# Chapter 2
# Introduction to Probability

## 2.2 Sample Spaces, Events, and Set Operations

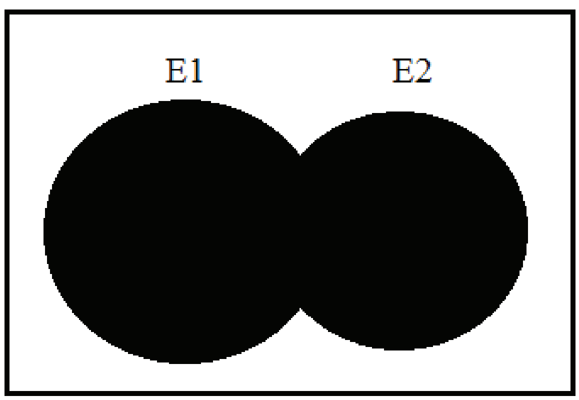
1. (a) The sample space is $\{(1,1),(1,2),\cdots,(1,6),\cdots,\cdots,(6,1),(6,2),\cdots,(6,6)\}$.

   (b) The sample space is $\{2,3,4,\cdots,12\}$.

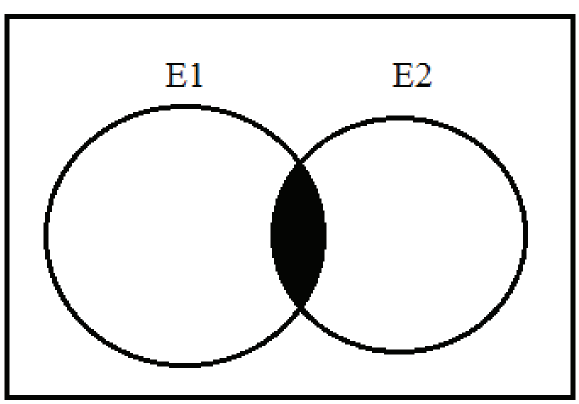
   (c) The sample space is $\{0,1,2,\cdots,6\}$.

   (d) The sample space is $\{1,2,3,\cdots\}$.

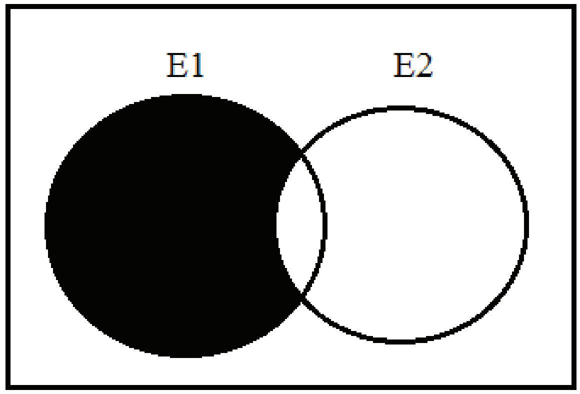
2. (a) The Venn diagram is shown as




   (b) The Venn diagram is shown as



(c)The Venn diagram is shown as



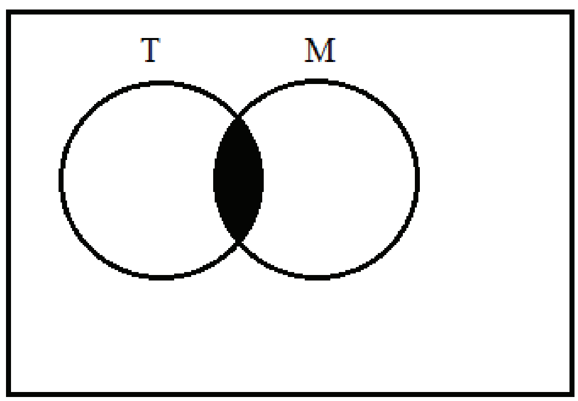

3. (a) The events are represented as

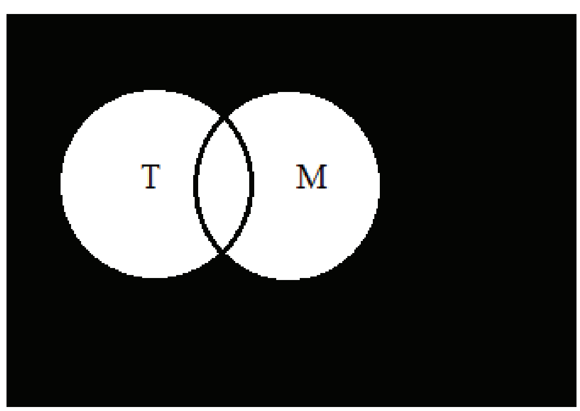
   (i) $T \cap M$

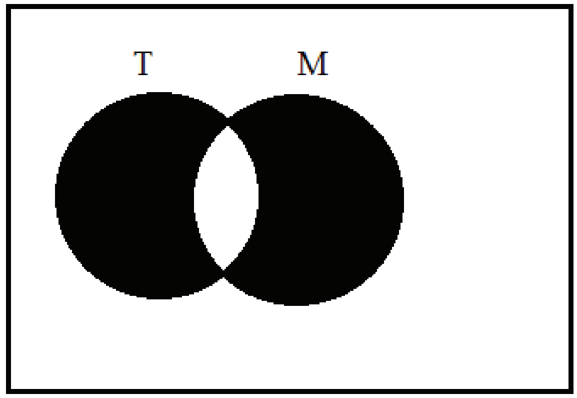
   (ii) $T^c \cap M^c$

   (iii) $(T \cap M^c) \cup (T^c \cap M)$

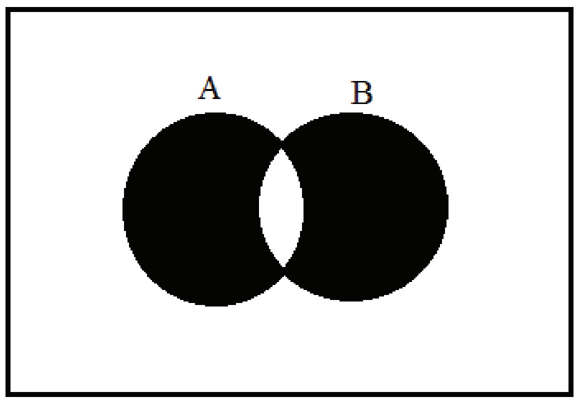
   (b) The Venn diagrams for part (a) are shown as



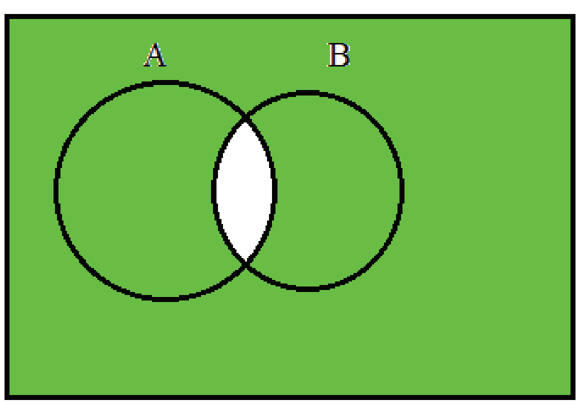



4. Both of the Venn diagrams should be similar to

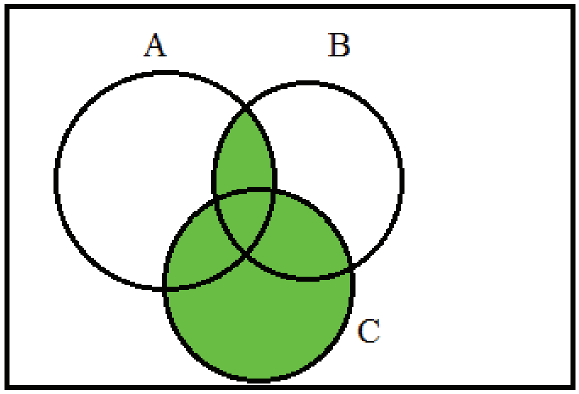



5. (a) $A^c = \{x | x \geq 75\}$, the component will last at least 75 time units.
   (b) $A \cap B = \{x | 53 < x < 75\}$, the component will last more than 53 units but less than 75 time units.
   (c) $A \cup B = S$, the sample space.
   (d) $(A - B) \cup (B - A) = \{x | x \geq 75 \text{ or } x \leq 53\}$, the component will last either at most 53 or at least 75 time units.

6. Both of the Venn diagrams should be similar to



7. Both of the Venn diagrams should be similar to




8. (a) Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$:

$$\begin{aligned} x \in (A-B)\cup(B-A) &\Leftrightarrow x \in A-B \text{ or } x \in B-A \\ &\Leftrightarrow x \in A \text{ but } x \notin B \text{ or } x \in B \text{ but } x \notin A \\ &\Leftrightarrow x \in A \text{ or } x \in B \text{ but not in both} \\ &\Leftrightarrow x \in A\cup B \text{ and } x \notin A\cap B \\ &\Leftrightarrow x \in (A\cup B)-(A\cap B). \end{aligned}$$

(b) Prove that $(A \cap B)^c = A^c \cup B^c$:

$$\begin{aligned} x \in (A\cap B)^c &\Leftrightarrow x \notin A\cap B \\ &\Leftrightarrow x \in A-B \text{ or } x \in B-A \text{ or } x \in (A\cup B)^c \\ &\Leftrightarrow [x \in A-B \text{ or } x \in (A\cup B)^c] \text{ or } [x \in B-A \text{ or } x \in (A\cup B)^c] \\ &\Leftrightarrow x \in B^c \text{ or } x \in A^c \Leftrightarrow x \in A^c \cup B^c. \end{aligned}$$

(c) Prove that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$:

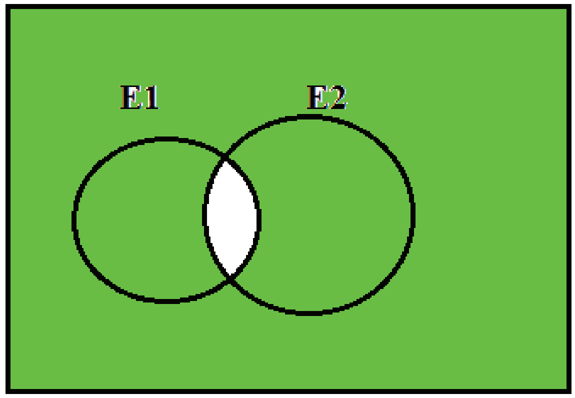
$$\begin{aligned} x \in (A\cap B)\cup C &\Leftrightarrow x \in A\cap B \text{ or } x \in C \\ &\Leftrightarrow x \in A \text{ and } x \in B \text{ or } x \in C \\ &\Leftrightarrow [x \in A \text{ or } x \in C] \text{ and } [x \in B \text{ or } x \in C] \\ &\Leftrightarrow x \in A\cup C \text{ and } x \in B\cup C \\ &\Leftrightarrow x \in (A\cup C)\cap(B\cup C). \end{aligned}$$

9. (a) The sample space is $S = \{(x_1, x_2, x_3, x_4, x_5)|x_i = 5.3, 5.4, 5.5, 5.6, 5.7, i = 1, 2, 3, 4, 5\}$. The size of the sample space is $5^5 = 3125$.

(b) The sample space is the collection of the distinct averages, $(x_1 + x_2 + x_3 + x_4 + x_5)/5$, formed from the elements of $S$. The R commands *s=c(5.3,5.4,5.5, 5.6,5.7); Sa= expand.grid(x1=s,x2=s,x3=s, x4=s,x5=s); length(table(rowSums(Sa)))* return 21 for the size of the sample space of the averages.

10. (a) The number of disks in $E_1$ is $5+16 = 21$, the number of disks in $E_2$ is $5+9 = 14$, and the number of disks in $E_3$ is $5 + 16 + 9 = 30$.

(b) Both of the Venn diagrams should be similar to




(c) $E_1 \cap E_2$ is the event that "the disk has low hardness and low shock absorption," $E_1 \cup E_2$ is the event that "the disk has low hardness or low shock absorption," $E_1 - E_2$ is the event that "the disk has low hardness but does not have low shock absorption," and $(E_1 - E_2) \cup (E_2 - E_1)$ is the event that "the disk has low hardness or low shock absorption but does not have low hardness and low shock absorption at the same time."

(d) The number of disks in $E_1 \cap E_2$ is 5, the number of disks in $E_1 \cup E_2$ is 30, the number of disks in $E_1 - E_2$ is 16, and the number of disks in $(E_1 - E_2) \cup (E_2 - E_1)$ is 25.

## 2.3 Experiments with Equally Likely Outcomes

1. $P(E_1) = 0.5$, $P(E_2) = 0.5$, $P(E_1 \cap E_2) = 0.3$, $P(E_1 \cup E_2) = 0.7$, $P(E_1 - E_2) = 0.2$, $P((E_1 - E_2) \cup (E_2 - E_1)) = 0.4$.

2. (a) If we select two wafers with replacement, then

   (i) The sample space for the experiment that records the doping type is {(n-type, n-type), (n-type, p-type), (p-type, n-type), (p-type, p-type)} and the corresponding probabilities are 0.25, 0.25, 0.25, and 0.25.

   (ii) The sample space for the experiment that records the number of n-type wafers is {0, 1, 2} and the corresponding probabilities are 0.25, 0.50, and 0.25.

   (b) If we select four wafers with replacement, then

   (i) The sample space for the experiment that records the doping type is all of the 4-component vectors, with each element being n-type or p-type. The size of the sample space can be found by the R commands

*G=expand.grid(W1=0:1,W2=0:1,W3=0:1,W4=0:1); length(G$W1)*

and the result is 16. The probability of each outcome is 1/16.

(ii) The sample space for the experiment that records the number of n-type wafer is {0, 1, 2, 3, 4}. The PMF is given by

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p(x)$ | 0.0625 | 0.2500 | 0.3750 | 0.2500 | 0.0625 |

(iii) The probability of at most one n-type wafer is 0.0625+0.25 = 0.3125.

3. $E_1 = \{6.8, 6.9, 7.0, 7.1\}$ and $E_2 = \{6.9, 7.0, 7.1, 7.2\}$. Thus, $P(E_1) = P(E_2) = 4/5$. $E_1 \cap E_2 = \{6.9, 7.0, 7.1\}$ and $P(E_1 \cap E_2) = 3/5$. $E_1 \cup E_2 = S$ and $P(E_1 \cup E_2) = 1$. $E_1 - E_2 = \{6.8\}$ and $P(E_1 - E_2) = 1/5$. Finally, $(E_1 - E_2) \cup (E_2 - E_1) = \{6.8, 7.2\}$, so $P(E_1 - E_2) \cup (E_2 - E_1)) = 2/5$.

4. (a) If the water PH level is measured over the next two irrigations, then

(i) The sample space is $S = \{(x_1, x_2) : x_1 = 6.8, 6.9, 7.0, 7.1, 7.2, \text{and } x_2 = 6.8, 6.9, 7.0, 7.1, 7.2\}$. The size of the sample space is 25.

(ii) The sample space of the experiment that records the average of the two PH measurements is $S = \{6.8, 6.85, 6.9, 6.95, 7, 7.05, 7.1, 7.15, 7.2\}$ and the PMF is

| $x$ | 6.8 | 6.85 | 6.9 | 6.95 | 7 | 7.05 | 7.1 | 7.15 | 7.2 |
|---|---|---|---|---|---|---|---|---|---|
| $p(x)$ | 0.04 | 0.08 | 0.12 | 0.16 | 0.20 | 0.16 | 0.12 | 0.08 | 0.04 |

(b) The probability mass function of the experiment that records the average of the PH measurements taken over the next five irrigations is

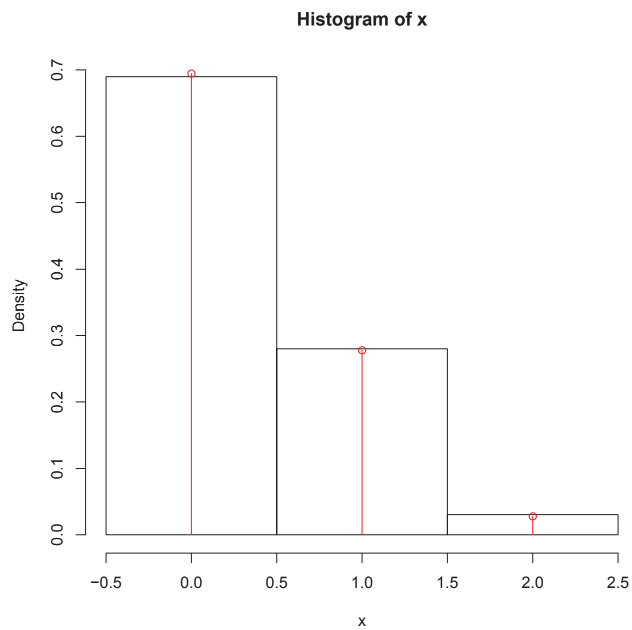
| $x$ | 6.8 | 6.82 | 6.84 | 6.86 | 6.88 | 6.9 | 6.92 |
|---|---|---|---|---|---|---|---|
| $p(x)$ | 0.00032 | 0.00160 | 0.00480 | 0.01120 | 0.02240 | 0.03872 | 0.05920 |
| $x$ | 6.94 | 6.96 | 6.98 | 7 | 7.02 | 7.04 | 7.06 |
| $p(x)$ | 0.08160 | 0.10240 | 0.11680 | 0.12192 | 0.11680 | 0.10240 | 0.08160 |
| $x$ | 7.08 | 7.1 | 7.12 | 7.14 | 7.16 | 7.18 | 7.2 |
| $p(x)$ | 0.05920 | 0.03872 | 0.02240 | 0.01120 | 0.00480 | 0.00160 | 0.00032 |

5. (a) The R command is *sample(0:2, size =10, replace=T, prob=pr)* and the following gives one possible result: 1, 0, 0, 1, 1, 0, 1, 0, 0, 0.

(b) The relative frequency based on 10,000 replications is

| 0 | 1 | 2 |
|---|---|---|
| 0.6897 | 0.2799 | 0.0304 |

(c) The histogram of the relative frequencies and line graph of the probability mass function is given on the next page.




This figure shows that all relative frequencies are good approximations to corresponding probabilities and we have empirical confirmation of the limiting relative frequency interpretation of probability.

6. (a) The number of ways to finish the test is $2^5 = 32$.

   (b) The sample space for the experiment that records the test score is $S = \{0, 1, 2, 3, 4, 5\}$.

   (c) The PMF of $X$ is given by

| $X$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p(x)$ | 0.03125 | 0.15625 | 0.31250 | 0.31250 | 0.15625 | 0.03125 |

7. The number of assignments is

$$\binom{4}{1, 1, 1, 1} = 24.$$

8. The probability is

$$\frac{26^2 \times 10^3}{26^3 \times 10^4} = 0.0038.$$

9. (a) The number of possible committees is

$$\binom{12}{4} = 495.$$

(b) The number of committees consisting of 2 biologists, 1 chemist, and 1 physicist is

$$\binom{5}{2}\binom{4}{1}\binom{3}{1} = 120.$$

(c) The probability is $120/495 = 0.2424$.

10. (a) The number of possible selections is

$$\binom{10}{5} = 252.$$

(b) The number of divisions of the 10 players into two teams of 5 is $252/2 = 126$.

(c) The number of handshakes is

$$\binom{12}{2} = 66.$$

11. (a) In order to go from the lower left corner to the upper right corner, we need to totally move 8 steps, with 4 steps to the right and 4 steps upwards. Thus, the total number of paths is

$$\binom{8}{4} = 70.$$

(b) We decompose the move as two stages: stage 1 is from lower left corner to circled point, which needs 5 steps with 3 steps to the right and 2 steps upwards; stage 2 is from the circled point to the upper right corner, which needs 3 steps with 1 step to the right and 2 steps upwards. Thus, the total number of paths passing the circled point is

$$\binom{5}{3}\binom{3}{1} = 30.$$

(c) The probability is $30/70 = 3/7$.

12. (a) In order to keep the system working, the nonfunctioning antennas cannot be next to each other. There are 8 antennas functioning; thus, the 5 nonfunctioning antennas must be in the 9 spaces created by the 8 functioning antennas. The number of arrangements is

$$\binom{9}{5} = 126.$$

(b) The total number of the 5 nonfunctioning antennas is $\binom{13}{5} = 1287$. Thus, the required probability is $126/1287 = 0.0979$.

13. (a) The total number of selections is

$$\binom{15}{5} = 3003.$$

(b) The number of selections containing three defective buses is

$$\binom{4}{3}\binom{11}{2} = 220.$$

(c) The asked probability is $220/3003 = 0.07326$.

(d) The probability all five buses are free of the defect is calculated as

$$\frac{\binom{11}{5}}{\binom{15}{5}} = 0.1538.$$

14. (a) The number of samples of size five is $\binom{30}{5} = 142506$.

(b) The number of samples that include two of the six tagged moose is $\binom{6}{2}\binom{24}{3} = 30360$.

(c)

(i) The probability is

$$\frac{\binom{6}{2}\binom{24}{3}}{\binom{30}{5}} = \frac{30360}{142506} = 0.213.$$

(ii) The probability is

$$\frac{\binom{24}{5}}{\binom{30}{5}} = \frac{30360}{142506} = 0.298.$$

15. (a) The probability is

$$\frac{48}{\binom{52}{5}} = 1.85 \times 10^{-5}.$$

(b) The probability is

$$\frac{\binom{4}{2}\binom{4}{2}44}{\binom{52}{5}} = 0.00061.$$

(c) The probability is

$$\frac{\binom{4}{3}\binom{12}{2}4^2}{\binom{52}{5}} = 0.0016.$$

16. The total number of possible assignments is $\binom{10}{2,2,2,2,2} = 113400$.

17. (a) There are $3^{15} = 14348907$ ways to classify the next 15 shingles in tow three grades.

(b) The number of ways to classify into three high, five medium and seven low grades is

$$\binom{15}{3,5,7} = 360360.$$

(c) The probability is $360360/14348907 = 0.0251$.

18. (a)

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

(b)

$$\begin{aligned}(a^2+b)^4 &= \binom{4}{0}(a^2)^0 b^{4-0} + \binom{4}{1}(a^2)^1 b^{4-1} + \binom{4}{2}(a^2)^2 b^{4-2} + \binom{4}{3}(a^2)^3 b^{4-3} \\ &\quad + \binom{4}{4}(a^2)^4 b^{4-4} \\ &= b^4 + 4a^2b^3 + 6a^4b^2 + 4a^6b + a^8\end{aligned}$$

19.

$$\begin{aligned}(a_1^2 + 2a_2 + a_3)^3 &= \binom{3}{0,0,3}(a_1^2)^0(2a_2)^0 a_3^3 + \binom{3}{0,1,2}(a_1^2)^0(2a_2)^1 a_3^2 \\ &\quad + \binom{3}{0,2,1}(a_1^2)^0(2a_2)^2 a_3^1 + \binom{3}{0,3,0}(a_1^2)^0(2a_2)^3 a_3^0 \\ &\quad + \binom{3}{1,0,2}(a_1^2)^1(2a_2)^0 a_3^2 + \binom{3}{1,1,1}(a_1^2)^1(2a_2)^1 a_3^1 \\ &\quad + \binom{3}{1,2,0}(a_1^2)^1(2a_2)^2 a_3^0 + \binom{3}{2,0,1}(a_1^2)^2(2a_2)^0 a_3^1 \\ &\quad + \binom{3}{2,1,0}(a_1^2)^2(2a_2)^1 a_3^0 + \binom{3}{3,0,0}(a_1^2)^3(2a_2)^0 a_3^0 \\ &= a_3^3 + 6a_2a_3^2 + 12a_2^2a_3 + 8a_2^3 + 3a_1^2a_3^2 + 12a_1^2a_2a_3 \\ &\qquad + 12a_1^2a_2^2 + 3a_1^4a_3 + 6a_1^4a_2 + a_1^6.\end{aligned}$$

## 2.4 Axioms and Properties of Probabilities

1. $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.37 + 0.23 - 0.47 = 0.13$.

2. (a) $P(A_1) = \cdots = P(A_m) = 1/m$.

   (b) If $m = 8$, $P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1) + P(A_2) + P(A_3) + P(A_4) = 4 \times 1/m = 1/2$.

3. (a) The R commands are

*t = c(50,51,52,53); G=expand.grid(X1=t,X2=t,X3=t); attach(G)*
*table((X1+X2+X3)/3)/length(X1)*

The resulting PMF is

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 50 | 50.33 | 50.67 | 51 | 51.33 |
| $p(x)$ | 0.015625 | 0.046875 | 0.093750 | 0.156250 | 0.187500 |
| $x$ | 51.67 | 52 | 52.33 | 52.67 | 53 |
| $p(x)$ | 0.187500 | 0.156250 | 0.093750 | 0.046875 | 0.015625 |

(b) The probability that the average gas mileage is at least 52 MPG is $0.156250 + 0.093750 + 0.046875 + 0.015625 = 0.3125$.

4. (a)

   (i) $E_1 = \{5, 6, 7, 8, 9, 10, 11, 12\}$. $P(E_1) = 4/36+5/36+6/36+5/36+4/36+3/36+2/36+1/36 = 5/6$.

   (ii) $E_2 = \{2, 3, 4, 5, 6, 7, 8\}$. $P(E_2) = 1/36 + 2/36 + 3/36 + 4/36 + 5/36 + 6/36 + 5/36 = 13/18$.

   (iii) $E_3 = E_1 \cup E_2 = \{2, \cdots, 12\}$, $P(E_3) = 1$. $E_4 = E_1 - E_2 = \{9, 10, 11, 12\}$, $P(E_4) = 4/36+3/36+2/36+1/36 = 5/18$. $E_5 = E_1^c \cap E_2^c = \emptyset$, $P(E_5) = 0$.

   (b) $P(E_3) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 30/36+26/36-(4/36+5/36 + 6/36 + 5/36) = 1$.

   (c) $P(E_5) = P(E_1^c \cap E_2^c) = P((E_1 \cup E_2)^c) = P(E_3^c) = 1 - P(E_3) = 1 - 1 = 0$.

5. (a)

   (i) $E_1 = \{(> 3, V), (< 3, V)\}$, $P(E_1) = 0.25 + 0.3 = 0.55$.

   (ii) $E_2 = \{(< 3, V), (< 3, D), (< 3, F)\}$, $P(E_2) = 0.3 + 0.15 + 0.13 = 0.58$.

   (iii) $E_3 = \{(> 3, D), (< 3, D)\}$, $P(E_3) = 0.1 + 0.15 = 0.25$.

   (iv) $E_4 = \{(> 3, V), (< 3, V), (< 3, D), (< 3, F)\}$, $P(E_4) = 0.25 + 0.3 + 0.15+0.13 = 0.83$. $E_5 = \{(> 3, V), (< 3, V), (< 3, D), (< 3, F), (> 3, D)\}$, $P(E_5) = 0.25 + 0.3 + 0.15 + 0.13 + 0.1 = 0.93$.

   (b) $P(E_4) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 0.55+0.58-0.3 = 0.83$.

   (c) $P(E_5) = P(E_1 \cup E_2 \cup E_3) = P(E_1)+P(E_2)+P(E_3)-P(E_1 \cap E_2)-P(E_1 \cap E_3)-P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) = 0.55+0.58+0.25-0.3-0-0.15+0 = 0.93$.

6. (a) The probability that, in any given hour, only machine A produces a batch with no defects is

$$P(E_1 \cap E_2^c) = P(E_1) - P(E_1 \cap E_2) = 0.95 - 0.88 = 0.07.$$

   (b) The probability, in that any given hour, only machine B produces a batch with no defects is

$$P(E_2 \cap E_1^c) = P(E_2) - P(E_1 \cap E_2) = 0.92 - 0.88 = 0.04.$$

(c) The probability that exactly one machine produces a batch with no defects is

$$P((E_1 \cap E_2^c) \cup (E_2 \cap E_1^c)) = P(E_1 \cap E_2^c) + P(E_2 \cap E_1^c) = 0.07 + 0.04 = 0.11.$$

(d) The probability that at least one machine produces a batch with no defects is

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 0.95 + 0.92 - 0.88 = 0.99.$$

7. The probability that at least one of the machines will produce a batch with no defectives is

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) \\ &\quad - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \\ &= 0.95 + 0.92 + 0.9 - 0.88 - 0.87 - 0.85 + 0.82 = 0.99. \end{aligned}$$

8. (a)

   (i) $P(E_1) = 0.10 + 0.04 + 0.02 + 0.08 + 0.30 + 0.06 = 0.6.$
   (ii) $P(E_2) = 0.10 + 0.08 + 0.06 + 0.04 + 0.30 + 0.14 = 0.72.$
   (iii) $P(E_1 \cap E_2) = 0.1 + 0.04 + 0.08 + 0.3 = 0.52.$

   (b) The probability mass function for the experiment that records only the online monthly volume of sales category is given as

| Online Sales | 0 | 1 | 2 |
|---|---|---|---|
| Probability | 0.16 | 0.44 | 0.4 |

9. Let

$$E_4 = \{\text{at least two of the original four components work}\},$$

and

$$\begin{aligned} E_5 = &\{\text{at least three of the original four components work}\} \\ &\cup \{\text{two of the original four components work} \\ &\quad \text{and the additional component works}\}. \end{aligned}$$

Then $E_4 \not\subset E_5$ because

$$\begin{aligned} B = &\{\text{exactly two of the original four components work} \\ &\quad \text{and the additional component does not work}\}, \end{aligned}$$

which is part of $E_4$, is not in $E_5$. Thus, $E_4 \not\subset E_5$ and, hence, it is not necessarily true that $P(E_4) \leq P(E_5)$.

10. (a) If two dice are rolled, there are a total of 36 possibilities, among which 6 are tied. Hence, the probability of tie is $6/36 = 1/6$.

(b) By symmetry of the game $P(A \text{ wins}) = P(B \text{ wins})$ and $P(A \text{ wins}) + P(B \text{ wins}) + P(\text{tie}) = 1$. Using the result of (a), we can solve that $P(A \text{ wins}) = P(B \text{ wins}) = 5/12$.

11. (a) $A > B = \{\text{die A results in 4}\}$, $B > C = \{\text{die C results in 2}\}$,
$C > D = \{\text{die C results in 6, or die C results in 2 and die D results in 1}\}$,
$D > A = \{\text{die D results in 5, or die D results in 1 and die A results in 0}\}$.

(b) $P(A > B) = 4/6$, $P(B > C) = 4/6$, $P(C > D) = 4/6$, $P(D > A) = 4/6$.

12. (a) If the participant sticks with the original choice, the probability of winning the big prize is 1/3.

(b) If the participant chooses to switch his/her choice, the probability of winning the big prize is 2/3. This is because that if the first choice was actually the minor prize, then, after switching, he/she will win the big prize. If the first choice was actually the big prize, after switching he/she will win the minor prize. While the first choice being the minor prize has a probability of 2/3, consequently, switching leads to a probability of 2/3 to win the big prize.

13. To prove that $P(\emptyset) = 0$, let $E_1 = S$ and $E_i = \emptyset$ for $i = 2, 3, \cdots$. Then $E_1, E_2, \cdots$ is a sequence of disjoint events. By Axiom 3, we have

$$P(S) = P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) = P(S) + \sum_{i=2}^{\infty} P(\emptyset),$$

which implies that $\sum_{i=2}^{\infty} P(\emptyset) = 0$, and we must have $P(\emptyset) = 0$.

To prove (2) of Proposition 2.4-1, let $E_i = \emptyset$ for $i = n+1, n+2, \cdots$. Then $E_1, E_2, \cdots$ is a sequence of disjoint events and $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{n} E_i$. By Axiom 3, we have

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} E_i\right) &= P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{n} P(E_i) + \sum_{i=n+1}^{\infty} P(E_i) \\ &= \sum_{i=1}^{n} P(E_i) + \sum_{i=n+1}^{\infty} P(\emptyset) = \sum_{i=1}^{n} P(E_i), \end{aligned}$$

which is what to be proved.

## 2.5 Conditional Probability

1. The probability can be calculated as

$$P(>3 | > 2) = \frac{P((>3) \cap (>2))}{P(>2)} = \frac{P(>3)}{P(>2)} = \frac{(1+3)^{-2}}{(1+2)^{2}} = 9/16.$$

2. Let $B = \{\text{system re-evaluation occurs}\}$ and $C = \{\text{a component is individually replaced}\}$. Consider a new experiment with reduced sample space $A = B \cup C$. The desired probability is the probability of $B$ in this new experiment, which is calculated as

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(B) + P(C)} = \frac{0.005}{0.005 + 0.1} = 0.048.$$

3. (a) $P(A) = 0.132 + 0.068 = 0.2$.

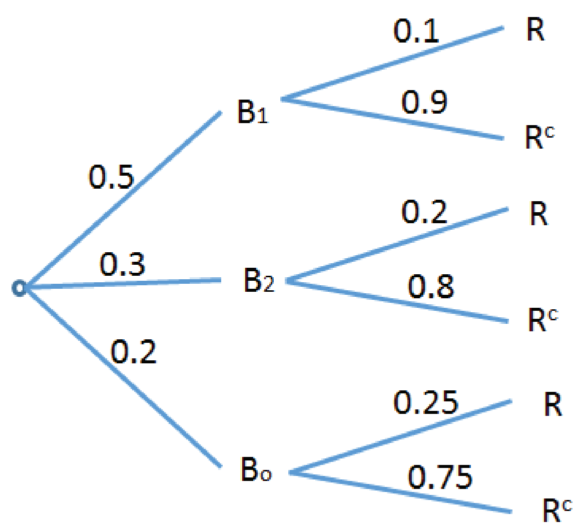
   (b) $P(A \cap B) = 0.132$, thus $P(B|A) = P(A \cap B)/P(A) = 0.132/0.2 = 0.66$.

   (c) $P(X = 1) = 0.2$, $P(X = 2) = 0.3$, $P(X = 3) = 0.5$.

4. We let $B_1, B_2, B_O$ be the event that the TV is brand 1, brand 2, and other brand, respectively. Let $R$ be the event that the TV needs warranty repair.

   (a) $P(B_1 \cap R) = P(B_1)P(R|B_1) = 0.5 \times 0.1 = 0.05$.

   (b) The tree diagram is

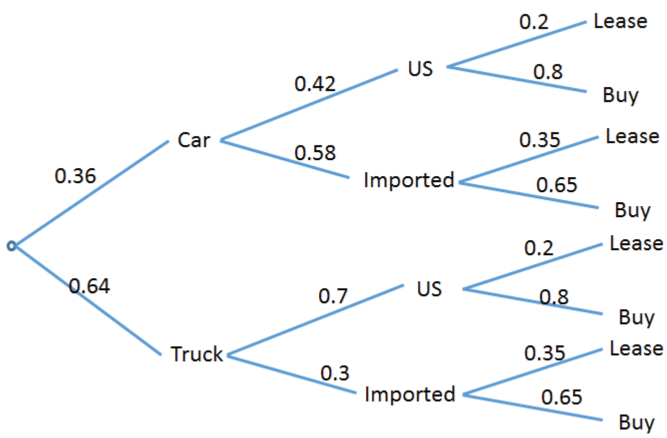



   (c) Using the diagram

$$\begin{aligned} P(R) &= P(B_1)P(R|B_1) + P(B_2)P(R|B_2) + P(B_O)P(R|B_O) \\ &= 0.5 \times 0.1 + 0.3 \times 0.2 + 0.2 \times 0.25 = 0.16. \end{aligned}$$

5. (a) The probability is $0.36 \times 0.58 = 0.2088$.

   (b) The tree diagram is




(c) By using the tree diagram, the probability that the next consumer will lease his/her vehicle is

$$0.36 \times 0.42 \times 0.2 + 0.36 \times 0.58 \times 0.35 + 0.64 \times 0.7 \times 0.2 + 0.64 \times 0.3 \times 0.35 = 0.26.$$

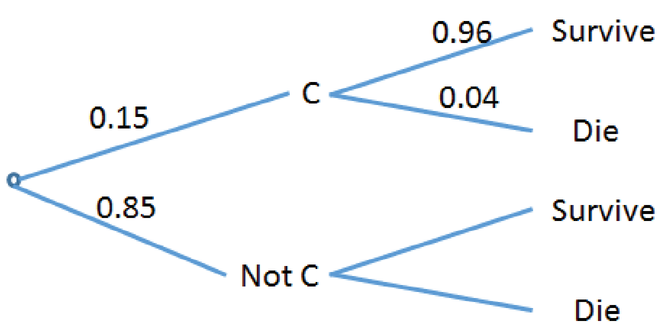
6. (a) $P(\text{no defect} \cap A) = P(\text{no defect}|A)P(A) = 0.99 \times 0.3 = 0.297.$

   (b) $P(\text{no defect} \cap B) = P(\text{no defect}|B)P(B) = 0.97 \times 0.3 = 0.291$, and $P(\text{no defect} \cap C) = P(\text{no defect}|C)P(C) = 0.92 \times 0.3 = 0.276.$

   (c) $P(\text{no defect}) = P(\text{no defect} \cap A) + P(\text{no defect} \cap B) + P(\text{no defect} \cap C) = 0.297 + 0.291 + 0.276 = 0.864.$

   (d) $P(C|\text{no defect}) = P(\text{no defect} \cap C)/P(\text{no defect}) = 0.276/0.864 = 0.3194.$

7. (a) The tree diagram is




   (b) From the given information, we have

$$P(\text{survive}) = 0.15 \times 0.96 + 0.85 \times P(\text{Survive}|\text{Not C-section}) = 0.98.$$

   Solving this equation gives us $P(\text{Survive}|\text{Not C-section}) = 0.984.$

8. Let $B$ be the event that the credit card holds monthly balance, then $P(B) = 0.7$ and $P(B^c) = 0.3$. Let $L$ be the event that the card holder has annual income less than \$20,000, then $P(L|B) = 0.3$ and $P(L|B^c) = 0.2$.

(a) $P(L) = P(L|B)P(B) + P(L|B^c)P(B^c) = 0.3 \times 0.7 + 0.2 \times 0.3 = 0.27$.

(b) $P(B|L) = P(L|B)P(B)/P(L) = 0.3 \times 0.7/0.27 = 0.778$.

9. Let $A$ be the event that the plant is alive and let $W$ be the roommate waters it. Then, from the given information, $P(W) = 0.85$ and $P(W^c) = 0.15$; $P(A|W) = 0.9$ and $P(A|W^c) = 0.2$.

   (a) $P(A) = P(A|W)P(W) + P(A|W^c)P(W^c) = 0.9 \times 0.85 + 0.2 \times 0.15 = 0.795$.

   (b) $P(W|A) = P(A|W)P(W)/P(A) = 0.9 \times 0.85/0.795 = 0.962$.

10. Let $D_1$ be the event that the first is defective and $D_2$ the event that the second is defective.

   (a) $P(\text{no defective}) = P(D_1^c \cap D_2^c) = P(D_2^c|D_1^c)P(D_1^c) = 6/9 \times 7/10 = 0.467$.

   (b) $X$ can be 0, 1, or 2. We already calculated $P(X = 0) = P(\text{no defective}) = 0.467$. $P(X = 2) = P(D_1 \cap D_2) = P(D_2|D_1)P(D_1) = 2/9 \times 3/10 = 0.067$. Thus, $P(X = 1) = 1 - P(X = 0) - P(X = 2) = 0.466$.

   (c) $P(D_1|X = 1) = P(D_1 \cap D_2^c)/P(X = 1) = P(D_2^c|D_1)P(D_1)/P(X = 1) = 7/9 \times 0.3/0.466 = 0.5$.

11. Let $L_1, L_2, L_3, L_4$ be the event that the radar traps are operated at the 4 locations, then $P(L_1) = 0.4, P(L_2) = 0.3, P(L_3) = 0.2, P(L_4) = 0.3$. Let $S$ be the person speeding to work, then $P(S|L_1) = 0.2, P(S|L_2) = 0.1, P(S|L_3) = 0.5, P(S|L_4) = 0.2$.

   (a) $P(S) = P(S|L_1)P(L_1) + P(S|L_2)P(L_2) + P(S|L_3)P(L_3) + P(S|L_4)P(L_4) = 0.2 \times 0.4 + 0.1 \times 0.3 + 0.5 \times 0.2 + 0.2 \times 0.3 = 0.27$.

   (b) $P(L_2|S) = P(S|L_2)P(L_2)/P(S) = 0.1 \times 0.3/0.27 = 0.11$.

12. Let $D$ be the event that the aircraft will be discovered, and $E$ be the event that it has an emergency locator. From the problem, $P(D) = 0.7$ and $P(D^c) = 0.3$; $P(E|D) = 0.6$ and $P(E|D^c) = 0.1$.

   (a) $P(E \cap D^c) = P(E|D^c)P(D^c) = 0.1 \times 0.3 = 0.03$.

   (b) $P(E) = P(E|D^c)P(D^c) + P(E|D)P(D) = 0.1 \times 0.3 + 0.6 \times 0.7 = 0.45$.

   (c) $P(D^c|E) = P(E \cap D^c)/P(E) = 0.03/0.45 = 0.067$.

13.

$$\begin{aligned}\text{R.H.S.} &= P(E_1)\frac{P(E_1 \cap E_2)}{P(E_1)}\frac{P(E_1 \cap E_2 \cap E_3)}{P(E_1 \cap E_2)} \cdots \frac{P(E_1 \cap E_2 \cap \cdots \cap E_{n-1} \cap E_n)}{P(E_1 \cap E_2 \cap \cdots \cap E_{n-1})} \\ &= P(E_1 \cap E_2 \cap \cdots \cap E_{n-1} \cap E_n) = \text{L.H.S.}\end{aligned}$$

## 2.6 Independent Events

1. From the given information $P(E_2) = 2/10$ and $P(E_2|E_1) = 2/9$, thus $P(E_2) \neq P(E_2|E_1)$. Consequently, $E_1$ and $E_2$ are not independent.

2. We can calculate from the table that $P(X = 1) = 0.132 + 0.068 = 0.2$ and $P(Y = 1) = 0.132 + 0.24 + 0.33 = 0.702$, thus $P(X = 1)P(Y = 1) = 0.2 \times 0.702 = 0.1404 \neq 0.132 = P(X = 1, Y = 1)$. Thus, the events $[X = 1]$ and $[Y = 1]$ are not independent.

3. (a) The probability is $0.9^{10} = 0.349$.

   (b) The probability is $0.1 \times 0.9^9 = 0.0387$.

   (c) The probability is $10 \times 0.1 \times 0.9^9 = 0.387$

4. A total of 8 fuses being inspected means that the first 7 are not defective and the 8th is defective, thus the probability is calculated as $0.99^7 \times 0.01 = 0.0093$.

5. Assuming the cars assembled on each line are independent, also assume that the two lines are independent. We have

   (a) The probability of finding zero nonconformance in the sample from line 1 is $0.8^4 = 0.410$.

   (b) The probability of finding zero nonconformance in the sample from line 1 is $0.9^3 = 0.729$.

   (c) The probability is $0.8^4 \times 0.9^3 = 0.2986$.

6. Yes. By the given information, $P(T|M) = P(T)$, we see that $T$ and $M$ are independent. Thus, $T$ and $F = M^c$ are also independent; that is, $P(T|F) = P(T)$.

7. (a) The completed table is given as

| | Football | Basketball | Track | Total |
|---|---|---|---|---|
| Male | 0.3 | 0.22 | 0.13 | 0.65 |
| Female | 0 | 0.28 | 0.07 | 0.35 |
| Total | 0.3 | 0.5 | 0.2 | 1 |

   (b) Let $B$ be the event that the student prefers basketball, then $P(F|B) = P(F \cap B)/P(B) = 0.28/0.5 = 0.56$.

   (c) $F$ and $B$ are not independent because $P(F|B) = 0.56 \neq 0.35 = P(F)$.

8. We can write

$$E_1 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\},$$

$$E_2 = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\},$$

and

$$E_3 = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}.$$

Thus, $E_1 \cap E_2 = E_1 \cap E_3 = E_2 \cap E_3 = \{(3,4)\}$, and $E_1 \cap E_2 \cap E_3 = \{(3,4)\}$. Hence, $P(E_1) = P(E_2) = P(E_3) = 1/6$, and $P(E_1 \cap E_2) = P(E_1 \cap E_3) = P(E_2 \cap E_3) = 1/36$, this shows that $E_1, E_2, E_3$ are pairwise independent. But $P(E_1 \cap E_2 \cap E_3) \neq P(E_1)P(E_2)P(E_3)$.

9. Since $E_1, E_2, E_3$ are independent, we have

$$\begin{aligned}
P(E_1 \cap (E_2 \cup E_3)) &= P((E_1 \cap E_2) \cup (E_1 \cap E_3)) = P(E_1 \cap E_2) + P(E_1 \cap E_3) \\
&\qquad - P(E_1 \cap E_2 \cap E_3) \\
&= P(E_1)P(E_2) + P(E_1)P(E_3) - P(E_1)P(E_2)P(E_3) \\
&= P(E_1)[P(E_2) + P(E_3) - P(E_2 \cap E_3)] = P(E_1)P(E_2 \cup E_3),
\end{aligned}$$

which proves the independence between $E_1$ and $E_2 \cup E_3$.

10. Let $E_1, E_2, E_3, E_4$ be the events that components 1, 2, 3, 4 function, respectively, then

$$\begin{aligned}
P(\text{system functions}) &= P((E_1 \cap E_2) \cup (E_3 \cap E_4)) = P(E_1 \cap E_2) + P(E_3 \cap E_4) \\
&\qquad - P(E_1 \cap E_2 \cap E_3 \cap E_4) \\
&= P(E_1)P(E_2) + P(E_3)P(E_4) - P(E_1)P(E_2)P(E_3)P(E_4) \\
&= 2 \times 0.9^2 - 0.9^4 = 0.9639.
\end{aligned}$$

11. Let $A$ denote the event that the system functions and $A_i$ denote the event that component $i$ functions, $i = 1, 2, 3, 4$. In mathematical notations

$$\begin{aligned}
A = (A_1 \cap A_2 \cap A_3 \cap A_4^c) \cup (A_1 \cap A_2 \cap A_3^c \cap A_4) \cup (A_1 \cap A_2^c \cap A_3 \cap A_4) \\
\cup (A_1^c \cap A_2 \cap A_3 \cap A_4) \cup (A_1 \cap A_2 \cap A_3 \cap A_4).
\end{aligned}$$

Thus

$$\begin{aligned}
P(A) &= P(A_1 \cap A_2 \cap A_3 \cap A_4^c) + P(A_1 \cap A_2 \cap A_3^c \cap A_4) + P(A_1 \cap A_2^c \cap A_3 \cap A_4) \\
&\qquad + P(A_1^c \cap A_2 \cap A_3 \cap A_4) + P(A_1 \cap A_2 \cap A_3 \cap A_4) \\
&= 4 \times 0.9^3 \times 0.1 + 0.9^4 = 0.9477.
\end{aligned}$$