

CHAPTER 2: Analytic Functions

EXERCISES 2.1: Functions of a Complex Variable

1.
 - a. $w = (3x^2 - 3y^2 + 5x + 1) + i(6xy + 5y + 1)$
 - b. $w = \frac{x}{x^2 + y^2} + i \left(-\frac{y}{x^2 + y^2} \right)$
 - c. $w = \frac{1}{z - i} = \frac{x}{x^2 + (y - 1)^2} + i \frac{-y + 1}{x^2 + (y - 1)^2}$
 - d. $w = \frac{2x^2 - 2y^2 + 3}{\sqrt{(x - 1)^2 + y^2}} + i \frac{4xy}{\sqrt{(x - 1)^2 + y^2}}$
 - e. $w = e^{3x} \cos 3y + i e^{3x} \sin 3y$
 - f. $w = (e^x + e^{-x}) \cos y + i(e^x - e^{-x}) \sin y$
 $= 2 \cosh x \cos y + i 2 \sinh x \sin y$
2.
 - a. \mathbb{C}
 - b. $\mathbb{C} \setminus \{0\}$
 - c. $\mathbb{C} \setminus \{i, -i\}$
 - d. $\mathbb{C} \setminus \{1\}$
 - e. \mathbb{C}
 - f. \mathbb{C}
3.
 - a. $\operatorname{Re} w > 5$
 - b. $\operatorname{Im} w \geq 0$
 - c. $|w| \geq 1$
 - d. The intersection of $|w| < 2$ and $-\pi < \operatorname{Arg} w < \pi/2$
4.
 - a. Taking θ from 0 to 2π , the points $z = re^{i\theta}$ traverse the circle $|z| = r$ exactly once in the counterclockwise direction. For the same values of θ the points $w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$ traverse the circle $|w| = \frac{1}{r}$ exactly once in the clockwise direction, hence the mapping is onto.
 - b. For $z = re^{i\theta_0}$ on the ray $\operatorname{Arg} z = \theta_0$, $w = \frac{1}{re^{i\theta_0}} = \frac{1}{r}e^{-i\theta_0}$ is on the ray $\operatorname{Arg} w = -\theta_0$. Taking values $0 < r < \infty$ shows that this mapping goes onto the ray $\operatorname{Arg} w = -\theta_0$.

- 4 (c) $|z-1| = 1$ $2\pi > \theta \geq 0 \Rightarrow z = 1 + e^{i\theta}$. $F(z) = 1/z = 1/(1 + e^{i\theta})$
 $= (1 + e^{-i\theta}) / \{2(1 + \cos\theta)\} = \frac{1}{2} - i(\frac{1}{2})\sin\theta / (1 + \cos\theta)$
 which is a vertical line at $x = \frac{1}{2}$.

5. a. domain: \mathbb{C}

range: $\mathbb{C} \setminus \{0\}$

b. $f(-z) = e^{-z} = \frac{1}{e^z} = \frac{1}{f(z)}$

c. circle $|w| = e$

d. ray $\text{Arg } w = \pi/4$

e. infinite sector $0 \leq \text{Arg } w \leq \pi/4$

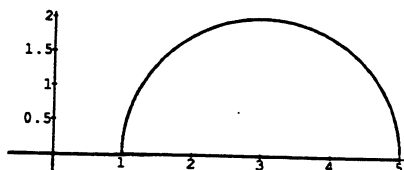
6. a. $J\left(\frac{1}{z}\right) = \frac{1}{2} \left(\frac{1}{z} + \frac{1}{1/z}\right) = \frac{1}{2} \left(z + \frac{1}{z}\right) = J(z)$

b. For $z = e^{i\theta}$ on the unit circle $|z| = 1$, $J(z) = \frac{1}{2} \left(e^{i\theta} + \frac{1}{e^{i\theta}}\right) = \cos \theta$.
 For all values of θ , this ranges over the real interval $[-1, 1]$.

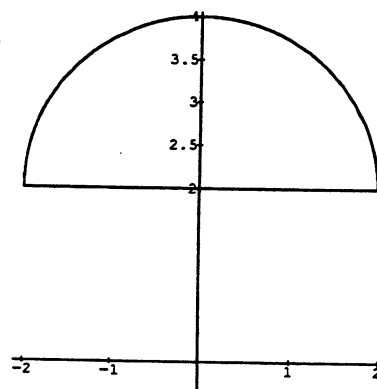
c. For $z = re^{i\theta}$ on the circle $|z| = r$, $J(z) = \frac{1}{2} \left(re^{i\theta} + \frac{1}{re^{i\theta}}\right) = \frac{1}{2} \left(r + \frac{1}{r}\right) \cos \theta + i \frac{1}{2} \left(r - \frac{1}{r}\right) \sin \theta$. Setting u and v equal to the real and imaginary parts of this expression, respectively, one gets a pair of parametric equations that are equivalent to the ellipse

$$\frac{u^2}{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]^2} = 1$$
, which has foci at ± 1 .

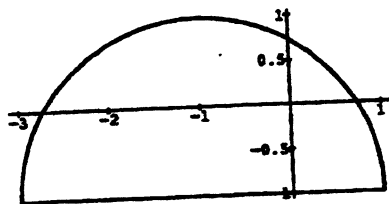
7. a.



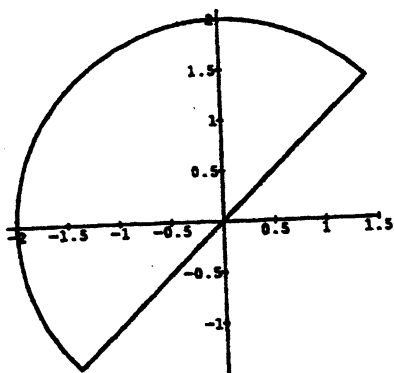
- b.



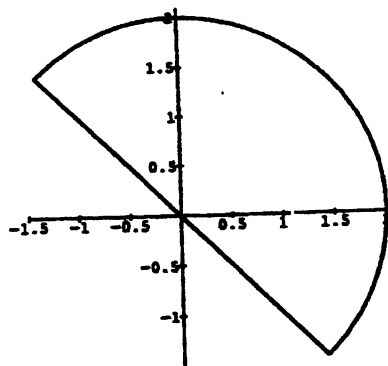
c.



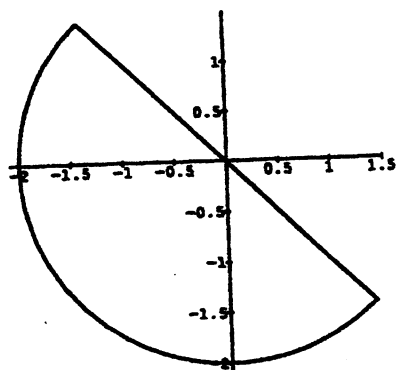
8. a.



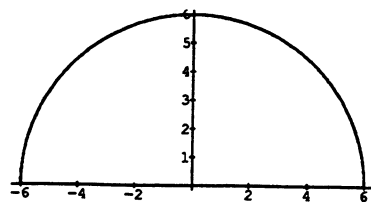
b.



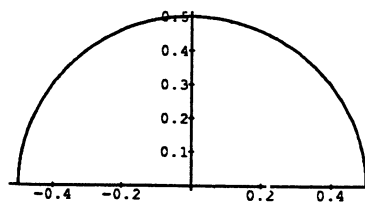
c.



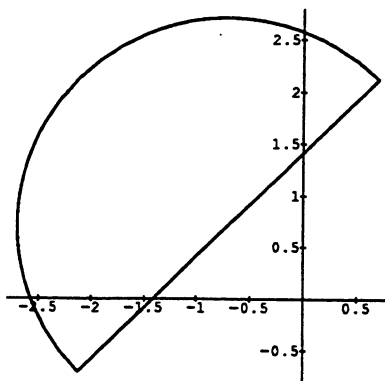
9. a.



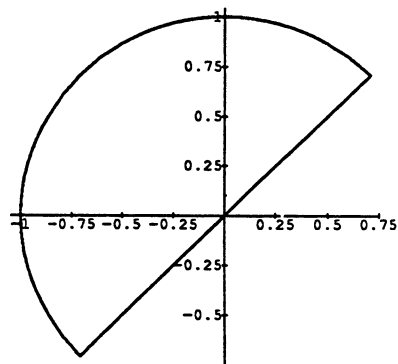
b.



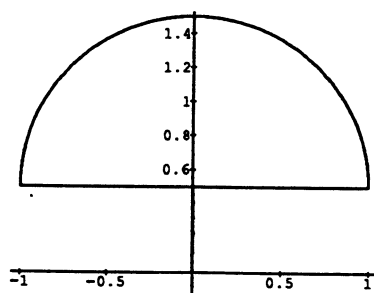
10. a. translate by i , rotate $\pi/4$



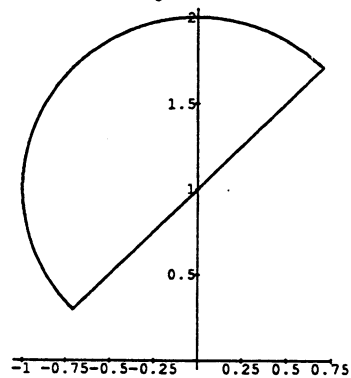
b. reduce by $1/2$, rotate $\pi/4$



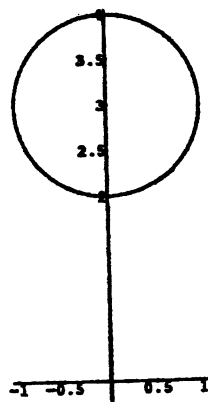
c. translate by i , reduce by $1/2$



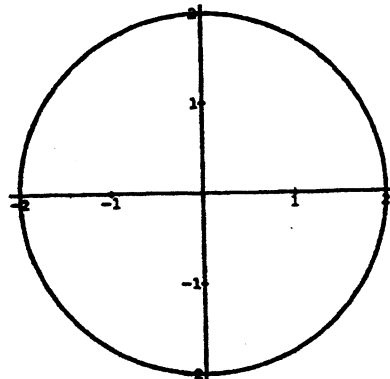
d. reduce by $1/2$, rotate $\pi/4$,
translate by i



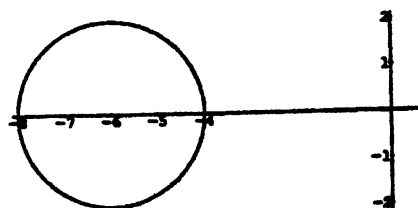
11. a. translate by -3 ,
rotate $-\pi/2$



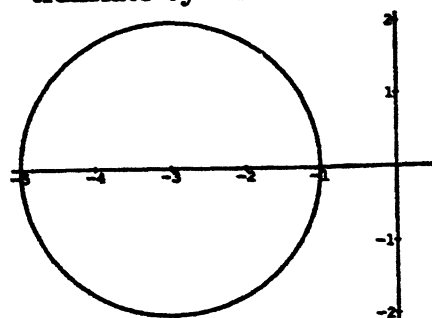
- b. magnify by 2,
rotate $-\pi/2$



- c. translate by -3 ,
magnify by 2



- d. magnify by 2, rotate $-\pi/2$,
translate by -3



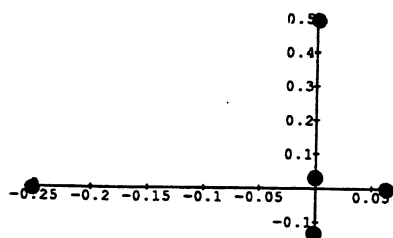
12. Let $a = \rho e^{i\phi}$, $F(z) = \rho z$, $G(z) = e^{i\phi} z$, and $H(z) = z + b$. Then $H(G(F(z))) = az + b$.

13. (a) $w = u + iv = z^2 = (1 + iy)^2 = 1 - y^2 + i2y$
 $u = 1 - y^2$, $v = 2y \Rightarrow y = v/2 \Rightarrow u = 1 - v^2/4$ a parabola in the w -plane.
 (b) $w = u + iv = z^2 = (x + iy)^2 = (x + i/x)^2 = x^2 - 1/x^2 + 2i$
 $u = x^2 - 1/x^2$, $v = 2$ a straight line in the w -plane.
 (c) $w = u + iv = z^2 = (1 + e^{i\theta})^2 = (1 + 2e^{i\theta} + e^{i2\theta}) = (e^{-i\theta} + 2 + e^{i\theta})e^{i\theta}$
 $= (2 + 2\cos\theta)e^{i\theta} = 2(1 + \cos\theta)e^{i\theta}$ a cardioid in the w -plane.
14. (a) $x_1 = 2x/(|z|^2 + 1)$, $x_2 = 2y/(|z|^2 + 1)$, $x_3 = (|z|^2 - 1)/(|z|^2 + 1)$
 $w = e^{i\phi} z = x\cos\phi - y\sin\phi + i(x\sin\phi + y\cos\phi)$, $|w| = |z|$
 $\underline{x}_1 = (x\cos\phi - y\sin\phi)/(|z|^2 + 1)$, $\underline{x}_2 = (x\sin\phi + y\cos\phi)/(|z|^2 + 1)$, $\underline{x}_3 = x_3$
 $\underline{x}_1 = (x_1\cos\phi - x_2\sin\phi)$, $\underline{x}_2 = (x_1\sin\phi + x_2\cos\phi)$, $\underline{x}_3 = x_3$ which corresponds to a rotation of an angle ϕ about the x_3 axis.
 (b) $w = -1/z$. $|w| = 1/|z|$. $w = -1/(x + iy) = -x/|z| + iy/|z|$
 $\underline{x}_1 = -x_1$, $\underline{x}_2 = x_2$, $\underline{x}_3 = -x_3$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained from (x_1, x_2, x_3) by a 180° rotation about the x_2 axis.
15. $w = (1+z)/(1-z) = (1+x+iy)/(1-x-iy) = (1-|z|^2 + i2y)/(1-2x+|z|^2)$
 $|w|^2 = (1+2x+|z|^2)/(1-2x+|z|^2)$.
 $(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, x_2, x_1)$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained by a 90° counterclockwise rotation about the x_2 axis.

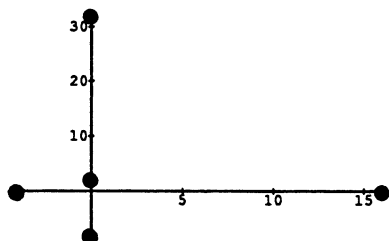
16. $w = (1 - iz)/(1 + iz) = (1 - ix + y)/(1 + ix - y) = (1 - |z|^2 + i2x)/(1 - 2y + |z|^2)$
 $|w|^2 = (1 + 2y + |z|^2)/(1 - 2y + |z|^2)$.
 $(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, -x_1, x_2)$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained as a 90° counterclockwise rotation about the x_2 axis followed by a 90° counterclockwise rotation about the x_3 axis.
17. Any circle or line in the z -plane corresponds to a line or circle on the stereographic projection onto the Riemann sphere. The function $w=1/z$ rotates the Riemann sphere 180° about the x_1 axis. Lines and circles on the rotated sphere project to lines and circles in the w -plane. As a result lines and circles in the z -plane map to lines and circles in the w -plane.

EXERCISES 2.2: Limits and Continuity

1. The first five terms are, respectively, $\frac{i}{2}$, $-\frac{1}{4}$, $-\frac{i}{8}$, $\frac{1}{16}$, and $\frac{i}{32}$. The sequence converges to 0 in a spiral-like fashion.



2. $2i, -4, -8i, 16, 32i$; divergent because terms grow in modulus without bound.



3. If $\lim_{n \rightarrow \infty} z_n = z_0$, then for any $\varepsilon > 0$, there is an integer N such that $|z_n - z_0| < \varepsilon$ for all $n > N$. For the same integer N we have $|x_n - x_0| \leq |z_n - z_0| < \varepsilon$ and $|y_n - y_0| \leq |z_n - z_0| < \varepsilon$ for all $n > N$. Therefore, $\lim_{n \rightarrow \infty} x_n = x_0$ and $\lim_{n \rightarrow \infty} y_n = y_0$.

If $\lim_{n \rightarrow \infty} x_n = x_0$ and $\lim_{n \rightarrow \infty} y_n = y_0$, then for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there are integers N_1 and N_2 such

$|x_n - x_0| < \varepsilon_1$ for all $n > N_1$ and $|y_n - y_0| < \varepsilon_2$ for all $n > N_2$. Given any $\varepsilon > 0$;

let $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = \varepsilon/2$. Then

$|z_n - z_0| \leq |x_n - x_0| + |y_n - y_0| < \varepsilon_1 + \varepsilon_2 = \varepsilon$ for all $n > \max(N_1, N_2)$.

Thus $\lim_{n \rightarrow \infty} z_n = z_0$.

4. If $z_n = x_n + iy_n \rightarrow z_0 = x_0 + iy_0$, then $x_n \rightarrow x_0$ and $y_n \rightarrow y_0$ (see Problem 3).
 $\underline{z}_n = x_n - iy_n \rightarrow x_0 - iy_0 = \underline{z}_0$.

If $\underline{z}_n = x_n - iy_n \rightarrow \underline{z}_0 = x_0 - iy_0$, then $x_n \rightarrow x_0$ and $y_n \rightarrow y_0$ (see Problem 3)..

$z_n = x_n + iy_n \rightarrow x_0 + iy_0 = z_0$. Thus $z_n \rightarrow z_0$ if and only if $\underline{z}_n \rightarrow \underline{z}_0$.

5. $\lim_{n \rightarrow \infty} |z_n| = 0 \implies$ There exists an integer N such that
 $||z_n| - 0| = |z_n| < \varepsilon$ whenever $n > N$. $\implies |z_n - 0| < \varepsilon$ whenever $n > N$. $\implies \lim_{n \rightarrow \infty} z_n = 0$, and conversely.

6. $z_0^n \rightarrow 0$ as $n \rightarrow \infty$ by problem 3, since the real-valued sequence $|z_0^n| \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, if $|z_0| > 1$, then $|z_0^n| \rightarrow \infty$ as $n \rightarrow \infty$ so z_0^n diverges.

7. a. converges to 0
 b. does not converge
 c. converges to π
 d. converges to $2 + i$
 e. converges to 0
 f. does not converge

8. Given $\varepsilon > 0$, choose $\delta = \varepsilon/6$. Then whenever $0 < |z - (1 + i)| < \delta$,

$$|6z - 4 - (2 + 6i)| = 6|z - (1 + i)| < 6(\varepsilon/6) = \varepsilon$$

9. Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{1 + \varepsilon}$. Whenever $0 < |z - (-i)| < \delta$ notice that $|z| > 1 - \delta$ and

$$\left| \frac{1}{z} - i \right| = \left| \left(-\frac{i}{z} \right) (i + z) \right| = \frac{1}{|z|} |z - (-i)| < \left(\frac{1}{1 - \delta} \right) \delta = \varepsilon$$

10. Given that f and g are continuous at z_0 ,

$$\lim_{z \rightarrow z_0} f(z) \pm g(z) = \lim_{z \rightarrow z_0} f(z) \pm \lim_{z \rightarrow z_0} g(z) = f(z_0) \pm g(z_0)$$

$\Rightarrow f(z) \pm g(z)$ is continuous at z_0 .

$$\lim_{z \rightarrow z_0} f(z)g(z) = \lim_{z \rightarrow z_0} f(z) \lim_{z \rightarrow z_0} g(z) = f(z_0)g(z_0)$$

$\Rightarrow f(z)g(z)$ is continuous at z_0 .

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \rightarrow z_0} f(z)}{\lim_{z \rightarrow z_0} g(z)} = \frac{f(z_0)}{g(z_0)}, \text{ provided } g(z_0) \neq 0$$

$\Rightarrow \frac{f(z)}{g(z)}$ is continuous at z_0 .

11. a. $-8i$
 b. $-\frac{7}{2}i$
 c. $6i$
 d. $-1/2$
 e. $2z_0$
 f. $4\sqrt{2}$

12. Clearly $\text{Arg } z$ is discontinuous at $z = 0$. Let $a > 0$ be any real number and consider the sequence

$$z_n = -a - i/n \quad n = 1, 2, \dots, \text{ which converges to } -a.$$

For each n , $-\pi < \text{Arg } z_n < -\pi/2$, but $\text{Arg } (-a) = \pi$.

13. $\lim_{z \rightarrow z_0} f(z)$ exists for all $z \neq -1$; f is continuous for all $z \neq 0, -1$; f has a removable discontinuity at $z = 0$.

14. Let z_0 be any complex number. Given $\varepsilon > 0$ choose $\delta = \varepsilon$. Then whenever $|z - z_0| < \delta$,

$$|g(z) - g(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \varepsilon.$$

15. Given $\varepsilon > 0$ choose δ so that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Then, whenever $|z - z_0| < \delta$:

- a. $|\overline{f(z)} - \overline{f(z_0)}| = |\overline{f(z) - f(z_0)}| = |f(z) - f(z_0)| < \varepsilon$
 b. $|\text{Re } f(z) - \text{Re } f(z_0)| = |\text{Re}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$
 c. $|\text{Im } f(z) - \text{Im } f(z_0)| = |\text{Im}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$
 d. $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \varepsilon$

16. Given $\varepsilon > 0$, choose $\delta_0 > 0$ such that $|f(g(z)) - f(g(z_0))| < \varepsilon$ whenever $|g(z) - g(z_0)| < \delta_0$. Now choose $\delta > 0$ such that $|g(z) - g(z_0)| < \delta_0$ whenever $|z - z_0| < \delta$. Then $|f(g(z)) - f(g(z_0))| < \varepsilon$ whenever $|z - z_0| < \delta$; hence $f(g(z))$ is continuous at z_0 .

17. No: Observe that although $\frac{1}{n} \rightarrow 0$ and $\frac{i}{n} \rightarrow 0$ as $n \rightarrow \infty$,
 $f\left(\frac{1}{n}\right) \rightarrow 1 + 2i$ and $f\left(\frac{i}{n}\right) \rightarrow 2i$; thus $\lim_{z \rightarrow 0} f(z)$ does not exist.

18. If $\lim_{z \rightarrow z_0} f(z) = w_0$, then given $\varepsilon > 0$ there exists $\delta > 0$ such that
 $|f(z) - w_0| < \varepsilon$ for all $|z - z_0| < \delta$. Notice that
 $|\underline{f(z)} - \underline{w_0}| = |\underline{f(z)} - \underline{w_0}| = |f(z) - w_0| < \varepsilon$ for all $|z - z_0| < \delta$. So that $\lim_{z \rightarrow z_0} \underline{f(z)} = \underline{w_0}$.
 $\lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) = \lim_{z \rightarrow z_0} ((f(z) + \underline{f(z)})/2) = (w_0 + \underline{w_0})/2 = \mu_0$.
 $\lim_{x \rightarrow x_0, y \rightarrow y_0} v(x, y) = \lim_{z \rightarrow z_0} ((f(z) - \underline{f(z)})/2i) = (w_0 - \underline{w_0})/2i = v_0$.
Thus, $\lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) = \mu_0$ and $\lim_{x \rightarrow x_0, y \rightarrow y_0} v(x, y) = v_0$.

Conversely, if $\lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) = \mu_0$ and $\lim_{x \rightarrow x_0, y \rightarrow y_0} v(x, y) = v_0$, then
(by Theorem 1.) $\mu_0 + iv_0 = \lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) + i \lim_{x \rightarrow x_0, y \rightarrow y_0} v(x, y) =$
 $\lim_{z \rightarrow z_0} ((f(z) + \underline{f(z)})/2) + \lim_{z \rightarrow z_0} ((f(z) - \underline{f(z)})/2) = \lim_{z \rightarrow z_0} f(z) = w_0$.
Also $\mu_0 - iv_0 = \lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) - i \lim_{x \rightarrow x_0, y \rightarrow y_0} v(x, y) = \lim_{z \rightarrow z_0}$
 $((f(z) + \underline{f(z)})/2) - \lim_{z \rightarrow z_0} ((f(z) - \underline{f(z)})/2) = \lim_{z \rightarrow z_0} f(z) = \underline{w_0}$.

Thus, $\lim_{z \rightarrow z_0} f(z) = w_0$.

19. $-\frac{1}{2} - i$, since $\lim_{x \rightarrow 1} \frac{x}{x^2 + 3y} = -\frac{1}{2}$ and $\lim_{y \rightarrow 1} xy = -1$.

20. For any z_0 in the complex plane,

$$\lim_{z \rightarrow z_0} e^z = \lim_{x \rightarrow x_0} e^x \cos y + i \lim_{y \rightarrow y_0} e^x \sin y = e^{x_0} \cos y_0 + i e^{x_0} \sin y_0 = e^{z_0}.$$

21. a. 1
b. 0
c. $-\pi/2 + i$
d. 1

22. By contradiction: Suppose $\lim_{z \rightarrow z_0} f(z) \neq w_0$. Then there is an $\varepsilon > 0$
for which there exists a sequence $\{z_n\}$ such that $|z_n - z_0| < \frac{1}{n}$ but
 $|f(z_n) - w_0| > \varepsilon$. For this sequence, $\lim_{n \rightarrow \infty} z_n = z_0$ but $\lim_{n \rightarrow \infty} f(z_n) \neq w_0$,
contrary to hypothesis.

23. If $z_n \rightarrow \infty$, then for any $M > 0$ there exist an integer N such $|z_n| > M$ for all $n > N$. Consider the chordal distance $\chi(z_n, \infty) = 2/\sqrt{|z_n|^2 + 1} < 2/\sqrt{|z_n|^2} = 2/|z_n| < 2/M < \varepsilon$ for all $n > N$. Thus $z_n \rightarrow \infty$ as $n \rightarrow \infty$ is equivalent to $\chi(z_n, \infty) \rightarrow 0$ as $n \rightarrow \infty$.
24. If $\lim_{z \rightarrow z_0} f(z) = \infty$, then for any $M > 0$ there exists $\delta > 0$ such that $|f(z)| > M$ for all $|z - z_0| < \delta$. Consider $\chi(f(z), \infty) = 2/\sqrt{|f(z)|^2 + 1} < 2/\sqrt{|f(z)|^2} = 2/|f(z)| < 2/M < \varepsilon$ for all $|z - z_0| < \delta$. Thus $\lim_{z \rightarrow z_0} f(z) = \infty$, is equivalent to $\lim_{z \rightarrow \infty} \chi(f(z), \infty) = 0$.
25. (a) ∞ (b) 3 (c) ∞ (d) ∞ (f) the limit does not exist.

EXERCISES 2.3: Analyticity

1. Let $\Delta z = z - z_0$ so that $\Delta z \rightarrow 0 \iff z \rightarrow z_0$. Then

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = L \iff$$

given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - L \right| < \varepsilon \text{ whenever } |\Delta z - 0| < \delta \iff$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - L \right| < \varepsilon \text{ whenever } |z - z_0| < \delta \iff$$

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = L.$$

2. If $\lambda(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$, then $\lambda(z) \rightarrow 0$ as $z \rightarrow z_0$ and $f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0) = f(z)$.
3. $\lim_{z \rightarrow z_0} f(z) = \lim_{z \rightarrow z_0} [f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0)]$
 $= f(z_0) + 0 + 0 = f(z_0).$

$$\begin{aligned}
4. \quad a. \quad \lim_{\Delta z \rightarrow 0} \frac{\operatorname{Re}(z + \Delta z) - \operatorname{Re}(z)}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{\operatorname{Re}(\Delta z)}{\Delta z} = \begin{cases} 1, & \text{if } \Delta z = \Delta x \\ 0, & \text{if } \Delta z = i\Delta y \end{cases} \\
b. \quad \lim_{\Delta z \rightarrow 0} \frac{\operatorname{Im}(z + \Delta z) - \operatorname{Im}(z)}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{\operatorname{Im}(\Delta z)}{\Delta z} = \begin{cases} 0, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = i\Delta y \end{cases} \\
c. \quad \text{Case 1, } z = 0.
\end{aligned}$$

$$\lim_{\Delta z \rightarrow 0} \frac{|0 + \Delta z| - |0|}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x + i\Delta y} = \begin{cases} \pm 1, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = \pm i\Delta y \end{cases}$$

Case 2, $z \neq 0$.

$$\begin{aligned}
&\lim_{\Delta z \rightarrow 0} \frac{|z + \Delta z| - |z|}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \frac{\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} - \sqrt{x^2 + y^2}}{\Delta x + i\Delta y} \\
&= \lim_{\Delta z \rightarrow 0} \frac{(x + \Delta x)^2 + (y + \Delta y)^2 - (x^2 + y^2)}{(\Delta x + i\Delta y)(\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} + \sqrt{x^2 + y^2})} \\
&= \lim_{\Delta z \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2}{(\Delta x + i\Delta y)(\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} + \sqrt{x^2 + y^2})} \\
&= \begin{cases} \frac{x}{\sqrt{x^2 + y^2}}, & \text{if } \Delta z = \Delta x, z \neq 0 \\ \frac{y}{i\sqrt{x^2 + y^2}}, & \text{if } \Delta z = i\Delta y, z \neq 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
5. \quad \text{Rule 5: } (f \pm g)'(z_0) &= \lim_{\Delta z \rightarrow 0} \frac{(f \pm g)(z_0 + \Delta z) - (f \pm g)(z_0)}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \pm \frac{g(z_0 + \Delta z) - g(z_0)}{\Delta z} \right] \\
&= f'(z_0) \pm g'(z_0)
\end{aligned}$$

$$\text{Rule 7: } (fg)'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{fg(z_0 + \Delta z) - fg(z_0)}{\Delta z}$$

$$\begin{aligned}
&= \lim_{\Delta z \rightarrow 0} \left\{ \frac{f(z_0 + \Delta z)g(z_0 + \Delta z) - f(z_0 + \Delta z)g(z_0)}{\Delta z} \right. \\
&\quad \left. + \frac{f(z_0 + \Delta z)g(z_0) - f(z_0)g(z_0)}{\Delta z} \right\} \\
&= \lim_{\Delta z \rightarrow 0} \left\{ f(z_0 + \Delta z) \frac{[g(z_0 + \Delta z) - g(z_0)]}{\Delta z} \right. \\
&\quad \left. + g(z_0) \frac{[f(z_0 + \Delta z) - f(z_0)]}{\Delta z} \right\} \\
&= f(z_0)g'(z_0) + g(z_0)f'(z_0)
\end{aligned}$$

6. Let $n > 0$ be an integer.

Then $\frac{d}{dz} z^{-n} = \frac{d}{dz} \left(\frac{1}{z^n} \right) = \frac{-nz^{n-1}}{z^{2n}}$ (using Rule 8) $= -nz^{-n-1}$.

7. a. $18z^2 + 16z + i$

b. $-12z(z^2 - 3i)^{-7}$

c. $\frac{-iz^4 + (2 + 27i)z^2 + 2\pi z + 18}{(iz^3 + 2z + \pi)^2}$

d. $\frac{-(z+2)^2(5z^2 + (16+i)z - 3 + 8i)}{(z^2 + iz + 1)^5}$

e. $24i(z^3 - 1)^3(z^2 + iz)^{99}(53z^4 + 28iz^3 - 50z - 25i)$

8. Let $z = z_0 + \Delta z$. Then

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \left| \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right| = |f'(z_0)|.$$

$$\begin{aligned}
\lim_{z \rightarrow z_0} \arg[f(z) - f(z_0)] - \arg(z - z_0) &= \lim_{z \rightarrow z_0} \arg \left[\frac{f(z) - f(z_0)}{z - z_0} \right] \\
\arg \left[\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right] &= \arg[f'(z_0)]
\end{aligned}$$

9. a. $2 - 3i$

b. $\pm i$

c. $\frac{-1 \pm i\sqrt{15}}{2}$

d. $\frac{1}{2}, 1$

10. $\lim_{\Delta z \rightarrow 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$

$$= \lim_{\Delta z \rightarrow 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \overline{\Delta z}) - z_0 \bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \left(\bar{z}_0 + \frac{\overline{\Delta z}}{\Delta z} z_0 + \overline{\Delta z} \right) = \begin{cases} \bar{z}_0 + z_0 & \text{if } \Delta z = \Delta x \\ \bar{z}_0 - z_0 & \text{if } \Delta z = i\Delta y \end{cases}$$

If $z_0 = 0$, then the difference quotient is

$$\lim_{\Delta z \rightarrow 0} (0 + 0 + \overline{\Delta z}) = 0.$$

11. a. nowhere analytic

b. nowhere analytic

c. analytic except at $z = 5$

d. everywhere analytic

e. nowhere analytic

f. analytic except at $z = 0$

g. nowhere analytic

h. nowhere analytic

12. The case when $n = 1$ is trivial. Assume that the result holds for all positive integers less than or equal to n and define

$Q(z) = P(z)(z - z_{n+1})$. Since $Q'(z) = P'(z)(z - z_{n+1}) + P(z)$, it follows that

$$\frac{Q'(z)}{Q(z)} = \frac{P'(z)}{P(z)} + \frac{1}{z - z_{n+1}} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_{n+1}}$$

13. a, b, d, f, and g are always true
14. $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow z_0} \frac{[f(z) - f(z_0)]/(z - z_0)}{[g(z) - g(z_0)]/(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}$
15. $\frac{3}{5}$
16. Any point on the line through z_1 and z_2 has the form
 $z = -\frac{1}{2} + i\sqrt{3}\left(\frac{1}{2} - t\right)$, t real (see Section 1.3, Exercise 18). However,
 $f(z_2) - f(z_1) = 0$ but $f'(w) = 3w^2 \neq 0$ on the line in question.
17. $F'(z_0) = f(z_0)(gh)'(z_0) + f'(z_0)gh(z_0)$
 $= f(z_0)[g(z_0)h'(z_0) + g'(z_0)h(z_0)] + f'(z_0)g(z_0)h(z_0)$
 $= f'(z_0)g(z_0)h(z_0) + f(z_0)g'(z_0)h(z_0) + f(z_0)g(z_0)h'(z_0)$

EXERCISES 2.4: The Cauchy-Riemann Equations

1. a. $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$
 b. $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$
 c. $\frac{\partial u}{\partial y} = 2 \neq -\frac{\partial v}{\partial x} = 1$
2. $\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3 = \frac{\partial v}{\partial y}$, but $\frac{\partial u}{\partial y} = 6xy = \frac{\partial v}{\partial x}$. Therefore $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ only when $x = 0$ or $y = 0$. This means h is differentiable on the axes but h is nowhere analytic since lines are not open sets in the complex plane.
3. $\frac{\partial u}{\partial x} = 6x + 2 = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -6y = -\frac{\partial v}{\partial x}$. Since these partial derivatives exist and are continuous for all x and y , g is analytic. g can be written as $g(z) = 3z^2 + 2z - 1$.

$$4. \frac{\partial u}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{u(\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{u(0, \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = 0.$$

Similarly $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$.

However, when $\Delta z \rightarrow 0$ through real values ($\Delta z = \Delta x$)

$$\lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = 0,$$

while along the real line $y = x$ ($\Delta z = \Delta x + i\Delta x$)

$$\lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^{4/3}(\Delta x)^{5/3} + i(\Delta x)^{5/3}(\Delta x)^{4/3}}{2(\Delta x)^2}}{\Delta x(1 + i)}$$

$$= \frac{1}{2}.$$

Therefore f is not differentiable at $z = 0$.

$$5. \frac{\partial u}{\partial x} = 2e^{x^2-y^2} [x \cos(2xy) - y \sin(2xy)] = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2} [y \cos(2xy) + x \sin(2xy)] = -\frac{\partial v}{\partial x}$$

f is entire because these first partials exist and are continuous for all x and y .

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 2e^{x^2-y^2} (x + iy) [\cos(2xy) + i \sin(2xy)]$$

$$= 2e^{(x^2-y^2)} e^{i2xy} (x + iy)$$

$$= 2ze^{z^2}$$

(This derivative could have been obtained directly, since $f(z) = e^{z^2}$.)

6. $z = re^{i\theta} \Rightarrow x = r \cos \theta$ and $y = r \sin \theta$ and

$$f(z) = u(x(r, \theta), y(r, \theta)) + iv(x(r, \theta), y(r, \theta))$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$$

Similar applications of the chain rule yield

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}(-r \sin \theta) + \frac{\partial u}{\partial y}r \cos \theta \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}(-r \sin \theta) + \frac{\partial v}{\partial y}r \cos \theta\end{aligned}$$

Replace the partial derivatives on the right sides of the equations for $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial r}$ by their Cauchy-Riemann counterparts to obtain:

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta = \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta = -\frac{1}{r} \frac{\partial u}{\partial \theta}\end{aligned}$$

7. Let $h(z) = f(z) - g(z)$. Then h is analytic in D and $h'(z) = 0$ so h is a constant function.

$$h(z) = c = f(z) - g(z) \implies f(z) = g(z) + c$$

8. $u(x, y) = c$ in $D \implies \frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$. Hence

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 \text{ so } f \text{ is constant in } D.$$

9. By contradiction. If f is analytic in a domain D then $v(x, y) = 0$ (a constant) $\implies f$ is constant (by condition 8) $\implies u$ is constant. (However, there is no open set in which $u(x, y) = |z^2 - z|$ is constant).

10. $\text{Im } f(z) = 0$ in $D \implies \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \implies \frac{\partial u}{\partial x} = 0$
 $\implies f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 \implies f$ is constant in D .

11. $\text{Re } f(z) = \frac{1}{2}[f(z) + \overline{f(z)}]$ is real valued and analytic if both f and \bar{f} are analytic. Hence $\text{Re } f(z)$ is constant by Exercise 10. It follows that $f(z)$ is constant by Exercise 8.

12. $|f(z)|$ constant in $D \implies |f(z)|^2 = u^2 + v^2$ is constant in D . If $u = 0$ or $v = 0$ in D , then f is constant by Exercises 8 and 10. Otherwise,

$$\begin{aligned}\frac{\partial |f|^2}{\partial x} &= 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \\ \frac{\partial |f|^2}{\partial y} &= 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = -2u \frac{\partial v}{\partial x} + 2v \frac{\partial u}{\partial x} = 0\end{aligned}$$

$$\implies \frac{1}{2}v \frac{\partial |f|^2}{\partial x} - \frac{1}{2}u \frac{\partial |f|^2}{\partial y} = 0 = (u^2 + v^2) \frac{\partial v}{\partial x}$$

$$\implies \frac{\partial v}{\partial x} = 0 \implies \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0$$

$$\implies f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0$$

$$\implies f \text{ is constant in } D.$$

13. $|f(z)|$ is analytic and real-valued, so the result follows from Exercises 10 and 12.

14. If the line is vertical then $\operatorname{Re} f(z)$ is constant and this reduces to Problem 8. If the line is not vertical, then $v(x, y) = mu(x, y) + b$, and

$$\begin{aligned}\frac{\partial v}{\partial x} &= m \frac{\partial u}{\partial x} = m \frac{\partial v}{\partial y}, \\ \frac{\partial v}{\partial y} &= m \frac{\partial u}{\partial y} = -m \frac{\partial v}{\partial x} = -m^2 \frac{\partial v}{\partial y}.\end{aligned}$$

It follows that

$$\frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ and } f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0.$$

Hence $f(z)$ is constant.

$$\begin{aligned}15. J(x_0, y_0) &= \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \Big|_{(x_0, y_0)} \\ &= \left[\frac{\partial u}{\partial x}(x_0, y_0) \right]^2 + \left[\frac{\partial v}{\partial x}(x_0, y_0) \right]^2 \\ &= |f'(z_0)|^2 \quad (\text{using Equation (1)})\end{aligned}$$

$$\begin{aligned}
16. \quad a. \quad \frac{\partial \tilde{f}}{\partial \xi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi} \\
&= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \frac{1}{2} + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \frac{1}{2i} \\
&= \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\
\frac{\partial \tilde{f}}{\partial \eta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta} \\
&= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \frac{1}{2} + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \left(\frac{-1}{2i} \right) \\
&= \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
b. \quad \frac{\partial \tilde{f}}{\partial \eta} = 0 &\Leftrightarrow 0 = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \text{ and } 0 = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
&\Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\end{aligned}$$

EXERCISES 2.5: Harmonic Functions

$$\begin{aligned}
1. \quad a. \quad u(x, y) &= x^2 - y^2 + 2x + 1, \quad \frac{\partial^2 u}{\partial x^2} = 2 = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \Delta u = 0 \\
v(x, y) &= 2xy + 2y, \quad \frac{\partial^2 v}{\partial x^2} = 0 = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \Delta v = 0 \\
b. \quad u(x, y) &= \frac{x}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \Delta u = 0 \\
v(x, y) &= -\frac{y}{x^2 + y^2}, \quad \frac{\partial^2 v}{\partial x^2} = \frac{-2y(3x^2 - y^2)}{(x^2 + y^2)^3} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \Delta v = 0 \\
c. \quad u(x, y) &= e^x \cos y, \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \Delta u = 0 \\
v(x, y) &= e^x \sin y, \quad \frac{\partial^2 v}{\partial x^2} = e^x \sin y = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \Delta v = 0 \\
2. \quad h(x, y) &= ax^2 + bxy - ay^2
\end{aligned}$$

3. a. $u = \operatorname{Re}(-iz)$, $v = -x + a$, where a is a constant
 b. $u = \operatorname{Re}(-ie^z)$, $v = -e^x \cos y + a$
 c. $u = \operatorname{Re}\left(\frac{-i}{2}z^2 - iz - z\right)$, $v = -\frac{1}{2}(x^2 - y^2) - (x + y) + a$
 d. It is straightforward to verify that $\Delta u = 0$.

$$\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}$$

$$\Rightarrow v(x, y) = \int \cos x \cosh y dy = \cos x \sinh y + \psi(x)$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x} = \sin x \sinh y + \psi'(x) \Rightarrow \psi(x) = a$$
 Thus, $v(x, y) = \cos x \sinh y + a$.
 e. It is straightforward to verify that $\Delta u = 0$.

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y} \Rightarrow$$

$$v(x, y) = \int \frac{x}{x^2 + y^2} dy = \tan^{-1}\left(\frac{y}{x}\right) + \psi(x)$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2} - \psi'(x) \Rightarrow \psi(x) = a$$
 Thus, $v(x, y) = \tan^{-1}\left(\frac{y}{x}\right) + a$.
 f. $u = \operatorname{Re}(-ie^{x^2})$, $v = -e^{x^2-y^2} \cos(2xy) + a$.

4. Suppose v and w are both harmonic conjugates of u , and consider $\phi(x, y) = w(x, y) - v(x, y)$. Then (using the Cauchy-Riemann equations for v and w),

$$\frac{\partial \phi}{\partial x} = \frac{\partial w}{\partial x} - \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} - \left(-\frac{\partial u}{\partial y}\right) = 0$$

and similarly $\frac{\partial \phi}{\partial y} = 0$. Hence $\phi(x, y) = a$, from which it follows that

$$w(x, y) = v(x, y) + a.$$

5. If $f(z) = u(x, y) + iv(x, y)$ is analytic then $-if(z) = v(x, y) - iu(x, y)$ is analytic. Thus $-u$ is a harmonic conjugate of v .

6. Since $f(z) = u + iv$ is analytic, $\frac{1}{2} [f(z)]^2 = \frac{1}{2}(u^2 - v^2) + iuv$ is analytic.

Thus $uv = \text{Im} \frac{1}{2} [f(z)]^2$ is harmonic.

7. $\phi(x, y) = x + 1$

8. a. Yes, because $\Delta(u + v) = \Delta u + \Delta v = 0$.

b. No. Take $u = x, v = x^2 - y^2$ as an example.

c. Yes, because $\Delta(u_x) = u_{xxx} + u_{xyy} = u_{xxx} + u_{yyx}$

$$= \frac{\partial}{\partial x}(\Delta u) = \frac{\partial}{\partial x}(0) = 0.$$

9. $\phi(x, y) = xy - 1$ (this is $\text{Im} \left(\frac{1}{2} z^2 - i \right)$)

10. Let $x = r \cos \theta$ and $y = r \sin \theta$.

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial \phi}{\partial x} \cos \theta + \frac{\partial \phi}{\partial y} \sin \theta \\ \frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial^2 \phi}{\partial x^2} \frac{\partial x}{\partial r} \cos \theta + \frac{\partial^2 \phi}{\partial y \partial x} \frac{\partial y}{\partial r} \cos \theta \\ &\quad + \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial x}{\partial r} \sin \theta + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial y}{\partial r} \sin \theta \\ &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \theta + \frac{\partial^2 \phi}{\partial y \partial x} 2 \sin \theta \cos \theta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \theta \\ \frac{\partial \phi}{\partial \theta} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial \phi}{\partial x} (-r \sin \theta) + \frac{\partial \phi}{\partial y} r \cos \theta \\ \frac{\partial^2 \phi}{\partial \theta^2} &= \frac{\partial^2 \phi}{\partial x^2} \frac{\partial x}{\partial \theta} (-r \sin \theta) + \frac{\partial^2 \phi}{\partial y \partial x} \frac{\partial y}{\partial \theta} (-r \sin \theta) + \frac{\partial \phi}{\partial x} (-r \cos \theta) \\ &\quad + \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial x}{\partial \theta} (r \cos \theta) + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial y}{\partial \theta} (r \cos \theta) + \frac{\partial \phi}{\partial y} (-r \sin \theta) \\ &= \frac{\partial^2 \phi}{\partial x^2} r^2 \sin^2 \theta + \frac{\partial^2 \phi}{\partial y \partial x} (-2r^2 \sin \theta \cos \theta) + \frac{\partial^2 \phi}{\partial y^2} r^2 \cos^2 \theta \\ &\quad + \frac{\partial \phi}{\partial x} (-r \cos \theta) + \frac{\partial \phi}{\partial y} (-r \sin \theta). \end{aligned}$$

Combining these partial derivatives, one gets

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

11. $\text{Im } f(z) = y - \frac{y}{x^2 + y^2} = 0 \implies yx^2 + y^3 - y = y(x^2 + y^2 - 1) = 0.$

The points satisfying $x^2 + y^2 - 1 = 0$ lie on the circle $|z| = 1$. The points (other than $z = 0$) satisfying $y = 0$ lie on the real axis.

12. $f(z) = z^n = r^n(\cos \theta + i \sin \theta)^n = r^n(\cos n\theta + i \sin n\theta) \implies$
 $\text{Re } f(z) = r^n \cos n\theta$ and $\text{Im } f(z) = r^n \sin n\theta$ are harmonic since f is analytic.

13. $\phi(x, y) = \text{Im } z^4 = r^4 \sin 4\theta = -4xy^3 + 4x^3y$

14. Let $\phi(x, y) = \ln |f(z)| = \frac{1}{2} \ln(u^2 + v^2)$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = \frac{u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}}{u^2 + v^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{(v^2 - u^2) \left[\left(\frac{\partial u}{\partial x} \right)^2 - \left(\frac{\partial v}{\partial x} \right)^2 \right] - 4uv \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}}{(u^2 + v^2)^2} + \frac{u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 v}{\partial x^2}}{u^2 + v^2}$$

A similar calculation yields $\frac{\partial^2 \phi}{\partial y^2}$. By applying Laplace's equation and the Cauchy-Riemann equations of u and v to $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$, the sum simplifies to reveal that $\Delta \phi = 0$.

15. Consider $\phi(z) = \text{Re}(Az^n + Bz^{-n}) + C$ which is harmonic for $1 \leq |z| \leq 2$. Consider the polar form for z . $z = re^{i\theta}$ and select $n=3$ to agree with the cosine argument. $\phi(re^{i\theta}) = Ar^3 \text{Re}(e^{i3\theta}) + Br^{-3} \text{Re}(e^{-i3\theta}) + C$.
 $\phi(re^{i\theta}) = Ar^3 \cos 3\theta + Br^{-3} \cos 3\theta + C = (Ar^3 + Br^{-3}) \cos 3\theta + C$.

$r=1 \implies (A+B) \cos 3\theta + C = 0 \implies A + B = 0, C = 0.$

$r=2 \implies (A \cdot 8 + B/8) \cos 3\theta = 5 \cos 3\theta. A = 40/63, B = -40/63$

$\phi(re^{i\theta}) = (40/63)(r^3 - r^{-3}) \cos 3\theta = (40/63) \text{Re}(z^3 - z^{-3}).$

16. $\phi(x, y) = \frac{1}{\ln 3} \ln |z| - 1$ or $\phi(x, y) = \ln \left| \frac{z}{3} \right|$ are two possibilities.

17. a. $\phi(x, y) = \operatorname{Re}(z^2 + 5z + 1) = x^2 - y^2 + 5x + 1$

b. $\phi(x, y) = 2\operatorname{Re}\left(\frac{z^2}{z + 2i}\right) = \frac{2x(x^2 + 4y + y^2)}{x^2 + y^2 + 4y + 4}$

18. Let $u = \phi_x$, $v = -\phi_y$. Then

$$\frac{\partial u}{\partial x} = \phi_{xx} = -\phi_{yy} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \phi_{xy} = -\frac{\partial v}{\partial x}$$

19. $\cos^2\theta = (\frac{1}{2})\cos 2\theta + \frac{1}{2} = \phi(z) = A\operatorname{Re}(r^{-2}e^{-i2\theta}) + B = Ar^{-2}\cos 2\theta + B$. In the limit as $r \rightarrow \infty$ $\phi(z) = \frac{1}{2} \Rightarrow B = \frac{1}{2}$. On the circle $|z|=1$, $r=1 \Rightarrow A = \frac{1}{2}$. $\phi(z) = (\frac{1}{2})r^{-2}\cos 2\theta + \frac{1}{2} = \operatorname{Re}[1/(2z^2)] + \frac{1}{2}$.

20. In order that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, let $v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, \eta) d\eta + \psi(x)$. Then

$$\begin{aligned} \frac{\partial v}{\partial x} &= \int_0^y \frac{\partial^2 u}{\partial x^2}(x, \eta) d\eta + \psi'(x) \\ &= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x, \eta) d\eta + \psi'(x) \quad (\text{because } u \text{ is harmonic}) \\ &= -\frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, 0) + \psi'(x). \end{aligned}$$

In order that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, it must be true that $\psi'(x) = -\frac{\partial u}{\partial y}(x, 0)$.

Thus,

$$\psi(x) = -\int_0^x \frac{\partial u}{\partial y}(\zeta, 0) d\zeta + a$$

and

$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, \eta) d\eta - \int_0^x \frac{\partial u}{\partial y}(\zeta, 0) d\zeta + a.$$

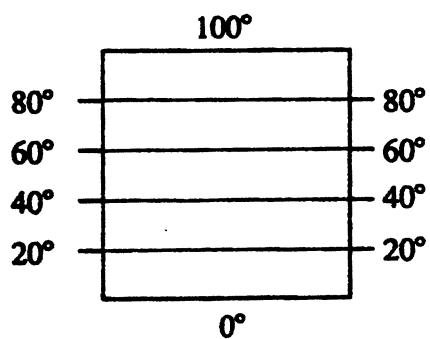
21. It is easily verified that $u = \ln|z|$ satisfies Laplace's equation on $\mathbb{C} \setminus \{0\}$ and that $u + iv = \ln|z| + i\operatorname{Arg}(z)$ satisfies the Cauchy-Riemann equations on the domain $D = \mathbb{C} \setminus \{\text{nonpositive real axis}\}$, so that

$\operatorname{Arg}(z)$ is a harmonic conjugate of u on D . By Problem 4, any harmonic conjugate of u has to be of the form $\operatorname{Arg}(z) + a$ in D . It is impossible to have a harmonic conjugate of this form that is continuous on $\mathbb{C} \setminus \{0\}$.

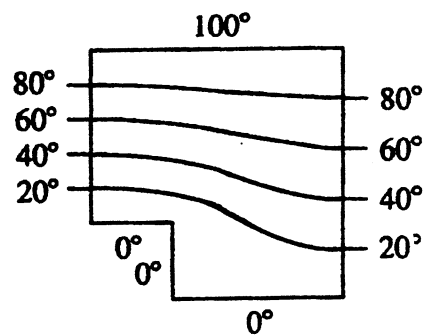
22.
$$\begin{aligned} \frac{\partial u}{\partial x} &= \phi_{xx}\phi_y + \phi_x\phi_{yx} + \psi_{xx}\psi_y + \psi_x\psi_{yx} \\ &= -\phi_{yy}\phi_y + \phi_x\phi_{yx} - \psi_{yy}\psi_y + \psi_x\psi_{yx} = \frac{\partial v}{\partial y} \end{aligned}$$

EXERCISES 2.6: Steady-State Temperature as a Harmonic Function.

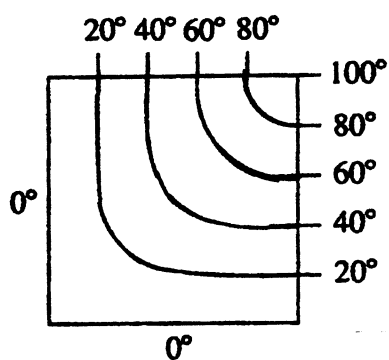
1. a.



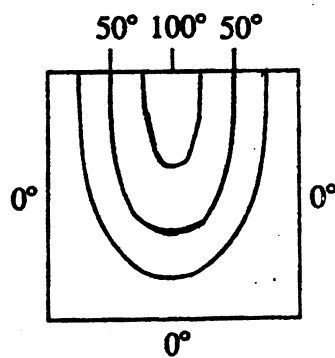
b.



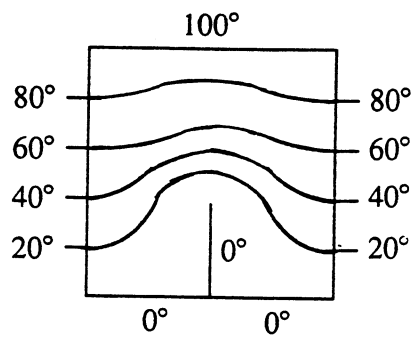
c.



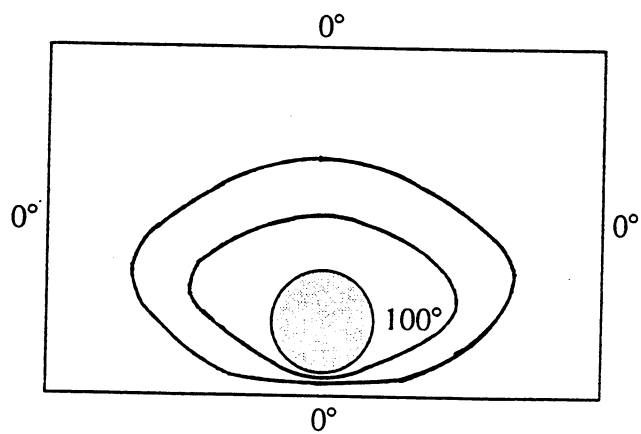
d.



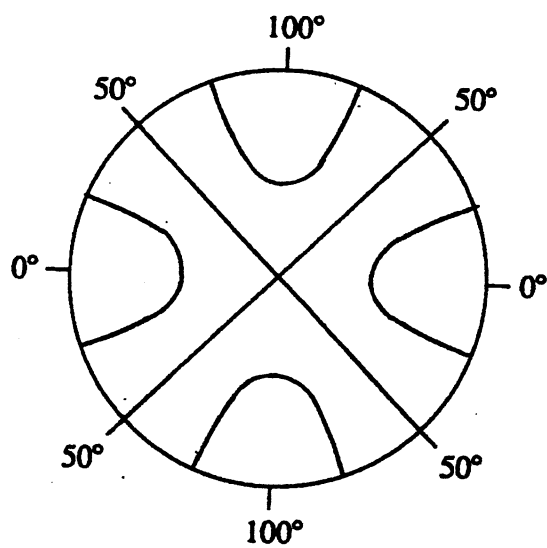
e.



2. This does not violate the maximum principle.



3. This does not violate the maximum principle.



Exercises 2.7

1. $f(z) = z^2 + c$ where c is a real constant.
 $\zeta_1 = (1 + \sqrt{1-4c})/2$, $\zeta_2 = (1 - \sqrt{1-4c})/2$
 Only ζ_2 is an attractor for $-3/4 < c < 1/4$.
2. $f(\zeta) = \zeta$ and $f'(\zeta) > 1$. Therefore we can pick a real number ρ between 1 and $|f'(\zeta)|$ such that $|f(z) - \zeta| = \rho|z - \zeta|$ for all z in a sufficiently small disk around ζ . If any point z_0 in this disk is the seed for an orbit $z_1 = f(z_0)$, $z_2 = f(z_1)$, ..., $z_n = f(z_{n-1})$, then we have $|z_n - \zeta| \geq \rho|z_{n-1} - \zeta| \geq \dots \geq \rho^n|z_0 - \zeta|$. Because $\rho > 1$, the point z_n moves away from ζ until the magnitude of the derivative becomes 1 or less. The orbit is out of the disk.
3. (a) Fixed points are $\zeta_1 = i$, $\zeta_2 = -i$. Both are repellers.
 (b) Fixed points are $\zeta_1 = 1/2$, $\zeta_2 = -1/2$, $\zeta_3 = -1$. Fixed points ζ_1 and ζ_3 are repellers, but fixed point ζ_2 is an attractor.
4. $z_0 = e^{i2\pi\alpha}$ with α an irrational real number. $z_n = e^{i2\pi\alpha 2^n}$. Because $|z_n| = 1$, the trajectory will follow the unit circle. If iterations p and q coincide, $2\pi\alpha 2^p - 2\pi\alpha 2^q = 2\pi\alpha(2^p - 2^q) = 2\pi k$ for some integer k . But because $(2^p - 2^q)$ is an integer that can be represented by m , the equation $2\pi\alpha m = 2\pi k$ is satisfied only if $k = \alpha m$ or $\alpha = k/m$. Because α is irrational it cannot be represented by a rational number and no iterations repeat.
5. Fixed points are $\zeta_1 = -1/2 + i\sqrt{5}/2$ (an attractor) and $\zeta_2 = -1/2 - i\sqrt{5}/2$ (a repeller).
6. $f(z) = z^2$. The seed is z_0 . $z_1 = z_0^2$, $z_2 = z_0^4$, ..., $z_n = z_0^{2^n}$. To have an n cycle $z_n = z_0 = z_0^{2^n}$. Or $z_n/z_0 = z_0^{2^n-1} = 1 = e^{i2\pi}$. Solving gives $z_0 = e^{i2\pi/(2^n-1)}$.
7. The cycle is 4. $2^4(2\pi/p) = 2\pi \pmod p \Rightarrow 2^4 = 1 \pmod p$. $p=3,5,15$. 3 will give repeated cycles of length 2. 5 and 15 will give the desired cycles of length 4.
8. Student Matlab: `n=100;c=.253; zo=0;y(1)=zo;`
`for k=1:n-1,y(k+1)=y(k)^2+c;end`
`plot(y)`
9. If $|\alpha| \leq 1$ the whole complex plane is the filled Julia set. If $|\alpha| \geq 1$ the origin is the filled Julia set.
10. $f(z) = z - F(z)/F'(z)$. $f(\zeta) = \zeta - F(\zeta)/F'(\zeta) = \zeta \Rightarrow F(\zeta)/F'(\zeta) = 0 \Rightarrow F(\zeta) = 0$ with the possible exception of the points where $F'(\zeta) = 0$.
 $f'(z) = 1 - F'(z)/F'(z) + F(z)F''(z)/(F'(z))^2 = F(z)F''(z)/(F'(z))^2$
 $f'(\zeta) = F(\zeta)F''(\zeta)/(F'(\zeta))^2 = 0$ where $F'(\zeta) \neq 0$ and every zero of $F(z)$ is an attractor as long as $F'(\zeta) \neq 0$.