## Chapter 1

# FUNDAMENTAL PRINCIPLES

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Problem 1.1 We are told that the scales of the two major terms in the two groups of terms in eq. (1.5) or eq. (1.6) are measured experimentally:

$$\underbrace{\frac{D\rho}{Dt}}_{\left(\sim u\,\frac{\partial \rho}{\partial x}\right)} + \underbrace{\rho \nabla \cdot \mathbf{v}}_{\left(\sim \rho\,\frac{\partial u}{\partial x}\right)} = 0 \quad \leftarrow \text{scales}$$

Therefore, if eq. (1.8) is to apply, then the first scale must be negligible,

$$u\,\frac{\partial \rho}{\partial x} < \rho\,\frac{\partial u}{\partial x}$$

in other words, the relationship between $\partial\rho/\partial x$ and $\partial u/\partial x$ must be

$$\frac{\partial \rho/\partial x}{\partial u/\partial x} < \frac{\rho}{u}$$

Note that "<" means "less than, in an order-of-magnitude sense", or "negligible with respect to". The scale analysis literature often uses "<<" to say the same thing; in the present treatment I use "<", because one sign is enough when we compare orders of magnitude (the use of multiple signs such as "<<" leads to the temptation to read too much in the length of the sign, for example, by using something like "<<<" to stress the word "negligible").

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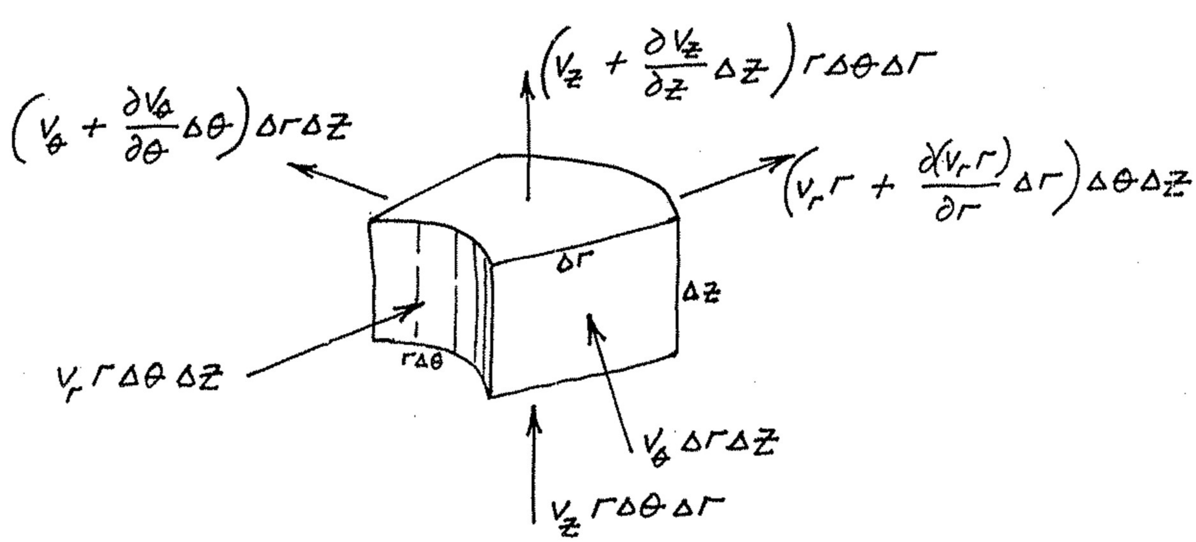
Problem 1.2. Consider the control volume $(\Delta r)(r\Delta\theta)(\Delta z)$ drawn around the point $(r,\theta,z)$ in Fig. 1.1. Around this control volume we write graphically eq. (1.1):



The term $\partial M_{cv}/\partial t$ is zero because $\rho$ is constant. Note also that the "in" arrows cancel, respectively, the leading terms of the "out" arrows. Dividing the three surviving terms by the control volume $r\Delta\theta\Delta r\Delta z$, we are left with

$$\frac{1}{r}\frac{\partial}{\partial r}(v_r r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0,$$

which is the same as eq. (1.9),

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

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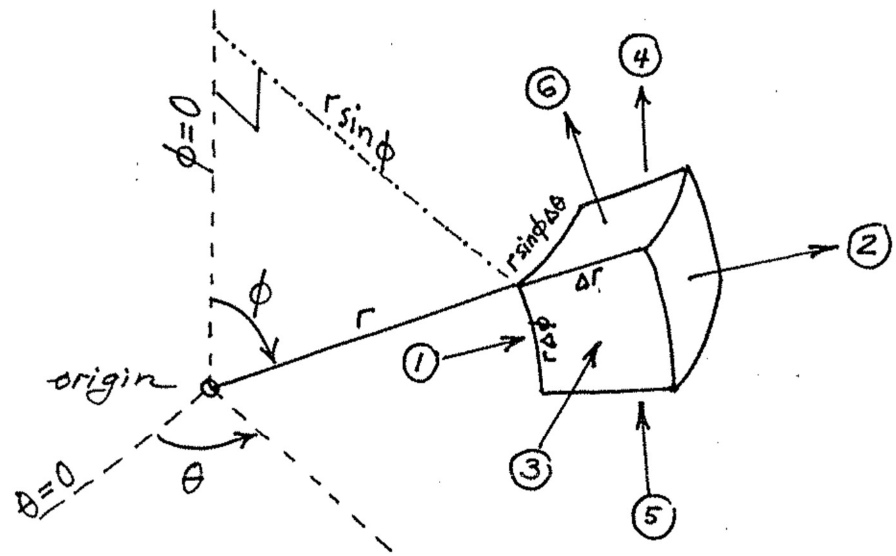
Problem 1.3. Consider the control volume described by the point $(r,\theta,\phi)$ in Fig. 1.1, as r, $\theta$ and $\phi$ change by $\Delta r$, $\Delta\theta$ and $\Delta\phi$, respectively




mass flowrates "in":

(1) $v_r r\Delta\phi \, r \sin\phi\Delta\theta$

(3) $v_\theta r\Delta\phi \, \Delta r$

(5) $v_\phi \Delta r \, r \sin\phi \, \Delta\theta$

mass flowrates "out":

(2) $\left(v_r r^2 + \frac{\partial}{\partial r}(v_r r^2)\Delta r\right) \sin\phi \, \Delta\phi \, \Delta\theta$

(4) $\left(v_\theta + \frac{\partial}{\partial \theta}(v_\theta)\Delta\theta\right) r\Delta\phi \, \Delta r$

(6) $\left(v_\phi \sin\phi + \frac{\partial}{\partial \phi}(v_\phi \sin\phi)\Delta\phi\right) r \, \Delta r \, \Delta\theta$

Since $\frac{\partial M_{cv}}{\partial t} = 0$, the six flowrates add up to

$$\frac{1}{r}\frac{\partial}{\partial r}(v_r r^2) + \frac{1}{\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}(v_\phi \sin\phi) = 0,$$

which is the same as eq. (1.10).

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Problem 1.4. The mass conservation equation for constant-density flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{or} \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

With this property in mind, the x-momentum equation (1.17) can be simplified:

$$\rho \frac{Du}{dt} = -\frac{\partial P}{\partial x} + \underbrace{\frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x}\right.}_{\mu \frac{\partial^2 u}{\partial x^2} - \underline{\mu \frac{\partial^2 v}{\partial x \partial y}}} \left. - \frac{2\mu}{3} \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{0}\right] + \underbrace{\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]}_{\mu \frac{\partial^2 u}{\partial y^2} + \underline{\mu \frac{\partial^2 v}{\partial y \partial x}}} + X$$

$$\longrightarrow 0 \longleftarrow$$

In conclusion, we obtain

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + X \qquad (1.18)$$

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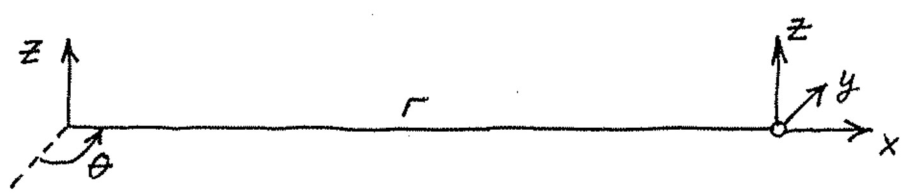
Problem 1.5. Graphically, the limit $r \to \infty$ and the transformation $\Delta r \to \Delta x$, $r\Delta\theta \to \Delta y$, $\Delta z \to \Delta z$ can be sketched as follows:




In eq. (1.9) we have

$$\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} + \underbrace{\frac{v_r}{r}}_{\infty} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

or, since $v_r \to u$, $v_\theta \to v$ and $v_z \to w$,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1.8)$$

The momentum equations (1.21) have the same property; for example, the r equation (1.21a) can be written as

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{\partial y} - \underbrace{\frac{v_\theta^2}{r}}_{\infty} + v_z \frac{\partial v_r}{\partial z}\right) =$$

$$= -\underbrace{\frac{\partial P}{\partial r}}_{\partial x} + \mu\left(\underbrace{\frac{\partial^2 v_r}{\partial r^2}}_{\partial x^2} + \underbrace{\frac{1}{r}}_{\infty}\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} - \underbrace{\frac{v_r}{r^2}}_{\infty^2} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{\partial y^2} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{\infty\ \partial y} + \frac{\partial^2 v_r}{\partial z^2}\right) + F_r,$$

in other words,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_r \qquad (1.19a)$$

The message of this exercise is that, through a simple transformation, the validity of equations in cylindrical coordinates may be tested based on the considerably more familiar Cartesian forms.

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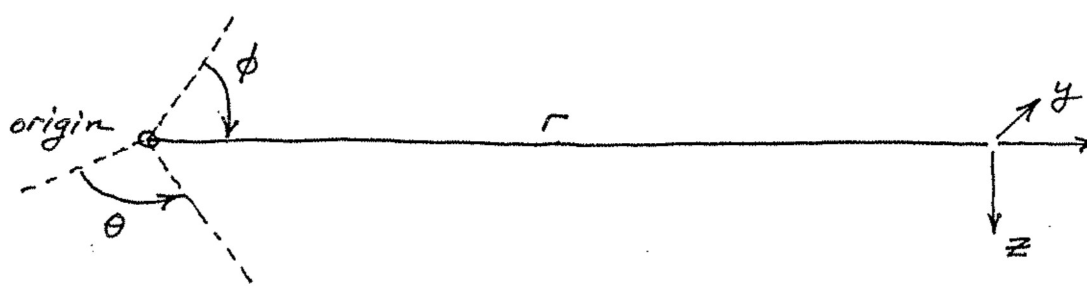
Problem 1.6. In the $r \to \infty$ limit, the spherical coordinates sketched in Fig. 1.1 become




in other words, $\Delta r \to \Delta x$, $r \sin\phi\, \Delta\theta \to \Delta y$ and $r\Delta\phi \to \Delta z$. The mass continuity equation (1.10) can be expanded as:

$$\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} + \underbrace{\frac{2 v_r}{r}}_{\infty} + \underbrace{\frac{1}{r}\frac{\partial v_\phi}{\partial \phi}}_{\partial z} + \underbrace{\frac{\cot\phi}{r}}_{\infty} + \underbrace{\frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta}}_{\partial y} = 0$$

Noting that $v_r \to u$, $v_\theta \to v$ and $v_\phi \to w$, the above equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \qquad (1.8)$$

Following the same procedure, the momentum equation (1.22a) reduces to eq. (1.19a).

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