Chapter 1

Probability Theory

"If any little problem comes your way, I shall be happy, if I can, to give you a hint or two as to its solution."

Sherlock Holmes
The Adventure of the Three Students

1.1 a. Each sample point describes the result of the toss (H or T) for each of the four tosses. So, for example THTT denotes T on 1st, H on 2nd, T on 3rd and T on 4th. There are $2^4 = 16$ such sample points.

b. The number of damaged leaves is a nonnegative integer. So we might use $S = \{0, 1, 2, \ldots\}$.

c. We might observe fractions of an hour. So we might use $S = \{t : t \geq 0\}$, that is, the half infinite interval $[0, \infty)$.

d. Suppose we weigh the rats in ounces. The weight must be greater than zero so we might use $S = (0, \infty)$. If we know no 10-day-old rat weighs more than 100 oz., we could use $S = (0, 100]$.

e. If n is the number of items in the shipment, then $S = \{0/n, 1/n, \ldots, 1\}$.

1.2 For each of these equalities, you must show containment in both directions.

a. $x \in A \backslash B \Leftrightarrow x \in A$ and $x \notin B \Leftrightarrow x \in A$ and $x \notin A \cap B \Leftrightarrow x \in A \backslash (A \cap B)$. Also, $x \in A$ and $x \notin B \Leftrightarrow x \in A$ and $x \in B^c \Leftrightarrow x \in A \cap B^c$.

b. Suppose $x \in B$. Then either $x \in A$ or $x \in A^c$. If $x \in A$, then $x \in B \cap A$, and, hence $x \in (B \cap A) \cup (B \cap A^c)$. Thus $B \subset (B \cap A) \cup (B \cap A^c)$. Now suppose $x \in (B \cap A) \cup (B \cap A^c)$. Then either $x \in (B \cap A)$ or $x \in (B \cap A^c)$. If $x \in (B \cap A)$, then $x \in B$. If $x \in (B \cap A^c)$, then $x \in B$. Thus $(B \cap A) \cup (B \cap A^c) \subset B$. Since the containment goes both ways, we have $B = (B \cap A) \cup (B \cap A^c)$. (Note, a more straightforward argument for this part simply uses the Distributive Law to state that $(B \cap A) \cup (B \cap A^c) = B \cap (A \cup A^c) = B \cap S = B$.)

c. Similar to part a).

d. From part b).
$A \cup B = A \cup [(B \cap A) \cup (B \cap A^c)] = A \cup (B \cap A) \cup A \cup (B \cap A^c) = A \cup [A \cup (B \cap A^c)] = A \cup (B \cap A^c)$.

1.3 a. $x \in A \cup B \;\Leftrightarrow\; x \in A$ or $x \in B \;\Leftrightarrow\; x \in B \cup A$
$x \in A \cap B \;\Leftrightarrow\; x \in A$ and $x \in B \;\Leftrightarrow\; x \in B \cap A$.

b. $x \in A \cup (B \cup C) \Leftrightarrow\; x \in A$ or $x \in B \cup C \Leftrightarrow\; x \in A \cup B$ or $x \in C \Leftrightarrow\; x \in (A \cup B) \cup C$.
(It can similarly be shown that $A \cup (B \cup C) = (A \cup C) \cup B$.)
$x \in A \cap (B \cap C) \Leftrightarrow\; x \in A$ and $x \in B$ and $x \in C \Leftrightarrow\; x \in (A \cap B) \cap C$.

c. $x \in (A \cup B)^c \Leftrightarrow\; x \notin A$ or $x \notin B \Leftrightarrow\; x \in A^c$ and $x \in B^c \Leftrightarrow\; x \in A^c \cap B^c$
$x \in (A \cap B)^c \Leftrightarrow\; x \notin A \cap B \Leftrightarrow\; x \notin A$ and $x \notin B \Leftrightarrow\; x \in A^c$ or $x \in B^c \Leftrightarrow\; x \in A^c \cup B^c$.

1.4 a. "A or B or both" is $A \cup B$. From Theorem 1.2.9b we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

b. "A or B but not both" is $(A \cap B^c) \cup (B \cap A^c)$. Thus we have

$$\begin{aligned} P((A \cap B^c) \cup (B \cap A^c)) &= P(A \cap B^c) + P(B \cap A^c) && \text{(disjoint union)} \\ &= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] && \text{(Theorem1.2.9a)} \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

c. "At least one of A or B" is $A \cup B$. So we get the same answer as in a).

d. "At most one of A or B" is $(A \cap B)^c$, and $P((A \cap B)^c) = 1 - P(A \cap B)$.

1.5 a. $A \cap B \cap C = \{$a U.S. birth results in identical twins that are female$\}$

b. $P(A \cap B \cap C) = \frac{1}{90} \times \frac{1}{3} \times \frac{1}{2}$

1.6

$$p_0 = (1-u)(1-w), \quad p_1 = u(1-w) + w(1-u), \quad p_2 = uw,$$

$$\begin{aligned} p_0 = p_2 &\Rightarrow u + w = 1 \\ p_1 = p_2 &\Rightarrow uw = 1/3. \end{aligned}$$

These two equations imply $u(1-u) = 1/3$, which has no solution in the real numbers. Thus, the probability assignment is not legitimate.

1.7 a.

$$P(\text{scoring } i \text{ points}) = \begin{cases} 1 - \frac{\pi r^2}{A} & \text{if } i = 0 \\ \frac{\pi r^2}{A}\left[\frac{(6-i)^2-(5-i)^2}{5^2}\right] & \text{if } i = 1, \ldots, 5. \end{cases}$$

b.

$$\begin{aligned} P(\text{scoring } i \text{ points}|\text{board is hit}) &= \frac{P(\text{scoring } i \text{ points} \cap \text{board is hit})}{P(\text{board is hit})} \\ P(\text{board is hit}) &= \frac{\pi r^2}{A} \\ P(\text{scoring } i \text{ points} \cap \text{board is hit}) &= \frac{\pi r^2}{A}\left[\frac{(6-i)^2-(5-i)^2}{5^2}\right] \qquad i = 1, \ldots, 5. \end{aligned}$$

Therefore,

$$P(\text{scoring } i \text{ points}|\text{board is hit}) = \frac{(6-i)^2-(5-i)^2}{5^2} \qquad i = 1, \ldots, 5$$

which is exactly the probability distribution of Example 1.2.7.

1.8 a. $P(\text{scoring exactly } i \text{ points}) = P(\text{inside circle } i) - P(\text{inside circle } i+1)$. Circle i has radius $(6-i)r/5$, so

$$P(\text{sscoring exactly } i \text{ points}) = \frac{\pi(6-i)^2 r^2}{5^2 \pi r^2} - \frac{\pi\left((6-(i+1))\right)^2 r^2}{5^2 \pi r^2} = \frac{(6-i)^2-(5-i)^2}{5^2}.$$

b. Expanding the squares in part a) we find $P(\text{scoring exactly } i \text{ points}) = \frac{11-2i}{25}$, which is decreasing in i.

c. Let $P(i) = \frac{11-2i}{25}$. Since $i \le 5$, $P(i) \ge 0$ for all i. $P(S) = P(\text{hitting the dartboard}) = 1$ by definition. Lastly, $P(i \cup j) = \text{area of } i \text{ ring} + \text{area of } j \text{ ring} = P(i) + P(j)$.

1.9 a. Suppose $x \in (\cup_\alpha A_\alpha)^c$, by the definition of complement $x \notin \cup_\alpha A_\alpha$, that is $x \notin A_\alpha$ for all $\alpha \in \Gamma$. Therefore $x \in A_\alpha^c$ for all $\alpha \in \Gamma$. Thus $x \in \cap_\alpha A_\alpha^c$ and, by the definition of intersection $x \in A_\alpha^c$ for all $\alpha \in \Gamma$. By the definition of complement $x \notin A_\alpha$ for all $\alpha \in \Gamma$. Therefore $x \notin \cup_\alpha A_\alpha$. Thus $x \in (\cup_\alpha A_\alpha)^c$.

b. Suppose $x \in (\cap_\alpha A_\alpha)^c$, by the definition of complement $x \notin (\cap_\alpha A_\alpha)$. Therefore $x \notin A_\alpha$ for some $\alpha \in \Gamma$. Therefore $x \in A_\alpha^c$ for some $\alpha \in \Gamma$. Thus $x \in \cup_\alpha A_\alpha^c$ and, by the definition of union, $x \in A_\alpha^c$ for some $\alpha \in \Gamma$. Therefore $x \notin A_\alpha$ for some $\alpha \in \Gamma$. Therefore $x \notin \cap_\alpha A_\alpha$. Thus $x \in (\cap_\alpha A_\alpha)^c$.

1.10 For $A_1, \ldots, A_n$

$$(i) \quad \left(\bigcup_{i=1}^n A_i\right)^c = \bigcap_{i=1}^n A_i^c \qquad (ii) \quad \left(\bigcap_{i=1}^n A_i\right)^c = \bigcup_{i=1}^n A_i^c$$

Proof of (i): If $x \in (\cup A_i)^c$, then $x \notin \cup A_i$. That implies $x \notin A_i$ for any i, so $x \in A_i^c$ for every i and $x \in \cap A_i$.

Proof of (ii): If $x \in (\cap A_i)^c$, then $x \notin \cap A_i$. That implies $x \in A_i^c$ for some i, so $x \in \cup A_i^c$.

1.11 We must verify each of the three properties in Definition 1.2.1.

a. (1) The empty set $\emptyset \in \{\emptyset, S\}$. Thus $\emptyset \in \mathcal{B}$. (2) $\emptyset^c = S \in \mathcal{B}$ and $S^c = \emptyset \in \mathcal{B}$. (3) $\emptyset \cup S = S \in \mathcal{B}$.

b. (1) The empty set $\emptyset$ is a subset of any set, in particular, $\emptyset \subset S$. Thus $\emptyset \in \mathcal{B}$. (2) If $A \in \mathcal{B}$, then $A \subset S$. By the definition of complementation, A^c is also a subset of S, and, hence, $A^c \in \mathcal{B}$. (3) If $A_1, A_2, \ldots \in \mathcal{B}$, then, for each i, $A_i \subset S$. By the definition of union, $\cup A_i \subset S$. Hence, $\cup A_i \in \mathcal{B}$.

c. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two sigma algebras. (1) $\emptyset \in \mathcal{B}_1$ and $\emptyset \in \mathcal{B}_2$ since $\mathcal{B}_1$ and $\mathcal{B}_2$ are sigma algebras. Thus $\emptyset \in \mathcal{B}_1 \cap \mathcal{B}_2$. (2) If $A \in \mathcal{B}_1 \cap \mathcal{B}_2$, then $A \in \mathcal{B}_1$ and $A \in \mathcal{B}_2$. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are both sigma algebra $A^c \in \mathcal{B}_1$ and $A^c \in \mathcal{B}_2$. Therefore $A^c \in \mathcal{B}_1 \cap \mathcal{B}_2$. (3) If $A_1, A_2, \ldots \in \mathcal{B}_1 \cap \mathcal{B}_2$, then $A_1, A_2, \ldots \in \mathcal{B}_1$ and $A_1, A_2, \ldots \in \mathcal{B}_2$. Therefore, since $\mathcal{B}_1$ and $\mathcal{B}_2$ are both sigma algebra, $\cup_{i=1}^\infty A_i \in \mathcal{B}_1$ and $\cup_{i=1}^\infty A_i \in \mathcal{B}_2$. Thus $\cup_{i=1}^\infty A_i \in \mathcal{B}_1 \cap \mathcal{B}_2$.

1.12 First write

$$\begin{aligned}
P\left(\bigcup_{i=1}^\infty A_i\right) &= P\left(\bigcup_{i=1}^n A_i \cup \bigcup_{i=n+1}^\infty A_i\right) \\
&= P\left(\bigcup_{i=1}^n A_i\right) + P\left(\bigcup_{i=n+1}^\infty A_i\right) \qquad (A_i\text{s are disjoint}) \\
&= \sum_{i=1}^n P(A_i) + P\left(\bigcup_{i=n+1}^\infty A_i\right) \qquad (\text{finite additivity})
\end{aligned}$$

Now define $B_k = \bigcup_{i=k}^\infty A_i$. Note that $B_{k+1} \subset B_k$ and $B_k \to \phi$ as $k \to \infty$. (Otherwise the sum of the probabilities would be infinite.) Thus

$$P\left(\bigcup_{i=1}^\infty A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^\infty A_i\right) = \lim_{n\to\infty}\left[\sum_{i=1}^n P(A_i) + P(B_{n+1})\right] = \sum_{i=1}^\infty P(A_i).$$

1.13 If A and B are disjoint, $P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{3}{4} = \frac{13}{12}$, which is impossible. More generally, if A and B are disjoint, then $A \subset B^c$ and $P(A) \leq P(B^c)$. But here $P(A) > P(B^c)$, so A and B cannot be disjoint.

1.14 If $S = \{s_1, \ldots, s_n\}$, then any subset of S can be constructed by either including or excluding s_i, for each i. Thus there are 2^n possible choices.

1.15 Proof by induction. The proof for $k = 2$ is given after Theorem 1.2.14. Assume true for k, that is, the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways. For $k+1$, the $k+1$th task can be done in n_{k+1} ways, and for each one of these ways we can complete the job by performing