pentagon.



29. $C = \frac{360}{7\frac{1}{5}} \cdot 5000 = 250,000$ stades. This value equals 129,175,000 feet or 24,465 miles. The diameter then equals 7,787 miles.