CHAPTER 2

MATHEMATICAL MODELS AND NUMERICAL METHODS

SECTION 2.1

POPULATION MODELS

Section 2.1 introduces the first of the two major classes of mathematical models studied in the textbook, and is a prerequisite to the discussion of equilibrium solutions and stability in Section 2.2.

In Problems 1–8 we outline the derivation of the desired particular solution, and then sketch some typical solution curves.

1. Noting that $x > 1$ because $x(0) = 2$, we write

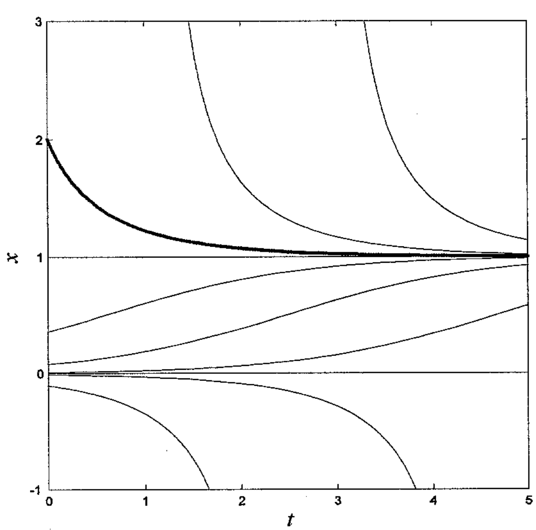
$$\int \frac{dx}{x(1-x)} = \int 1\,dt; \qquad \int\left(\frac{1}{x} - \frac{1}{x-1}\right)dx = \int 1\,dt$$

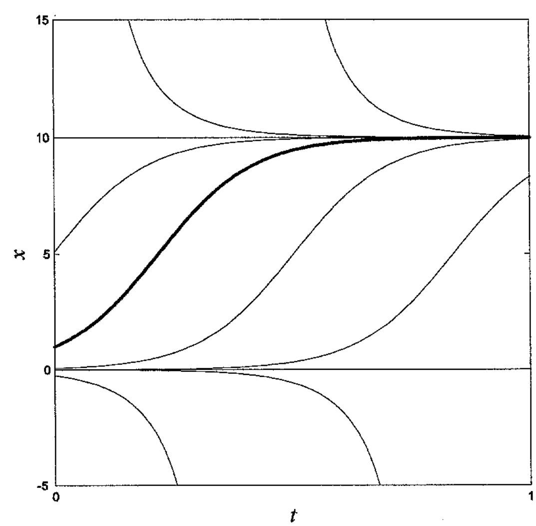
$$\ln x - \ln(x-1) = t + \ln C; \qquad \frac{x}{x-1} = Ce^t$$

$$x(0) = 2 \text{ implies } C = 2; \qquad x = 2(x-1)e^t$$

$$x(t) = \frac{2e^t}{2e^t - 1} = \frac{2}{2 - e^{-t}}.$$

Typical solution curves are shown in the figure on the left below.





2. Noting that $x < 10$ because $x(0) = 1$, we write

$$\int \frac{dx}{x(10-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x} + \frac{1}{10-x}\right) dx = \int 10\,dt$$

$$\ln x - \ln(10-x) = 10t + \ln C; \qquad \frac{x}{10-x} = C e^{10t}$$

$$x(0) = 1 \text{ implies } C = \frac{1}{9}; \qquad 9x = (10-x)e^{10t}$$

$$x(t) = \frac{10e^{10t}}{9+e^{10t}} = \frac{10}{1+9e^{-10t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

3. Noting that $x > 1$ because $x(0) = 3$, we write

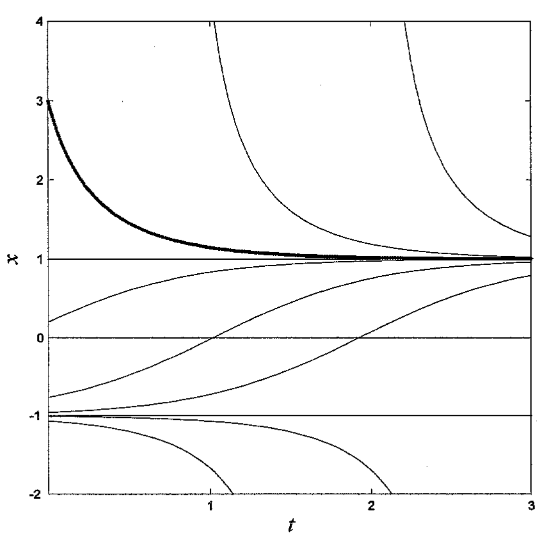
$$\int \frac{dx}{(1+x)(1-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \int (-2)\,dt$$

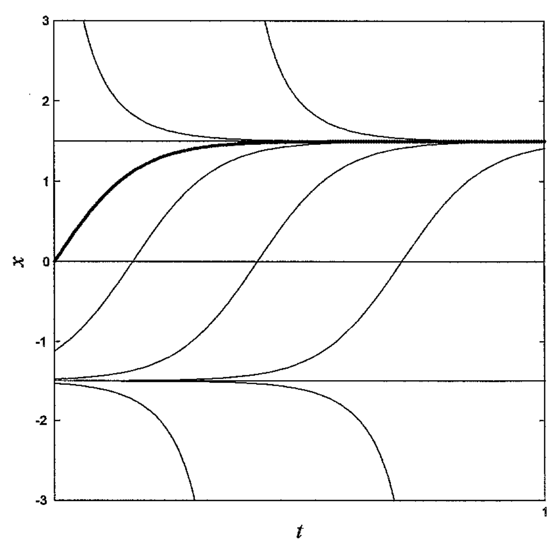
$$\ln(x-1) - \ln(x+1) = -2t + \ln C; \qquad \frac{x-1}{x+1} = C e^{-2t}$$

$$x(0) = 3 \text{ implies } C = \frac{1}{2}; \qquad 2(x-1) = (x+1)e^{-2t}$$

$$x(t) = \frac{2+e^{-2t}}{2-e^{-2t}} = \frac{2e^{2t}+1}{2e^{2t}-1}.$$

Typical solution curves are shown in the figure on the left below.





4. Noting that $|x| < \frac{3}{2}$ because $x(0) = 0$, we write

$$\int \frac{dx}{(3+2x)(3-2x)} = \int 1\,dt; \qquad \int\left(\frac{1}{3+2x} + \frac{1}{3-2x}\right)dx = \int 6\,dt$$

$$\frac{1}{2}\ln(3+2x) - \frac{1}{2}\ln(3-2x) = 6t + \frac{1}{2}\ln C; \qquad \frac{3+2x}{3-2x} = Ce^{12t}$$

$$x(0) = 0 \text{ implies } C = 1; \qquad 3+2x = (3-2x)e^{12t}$$

$$x(t) = \frac{3e^{12t} - 3}{2e^{12t} + 2} = \frac{3\left(e^{12t} - 1\right)}{2\left(e^{12t} + 1\right)}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

5. Noting that $x > 5$ because $x(0) = 8$, we write

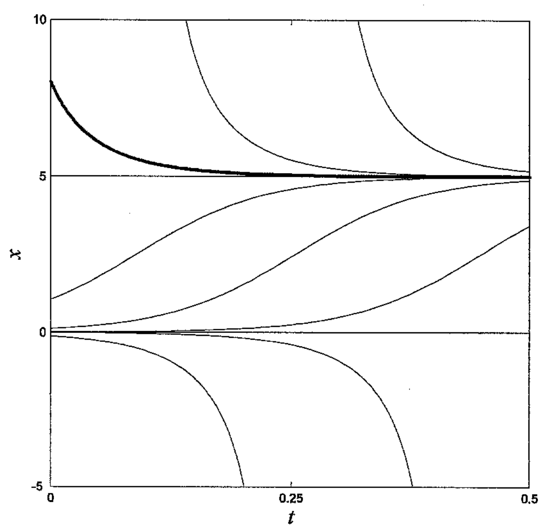
$$\int \frac{dx}{x(x-5)} = \int(-3)\,dt; \qquad \int\left(\frac{1}{x} - \frac{1}{x-5}\right)dx = \int 15\,dt$$

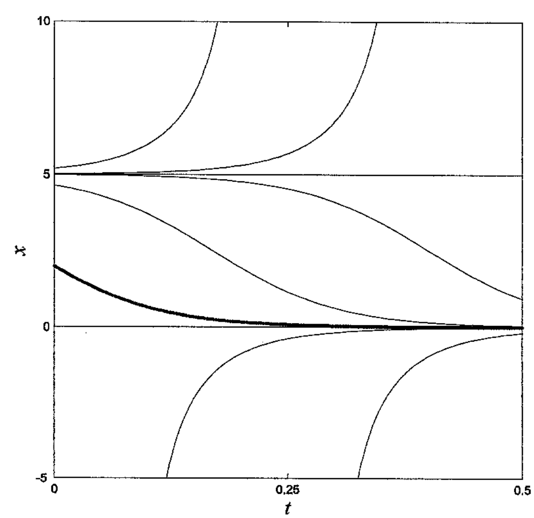
$$\ln x - \ln(x-5) = 15t + \ln C; \qquad \frac{x}{x-5} = Ce^{15t}$$

$$x(0) = 8 \text{ implies } C = 8/3; \qquad 3x = 8(x-5)e^{15t}$$

$$x(t) = \frac{-40e^{15t}}{3 - 8e^{15t}} = \frac{40}{8 - 3e^{-15t}}.$$

Typical solution curves are shown in the figure on the left below.





6. Noting that $x < 5$ because $x(0) = 2$, we write

$$\int \frac{dx}{x(5-x)} = \int (-3)\,dt; \qquad \int \left(\frac{1}{x} + \frac{1}{5-x}\right) dx = \int (-15)\,dt$$

$$\ln x - \ln(5-x) = -15t + \ln C; \qquad \frac{x}{5-x} = C e^{-15t}$$

$$x(0) = 2 \text{ implies } C = 2/3; \qquad 3x = 2(5-x)e^{-15t}$$

$$x(t) = \frac{10e^{-15t}}{3+2e^{-15t}} = \frac{10}{2+3e^{15t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

7. Noting that $x > 7$ because $x(0) = 11$, we write

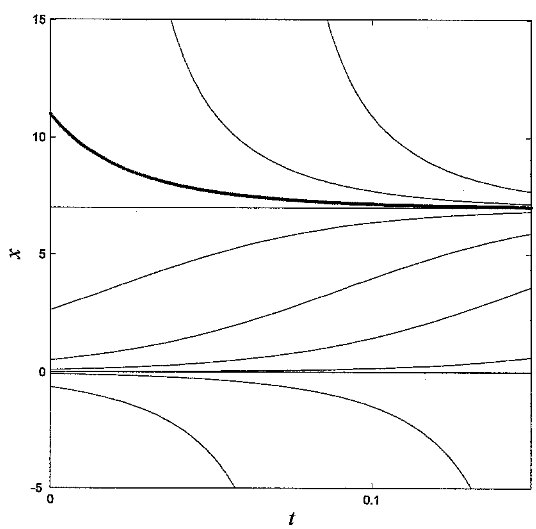
$$\int \frac{dx}{x(x-7)} = \int (-4)\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-7}\right) dx = \int 28\,dt$$

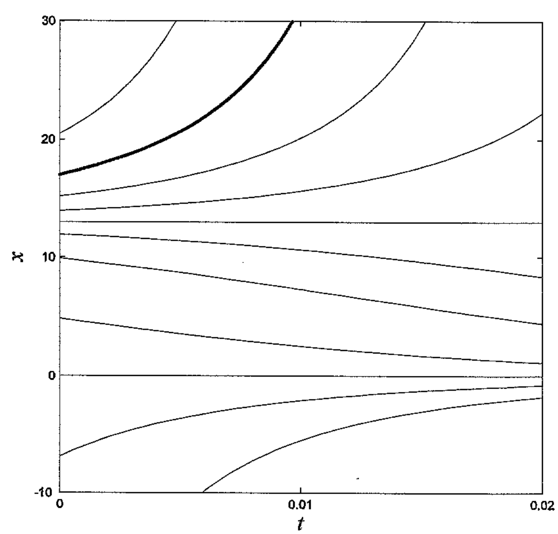
$$\ln x - \ln(x-7) = 28t + \ln C; \qquad \frac{x}{x-7} = C e^{28t}$$

$$x(0) = 11 \text{ implies } C = 11/4; \qquad 4x = 11(x-17)e^{28t}$$

$$x(t) = \frac{-77e^{28t}}{4-11e^{28t}} = \frac{77}{11-4e^{-28t}}.$$

Typical solution curves are shown in the figure on the left below.





8. Noting that $x > 13$ because $x(0) = 17$, we write

$$\int \frac{dx}{x(x-13)} = \int 7\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-13}\right) dx = \int (-91)\,dt$$

$$\ln x - \ln(x-13) = -91t + \ln C; \qquad \frac{x}{x-13} = C e^{-91t}$$

$$x(0) = 17 \text{ implies } C = 17/4; \qquad 4x = 17(x-13)e^{-91t}$$

$$x(t) = \frac{-221 e^{-91t}}{4 - 17 e^{-91t}} = \frac{221}{17 - 4e^{91t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

9. Substitution of $P(0) = 100$ and $P'(0) = 20$ into $P' = k\sqrt{P}$ yields $k = 2$, so the differential equation is $P' = 2\sqrt{P}$. Separation of variables and integration, $\int dP/2\sqrt{P} = \int dt$, gives $\sqrt{P} = t + C$. Then $P(0) = 100$ implies $C = 10$, so $P(t) = (t + 10)^2$. Hence the number of rabbits after one year is $P(12) = 484$.

10. Given $P' = -\delta P = -(k/\sqrt{P})P = -k\sqrt{P}$, separation of variables and integration as in Problem 9 yields $2\sqrt{P} = -kt + C$. The initial condition $P(0) = 900$ gives $C = 60$, and then the condition $P(6) = 441$ implies that $k = 3$. Therefore $2\sqrt{P} = -3t + 60$, so $P = 0$ after $t = 20$ weeks.

11. **(a)** Starting with $dP/dt = k\sqrt{P}$, $dP/dt = k\sqrt{P}$, we separate the variables and integrate to get $P(t) = (kt/2 + C)^2$. Clearly $P(0) = P_0$ implies $C = \sqrt{P_0}$.

(b) If $P(t) = (kt/2 + 10)^2$, then $P(6) = 169$ implies that $k = 1$. Hence $P(t) = (t/2 + 10)^2$, so there are 256 fish after 12 months.

12. Solution of the equation $P' = kP^2$ by separation of variables and integration,

$$\int \frac{dP}{P^2} = \int k\,dt; \qquad -\frac{1}{P} = kt - C,$$

gives $P(t) = 1/(C - kt)$. Now $P(0) = 12$ implies that $C = 1/12$, so now $P(t) = 12/(1 - 12kt)$. Then $P(10) = 24$ implies that $k = 1/240$, so finally $P(t) = 240/(20 - t)$. Hence $P = 48$ when $t = 15$, that is, in the year 2003. And obviously $P \to \infty$ as $t \to 20$.

13. **(a)** If the birth and death rates both are proportional to P^2 and $\beta > \delta$, then Eq. (1) in this section gives $P' = kP^2$ with k positive. Separating variables and integrating as in Problem 12, we find that $P(t) = 1/(C - kt)$. The initial condition $P(0) = P_0$ then gives $C = 1/P_0$, so $P(t) = 1/(1/P_0 - kt) = P_0/(1 - kP_0 t)$.

(b) If $P_0 = 6$ then $P(t) = 6/(1 - 6kt)$. Now the fact that $P(10) = 9$ implies that $k = 180$, so $P(t) = 6/(1 - t/30) = 180/(30 - t)$. Hence it is clear that $P \to \infty$ as $t \to 30$ (doomsday).

14. Now $dP/dt = -kP^2$ with $k > 0$, and separation of variables yields $P(t) = 1/(kt + C)$. Clearly $C = 1/P_0$ as in Problem 13, so $P(t) = P_0/(1 + kP_0 t)$. Therefore it is clear that $P(t) \to 0$ as $t \to \infty$, so the population dies out in the long run.

15. If we write $P' = bP(a/b - P)$ we see that $M = a/b$. Hence

$$\frac{B_0 P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M.$$

Note also (for Problems 16 and 17) that $a = B_0/P_0$ and $b = D_0/P_0^2 = k$.

16. The relations in Problem 15 give $k = 1/2400$ and $M = 160$. The solution is $P(t) = 19200/(120 + 40e^{-t/15})$. We find that $P = 0.95M$ after about 27.69 months.

17. The relations in Problem 15 give $k = 1/2400$ and $M = 180$. The solution is $P(t) = 43200/(240 - 60e^{-3t/80})$. We find that $P = 1.05M$ after about 44.22 months.

18. If we write $P' = aP(P - b/a)$ we see that $M = b/a$. Hence

$$\frac{D_0 P_0}{B_0} = \frac{(bP_0)P_0}{aP_0^2} = \frac{b}{a} = M.$$

Note also (for Problems 19 and 20) that $b = D_0/P_0$ and $a = B_0/P_0^2 = k$.

19. The relations in Problem 18 give $k = 1/1000$ and $M = 90$. The solution is $P(t) = 9000/(100 - 10e^{9t/100})$. We find that $P = 10M$ after about 24.41 months.

20. The relations in Problem 18 give $k = 1/1100$ and $M = 120$. The solution is $P(t) = 13200/(110 + 10e^{6t/55})$. We find that $P = 0.1M$ after about 42.12 months.

21. Starting with the differential equation $dP/dt = kP(200 - P)$, we separate variables and integrate, noting that $P < 200$ because $P_0 = 100$:

$$\int \frac{dP}{P(200-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 200k\,dt;$$

$$\ln \frac{P}{200-P} = 200kt + \ln C \quad \Rightarrow \quad \frac{P}{200-P} = Ce^{200kt}.$$

Now $P(0) = 100$ gives $C = 1$, and $P'(0) = 1$ implies that $1 = k \cdot 100(200-100)$, so we find that $k = 1/10000$. Substitution of these numerical values gives

$$\frac{P}{200-P} = e^{200t/10000} = e^{t/50},$$

and we solve readily for $P(t) = 200/\left(1+e^{-t/50}\right)$. Finally, $P(60) = 200/\left(1+e^{-6/5}\right) \approx 153.7$ million.

22. We work in thousands of persons, so $M = 100$ for the total fixed population. We substitute $M = 100$, $P'(0) = 1$, and $P_0 = 50$ in the logistic equation, and thereby obtain

$$1 = k(50)(100-50), \qquad \text{so} \qquad k = 0.0004.$$

If t denotes the number of days until 80 thousand people have heard the rumor, then Eq. (7) in the text gives

$$80 = \frac{50 \times 100}{50 + (100-50)e^{-0.04t}},$$

and we solve this equation for $t \approx 34.66$. Thus the rumor will have spread to 80% of the population in a little less than 35 days.

23. **(a)** $x' = 0.8x - 0.004x^2 = 0.004x(200-x)$, so the maximum amount that will dissolve is $M = 200$ g.

(b) With $M = 200$, $P_0 = 50$, and $k = 0.004$, Equation (4) in the text yields the solution

$$x(t) = \frac{10000}{50 + 150e^{-0.08t}}.$$

Substituting $x = 100$ on the left, we solve for $t = 1.25 \ln 3 \approx 1.37$ sec.

24. The differential equation for $N(t)$ is $N'(t) = kN(15-N)$. When we substitute $N(0) = 5$ (thousands) and $N'(0) = 0.5$ (thousands/day) we find that $k = 0.01$. With N in place of P, this is the logistic equation in Eq. (3) of the text, so its solution is given by Equation (7):

$$N(t) = \frac{15\times 5}{5+10\exp[-(0.01)(15)t]} = \frac{15}{1+2\,e^{-0.15t}}.$$

Upon substituting $N = 10$ on the left, we solve for $t = (\ln 4)/(0.15) \approx 9.24$ days.

25. Proceeding as in Example 3 in the text, we solve the equations

$$25.00k(M-25.00) = 3/8, \qquad 47.54k(M-47.54) = 1/2$$

for $M = 100$ and $k = 0.0002$. Then Equation (4) gives the population function

$$P(t) = \frac{2500}{25+75e^{-0.02t}}.$$

We find that $P = 75$ when $t = 50 \ln 9 \approx 110$, that is, in 2035 A. D.

26. The differential equation for $P(t)$ is

$$P'(t) = 0.001P^2 - \delta P.$$

When we substitute $P(0) = 100$ and $P'(0) = 8$ we find that $\delta = 0.02$, so

$$\frac{dP}{dt} = 0.001P^2 - 0.02P = 0.001P(P-20).$$

We separate variables and integrate, noting that $P > 20$ because $P_0 = 100$:

$$\int \frac{dP}{P(P-20)} = \int 0.001\,dt \quad\Rightarrow\quad \int\left(\frac{1}{P-20}-\frac{1}{P}\right)dP = \int 0.02\,dt;$$

$$\ln\frac{P-20}{P} = \frac{1}{50}t + \ln C \quad\Rightarrow\quad \frac{P-20}{P} = Ce^{t/50}.$$

Now $P(0) = 100$ gives $C = 4/5$, hence

$$5(P-20) = 4P\,e^{t/50} \quad\Rightarrow\quad P(t) = \frac{100}{5-4e^{t/50}}.$$

It follows readily that $P = 200$ when $t = 50 \ln(9/8) \approx 5.89$ months.

27. We are given that

$$P' = kP^2 - 0.01P,$$

When we substitute $P(0) = 200$ and $P'(0) = 2$ we find that $k = 0.0001$, so

$$\frac{dP}{dt} = 0.0001P^2 - 0.01P = 0.0001P(P-100).$$

We separate variables and integrate, noting that $P > 100$ because $P_0 = 200$:

$$\int \frac{dP}{P(P-100)} = \int 0.0001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-100} - \frac{1}{P}\right) dP = \int 0.01\,dt;$$

$$\ln\frac{P-100}{P} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{P-100}{P} = Ce^{t/100}.$$

Now $P(0) = 100$ gives $C = 1/2$, hence

$$2(P-100) = Pe^{t/100} \quad \Rightarrow \quad P(t) = \frac{200}{2 - e^{t/100}}.$$

(a) $P = 1000$ when $t = 100\ln(9/5) \approx 58.78$.

(b) $P \to \infty$ as $t \to 100\ln 2 \approx 69.31$.

28. Our alligator population satisfies the equation

$$\frac{dP}{dt} = 0.0001x^2 - 0.01x = 0.0001x(x-100).$$

With x in place of P, this is the same differential equation as in Problem 27, but now we use absolute values to allow both possibilities $x < 100$ and $x > 100$:

$$\int \frac{dx}{x(x-100)} = \int 0.0001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{x-100} - \frac{1}{x}\right) dP = \int 0.01\,dt;$$

$$\ln\frac{|x-100|}{x} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{|x-100|}{x} = Ce^{t/100}. \qquad (*)$$

(a) If $x(0) = 25$ then $x < 100$ and $|x-100| = 100 - x$, so (*) gives $C = 3$ and hence

$$100 - x = 3xe^{t/100} \quad \Rightarrow \quad x(t) = \frac{100}{1 + 3e^{t/100}}.$$

We therefore see that $x(t) \to 0$ as $t \to \infty$.

(b) But if $x(0) = 150$ then $x > 100$ and $|x-100| = x-100$, so (*) gives $C = 1/3$ and hence

$$3(x-100) = xe^{t/100} \quad \Rightarrow \quad x(t) = \frac{300}{3-e^{t/100}}.$$

Now $x(t) \to +\infty$ as $t \to (100\ln 3)^-$, so doomsday occurs after about 109.86 months.

29. Here we have the logistic equation

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2 = 0.0001489P(210.544 - P)$$

where $k = 0.0001489$ and $P = 210.544$. With $P_0 = 3.9$ also, Eq. (7) in the text gives

$$P(t) = \frac{(210.544)(3.9)}{(3.9)+(210.544-3.9)e^{-(0.0001489)(210.544)t}} = \frac{821.122}{3.9+206.644e^{-0.03135t}}.$$

(a) This solution gives $P(140) \approx 127.008$, fairly close to the actual 1930 U.S. census population of 123.2 million.

(b) The limiting population as $t \to \infty$ is $821.122/3.9 = 210.544$ million.

(c) Since the actual U.S. population in 200 was about 281 million — already exceeding the maximum population predicted by the logistic equation — we see that that this model did *not* continue to hold throughout the 20th century.

30. The equation is separable, so we have

$$\int \frac{dP}{P} = \int \beta_0 e^{-\alpha t}dt, \quad \text{so} \quad \ln P = -\frac{\beta_0}{\alpha}e^{-\alpha t} + C.$$

The initial condition $P(0) = P_0$ gives $C = \ln P_0 + \beta_0/\alpha$, so

$$P(t) = P_0 \exp\left[\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)\right].$$

31. If we substitute $P(0) = 10^6$ and $P'(0) = 3\times 10^5$ into the differential equation

$$P'(t) = \beta_0 e^{-\alpha t}P,$$

we find that $\beta_0 = 0.3$. Hence the solution given in Problem 30 is

$$P(t) = P_0 \exp[(0.3/\alpha)(1-e^{-\alpha t})].$$