

2

Signals and Systems

SIGNALS AND THEIR PROPERTIES

Solution 2.1

- (a) $\delta_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) = \sum_{m=-\infty}^{\infty} \delta(x - m) \cdot \sum_{n=-\infty}^{\infty} \delta(y - n)$, therefore it is a separable signal.
- (b) $\delta_l(x, y)$ is separable if $\sin(2\theta) = 0$. In this case, either $\sin \theta = 0$ or $\cos \theta = 0$, $\delta_l(x, y)$ is a product of a constant function in one axis and a 1-D *delta* function in another. But in general, $\delta_l(x, y)$ is not separable.
- (c) $e(x, y) = \exp[j2\pi(u_0x + v_0y)] = \exp(j2\pi u_0x) \cdot \exp(j2\pi v_0y) = e_{1D}(x; u_0) \cdot e_{1D}(y; v_0)$, where $e_{1D}(t; \omega) = \exp(j2\pi \omega t)$. Therefore, $e(x, y)$ is a separable signal.
- (d) $s(x, y)$ is a separable signal when $u_0v_0 = 0$. For example, if $u_0 = 0$, $s(x, y) = \sin(2\pi v_0y)$ is the product of a constant signal in x and a 1-D sinusoidal signal in y . But in general, when both u_0 and v_0 are nonzero, $s(x, y)$ is not separable.

Solution 2.2

- (a) Not periodic. $\delta(x, y)$ is non-zero only when $x = y = 0$.
- (b) Periodic.

$$\text{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n)$$

For arbitrary integers M and N

$$\begin{aligned} \text{comb}(x + M, y + N) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m + M, y - n + N) \\ &= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta(x - p, y - q) \text{ [let } p = m - M, q = n - N\text{]} \\ &= \text{comb}(x, y) \end{aligned}$$

So the smallest period is 1 in both x and y directions.

(c) Periodic. Let $f(x + T_x, y) = f(x, y)$, we have

$$\sin(2\pi x) \cos(4\pi y) = \sin(2\pi(x + T_x)) \cos(4\pi y).$$

Solve the above equation, we have $2\pi T_x = 2k\pi$ for arbitrary integer k . So the smallest period for x is $T_{x0} = 1$. Similarly we can find the smallest period for y is $T_{y0} = 1/2$.

(d) Periodic. Let $f(x + T_x, y) = f(x, y)$, we have

$$\sin(2\pi(x + y)) = \sin(2\pi(x + T_x + y)).$$

So the smallest period for x is $T_{x0} = 1$ and the smallest period for y is $T_{y0} = 1$.

(e) Not periodic. We can see this by contradiction. Suppose $f(x, y) = \sin(2\pi(x^2 + y^2))$ is periodic, then there exist some T_x such that $f(x + T_x, y) = f(x, y)$:

$$\begin{aligned} \sin(2\pi(x^2 + y^2)) &= \sin(2\pi((x + T_x)^2 + y^2)) \\ &= \sin(2\pi(x^2 + y^2 + 2xT_x + T_x^2)) \end{aligned}$$

In order for the above equation to hold, we must have $2xT_x + T_x^2 = k$ for some integer k . The solution for T_x depends on x . So $f(x, y) = \sin(2\pi(x^2 + y^2))$ is not periodic.

(f) Periodic. Let $f_d(m + M, n) = f_d(m, n)$, we have

$$\sin\left(\frac{\pi}{5}m\right) \cos\left(\frac{\pi}{5}n\right) = \sin\left(\frac{\pi}{5}(m + M)\right) \cos\left(\frac{\pi}{5}n\right)$$

Solve for M , we have $M = 10k$ for any integer k . The smallest period for both m and n is 10.

(g) Not periodic. Analog to part (f), by letting $f_d(m + M, n) = f_d(m, n)$, we have

$$\sin\left(\frac{1}{5}m\right) \cos\left(\frac{1}{5}n\right) = \sin\left(\frac{1}{5}(m + M)\right) \cos\left(\frac{1}{5}n\right)$$

The solution for M is $M = 10k\pi$. Since $f_d(m, n)$ is a discrete signal, its period must be an integer if it is periodic. There is no integer k that solves the equality for $M = 10k\pi$ for some M . So, $f_d(m, n) = \sin\left(\frac{1}{5}m\right) \cos\left(\frac{1}{5}n\right)$ is not periodic.

Solution 2.3

(a)

$$\begin{aligned} E_\infty(\delta_s) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_s^2(x, y) dx dy \\ &= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \int_{-X}^X \int_{-Y}^Y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) dx dy \\ &= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} (2\lfloor X \rfloor + 1)(2\lfloor Y \rfloor + 1) \\ &= \infty \end{aligned}$$

where $\lfloor X \rfloor$ is the greatest integer that is smaller than or equal to X .

$$\begin{aligned}
P_\infty(\delta_s) &= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y \delta_s^2(x, y) dx dy \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n) dx dy \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{(2[X] + 1)(2[Y] + 1)}{4XY} \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \left\{ \frac{4[X][Y]}{4XY} + \frac{2[X] + 2[Y]}{4XY} + \frac{1}{4XY} \right\} \\
P_\infty(\delta_s) &= 1
\end{aligned}$$

(b)

$$\begin{aligned}
E_\infty(\delta_l) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\delta(x \cos \theta + y \sin \theta - l)|^2 dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x \cos \theta + y \sin \theta - l) dx dy \\
&\stackrel{\textcircled{1}}{=} \begin{cases} \int_{-\infty}^{\infty} \frac{1}{|\sin \theta|} dx, & \sin \theta \neq 0 \\ \int_{-\infty}^{\infty} \frac{1}{|\cos \theta|} dy, & \cos \theta \neq 0 \end{cases} \\
E_\infty(\delta_l) &= \infty
\end{aligned}$$

① comes from the scaling property of the point impulse. The 1-D version of Eq. (2.8) in the text is $\delta(ax) = \frac{1}{|a|} \delta(x)$.

Suppose $\cos \theta \neq 0$,

$$\delta(x \cos \theta + y \sin \theta - l) = \frac{1}{|\cos \theta|} \delta\left(x + y \frac{\sin \theta}{\cos \theta} - \frac{l}{\cos \theta}\right)$$

therefore,

$$\int_{-\infty}^{\infty} \delta(x \cos \theta + y \sin \theta - l) dx = \frac{1}{|\cos \theta|}.$$

$$\begin{aligned}
P_\infty(\delta_l) &= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y |\delta(x \cos \theta + y \sin \theta - l)|^2 dx dy \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y \delta(x \cos \theta + y \sin \theta - l) dx dy
\end{aligned}$$

without loss of generality, assume $\theta = 0$ and $l = 0$, so we have $\sin \theta = 0$ and $\cos \theta = 1$. Therefore:

$$P_\infty(\delta_l) = \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y \delta(x) dx dy$$

$$\begin{aligned}
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-Y}^Y \left\{ \int_{-X}^X \delta(x) dx \right\} dy \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-Y}^Y 1 dx \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{2Y}{4XY} \\
&= \lim_{X \rightarrow \infty} \frac{1}{2X} \\
P_{\infty}(\delta_t) &= 0
\end{aligned}$$

(c)

$$\begin{aligned}
E_{\infty}(e) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\exp[j2\pi(u_0x + v_0y)]|^2 dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1 dx dy \\
E_{\infty}(e) &= \infty \\
P_{\infty}(e) &= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y |\exp[j2\pi(u_0x + v_0y)]|^2 dx dy \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y 1 dx dy \\
P_{\infty}(e) &= 1
\end{aligned}$$

(d)

$$\begin{aligned}
E_{\infty}(s) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2[2\pi(u_0x + v_0y)] dx dy \\
&\stackrel{\textcircled{2}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1 - \cos[4\pi(u_0x + v_0y)]}{2} dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} dx dy - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos[4\pi(u_0x + v_0y)]}{2} dx dy \\
E_{\infty}(s) &\stackrel{\textcircled{3}}{=} \infty
\end{aligned}$$

② comes from the trigonometric identity: $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

③ holds because the first integral goes to infinity. The absolute value of the second integral is bounded, although it does not converge as X and Y go to infinity.

$$\begin{aligned}
P_{\infty}(s) &= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-X}^X \int_{-Y}^Y \sin^2[2\pi(u_0x + v_0y)] dx dy \\
&= \lim_{X \rightarrow \infty} \lim_{Y \rightarrow \infty} \frac{1}{4XY} \int_{-Y}^Y \left\{ \int_{-X}^X \frac{1 - \cos[4\pi(u_0x + v_0y)]}{2} dx \right\} dy
\end{aligned}$$