

CHAPTER 2

Matrices

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CHAPTER 2

Matrices

Section 2.1 Operations with Matrices

2. $x = 13, y = 12$

4. $x + 2 = 2x + 6$ $2y = 18$
 $-4 = x$ $y = 9$

$2x = -8$ $y + 2 = 11$
 $x = -4$ $y = 9$

6. (a) $A + B = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 6+1 & -1+4 \\ 2+(-1) & 4+5 \\ -3+1 & 5+10 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 1 & 9 \\ -2 & 15 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 6-1 & -1-4 \\ 2-(-1) & 4-5 \\ -3-1 & 5-10 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 3 & -1 \\ -4 & -5 \end{bmatrix}$

(c) $2A = 2 \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2(6) & 2(-1) \\ 2(2) & 2(4) \\ 2(-3) & 2(5) \end{bmatrix} = \begin{bmatrix} 12 & -2 \\ 4 & 8 \\ -6 & 10 \end{bmatrix}$

(d) $2A - B = \begin{bmatrix} 12 & -2 \\ 4 & 8 \\ -6 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 12-1 & -2-4 \\ 4-(-1) & 8-5 \\ -6-1 & 10-10 \end{bmatrix} = \begin{bmatrix} 11 & -6 \\ 5 & 3 \\ -7 & 0 \end{bmatrix}$

(e) $B + \frac{1}{2}A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} + \begin{bmatrix} 3 & -\frac{1}{2} \\ 1 & 2 \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 4 & \frac{7}{2} \\ 0 & 7 \\ -\frac{1}{2} & \frac{25}{2} \end{bmatrix}$

8. (a) $A + B = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3+0 & 2+2 & -1+1 \\ 2+5 & 4+4 & 5+2 \\ 0+2 & 1+1 & 2+0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 8 & 7 \\ 2 & 2 & 2 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3-0 & 2-2 & -1-1 \\ 2-5 & 4-4 & 5-2 \\ 0-2 & 1-1 & 2-0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -2 \\ -3 & 0 & 3 \\ -2 & 0 & 2 \end{bmatrix}$

(c) $2A = 2 \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(3) & 2(2) & 2(-1) \\ 2(2) & 2(4) & 2(5) \\ 2(0) & 2(1) & 2(2) \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 4 & 8 & 10 \\ 0 & 2 & 4 \end{bmatrix}$

(d) $2A - B = 2 \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 4 & 8 & 10 \\ 0 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -3 \\ -1 & 4 & 8 \\ -2 & 1 & 4 \end{bmatrix}$

(e) $B + \frac{1}{2}A = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 1 & -\frac{1}{2} \\ 1 & 2 & \frac{5}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 3 & \frac{1}{2} \\ 6 & 6 & \frac{9}{2} \\ 2 & \frac{3}{2} & 1 \end{bmatrix}$

10. (a) $A + B$ is not possible. A and B have different sizes.

(b) $A - B$ is not possible. A and B have different sizes.

$$(c) 2A = 2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

(d) $2A - B$ is not possible. A and B have different sizes.

(e) $B + \frac{1}{2}A$ is not possible. A and B have different sizes.

$$12. (a) c_{23} = 5a_{23} + 2b_{23} = 5(2) + 2(11) = 32$$

$$(b) c_{32} = 5a_{32} + 2b_{32} = 5(1) + 2(4) = 13$$

$$16. (a) AB = \begin{bmatrix} 2 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2(4) + (-2)(2) & 2(1) + (-2)(-2) \\ -1(4) + 4(2) & -1(1) + 4(-2) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & -9 \end{bmatrix}$$

$$(b) BA = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 4(2) + 1(-1) & 4(-2) + 1(4) \\ 2(2) + (-2)(-1) & 2(-2) + (-2)(4) \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 6 & -12 \end{bmatrix}$$

$$18. (a) AB = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)(2) + 7(1) & 1(1) + (-1)(1) + 7(-3) & 1(2) + (-1)(1) + 7(2) \\ 2(1) + (-1)(2) + 8(1) & 2(1) + (-1)(1) + 8(-3) & 2(2) + (-1)(1) + 8(2) \\ 3(1) + 1(2) + (-1)(1) & 3(1) + 1(1) + (-1)(-3) & 3(2) + 1(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 7 & 5 \end{bmatrix}$$

$$(b) BA = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(2) + 2(3) & 1(-1) + 1(-1) + 2(1) & 1(7) + 1(8) + 2(-1) \\ 2(1) + 1(2) + 1(3) & 2(-1) + 1(-1) + 1(1) & 2(7) + 1(8) + 1(-1) \\ 1(1) + (-3)(2) + 2(3) & 1(-1) + (-3)(-1) + 2(1) & 1(7) + (-3)(8) + 2(-1) \end{bmatrix} = \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}$$

$$20. (a) AB = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 0 & 4 \\ 4 & -2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3(1) + 2(2) + 1(1) & 3(2) + 2(-1) + 1(-2) \\ -3(1) + 0(2) + 4(1) & -3(2) + 0(-1) + 4(-2) \\ 4(1) + (-2)(2) + (-4)(1) & 4(2) + (-2)(-1) + (-4)(-2) \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 1 & -14 \\ -4 & 18 \end{bmatrix}$$

(b) BA is not defined because B is 3×2 and A is 3×3 .

$$22. (a) AB = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -1(2) & -1(1) & -1(3) & -1(2) \\ 2(2) & 2(1) & 2(3) & 2(2) \\ -2(2) & -2(1) & -2(3) & -2(2) \\ 1(2) & 1(1) & 1(3) & 1(2) \end{bmatrix} = \begin{bmatrix} -2 & -1 & -3 & -2 \\ 4 & 2 & 6 & 4 \\ -4 & -2 & -6 & -4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

$$(b) BA = \begin{bmatrix} 2 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1) + 1(2) + 3(-2) + 2(1) \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix}$$

24. (a) AB is not defined because A is 2×2 and B is 3×2 .

$$(b) BA = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2(2) + 1(5) & 2(-3) + 1(2) \\ 1(2) + 3(5) & 1(-3) + 3(2) \\ 2(2) + (-1)(5) & 2(-3) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ 17 & 3 \\ -1 & -8 \end{bmatrix}$$

14. Simplifying the right side of the equation produces

$$\begin{bmatrix} w & x \\ y & x \end{bmatrix} = \begin{bmatrix} -4 + 2y & 3 + 2w \\ 2 + 2z & -1 + 2x \end{bmatrix}$$

By setting corresponding entries equal to each other, you obtain four equations.

$$\begin{aligned} w &= -4 + 2y \\ x &= 3 + 2w \\ y &= 2 + 2z \\ x &= -1 + 2x \end{aligned} \Rightarrow \begin{cases} -2y + w = -4 \\ x - 2w = 3 \\ y - 2z = 2 \\ x = 1 \end{cases}$$

The solution to this linear system is: $x = 1$, $y = \frac{3}{2}$,

$z = -\frac{1}{4}$, and $w = -1$.

$$\begin{aligned}
26. (a) AB &= \begin{bmatrix} 2 & 1 & 2 \\ 3 & -1 & -2 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 & 3 \\ -1 & 2 & -3 & -1 \\ -2 & 1 & 4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 2(4) + 1(-1) + 2(-2) & 2(0) + 1(2) + 2(1) & 2(1) + 1(-3) + 2(4) & 2(3) + 1(-1) + 2(3) \\ 3(4) + (-1)(-1) + (-2)(-2) & 3(0) + (-1)(2) + (-2)(1) & 3(1) + (-1)(-3) + (-2)(4) & 3(3) + (-1)(-1) + (-2)(3) \\ -2(4) + 1(-1) + (-2)(-2) & -2(0) + 1(2) + (-2)(1) & -2(1) + 1(-3) + (-2)(4) & -2(3) + 1(-1) + (-2)(3) \end{bmatrix} \\
&= \begin{bmatrix} 3 & 4 & 7 & 11 \\ 17 & -4 & -2 & 4 \\ -5 & 0 & -13 & -13 \end{bmatrix}
\end{aligned}$$

(b) BA is not defined because B is 3×4 and A is 3×3 .

28. (a) AB is not defined because A is 2×5 and B is 2×2 .

$$\begin{aligned}
(b) BA &= \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 20 \end{bmatrix} \\
&= \begin{bmatrix} 1(1) + 6(6) & 1(0) + 6(13) & 1(3) + 6(8) & 1(-2) + 6(-17) & 1(4) + 6(20) \\ 4(1) + 2(6) & 4(0) + 2(13) & 4(3) + 2(8) & 4(-2) + 2(-17) & 4(4) + 2(20) \end{bmatrix} \\
&= \begin{bmatrix} 37 & 78 & 51 & -104 & 124 \\ 16 & 26 & 28 & -42 & 56 \end{bmatrix}
\end{aligned}$$

30. $C + E$ is not defined because C and E have different sizes.

32. $-4A$ is defined and has size 3×4 because A has size 3×4 .

34. BE is defined. Because B has size 3×4 and E has size 4×3 , the size of BE is 3×3 .

36. $2D + C$ is defined and has size 4×2 because $2D$ and C have size 4×2 .

38. As a system of linear equations, $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 3x_4 &= 0 \\
x_1 - x_2 + x_4 &= 0 \\
x_2 - x_3 + 2x_4 &= 0
\end{aligned}$$

Use Gauss-Jordan elimination on the augmented matrix for this system.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Choosing $x_4 = t$, the solution is

$x_1 = -2t$, $x_2 = -t$, $x_3 = t$, and $x_4 = t$, where t is any real number.

40. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

42. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} -4 & 9 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -13 \\ 12 \end{bmatrix}$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} -4 & 9 & -13 \\ 1 & -3 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -23 \\ 0 & 1 & -\frac{35}{3} \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -23 \\ -\frac{35}{3} \end{bmatrix}$.

44. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & -1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 2 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 0 & 0 \end{array} \right]$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}.$

46. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & 0 \\ 0 & -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -11 \\ 40 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 17 & 0 & 0 \\ 1 & 3 & 0 & -11 & 0 & 0 \\ 0 & -6 & 5 & 40 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{array} \right]$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}.$

48. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\left[\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 5 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$

50. The augmented matrix row reduces as follows.

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 4 & 1 & 1 & 0 & -2 & -3 \\ -1 & 0 & 2 & 3 & 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 & 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & -2 & -3 & 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 2 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

There are an infinite number of solutions. For example, $x_3 = 0$, $x_2 = 2$, $x_1 = -3$.

So, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}.$

52. The augmented matrix row reduces as follows.

$$\left[\begin{array}{ccc|ccc} -3 & 5 & -22 & 1 & -3 & 10 \\ 3 & 4 & 4 & 0 & 9 & -18 \\ 4 & -8 & 32 & 0 & -4 & 8 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & 10 & 1 & -3 & 10 \\ 0 & 9 & -18 & 0 & 9 & -18 \\ 0 & -4 & 8 & 0 & -4 & 8 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & 10 & 1 & 0 & 4 \\ 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & 0 & 0 \end{array} \right]$$

So,

$$\mathbf{b} = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix} = 4 \begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}.$$

54. Expanding the left side of the equation produces

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2a_{11} - a_{21} & 2a_{12} - a_{22} \\ 3a_{11} - 2a_{21} & 3a_{12} - 2a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and you obtain the system

$$\begin{aligned} 2a_{11} - a_{21} &= 1 \\ 2a_{12} - a_{22} &= 0 \\ 3a_{11} - 2a_{21} &= 0 \\ 3a_{12} - 2a_{22} &= 1. \end{aligned}$$

Solving by Gauss-Jordan elimination yields

$$a_{11} = 2, a_{12} = -1, a_{21} = 3, \text{ and } a_{22} = -2.$$

So, you have $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$

56. Expanding the left side of the matrix equation produces

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2a + 3b & a + b \\ 2c + 3d & c + d \end{bmatrix} = \begin{bmatrix} 3 & 17 \\ 4 & -1 \end{bmatrix}.$$

You obtain two systems of linear equations (one involving a and b and the other involving c and d).

$$2a + 3b = 3$$

$$a + b = 17,$$

and

$$2c + 3d = 4$$

$$c + d = -1.$$

Solving by Gauss-Jordan elimination yields $a = 48$,

$$b = -31, c = -7, \text{ and } d = 6.$$

$$58. AA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$62. (a) AB = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{bmatrix}$$

The i th row of B has been multiplied by a_{ii} , the i th diagonal entry of A .

$$(b) BA = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{22}b_{12} & a_{33}b_{13} \\ a_{11}b_{21} & a_{22}b_{22} & a_{33}b_{23} \\ a_{11}b_{31} & a_{22}b_{32} & a_{33}b_{33} \end{bmatrix}$$

The i th column of B has been multiplied by a_{ii} , the i th diagonal entry of A .

(c) If $a_{11} = a_{22} = a_{33}$, then $AB = a_{11}B = BA$.

64. The trace is the sum of the elements on the main diagonal.

$$1 + 1 + 1 = 3$$

$$60. AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-7) + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + (-5)4 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly,

$$BA = \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

66. The trace is the sum of the elements on the main diagonal.

$$1 + 0 + 2 + (-3) = 0$$

68. Let $AB = [c_{ij}]$, where $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$. Then, $Tr(AB) = \sum_{i=1}^n c_{ii} = \sum_{i=1}^n \left(\sum_{k=1}^n a_{ik}b_{ki} \right)$.

Similarly, if $BA = [d_{ij}]$, $d_{ij} = \sum_{k=1}^n b_{ik}a_{kj}$. Then $Tr(BA) = \sum_{i=1}^n d_{ii} = \sum_{i=1}^n \left(\sum_{k=1}^n b_{ik}a_{ki} \right) = Tr(AB)$.

$$70. AB = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & \cos \alpha(-\sin \beta) - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & \sin \alpha(-\sin \beta) + \cos \alpha \cos \beta \end{bmatrix}$$

$$BA = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} \cos \beta \cos \alpha - \sin \beta \sin \alpha & \cos \beta(-\sin \alpha) - \sin \beta \cos \alpha \\ \sin \beta \cos \alpha + \cos \beta \sin \alpha & \sin \beta(-\sin \alpha) + \cos \beta \cos \alpha \end{bmatrix}$$

$$\text{So, you see that } AB = BA = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}.$$

72. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Then the matrix equation $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is equivalent to

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This equation implies that

$$a_{11}b_{11} + a_{12}b_{21} - b_{11}a_{11} - b_{12}a_{21} = a_{12}b_{21} - b_{12}a_{21} = 1$$

$$a_{21}b_{12} + a_{22}b_{22} - b_{21}a_{12} - b_{22}a_{22} = a_{21}b_{12} - b_{21}a_{12} = 1$$

which is impossible. So, the original equation has no solution.

74. Assume that A is an $m \times n$ matrix and B is a $p \times q$ matrix. Because the product AB is defined, you know that $n = p$. Moreover, because AB is square, you know that $m = q$. Therefore, B must be of order $n \times m$, which implies that the product BA is defined.

76. Let rows s and t be identical in the matrix A . So, $a_{sj} = a_{tj}$ for $j = 1, \dots, n$. Let $AB = [c_{ij}]$, where

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}. \text{ Then, } c_{sj} = \sum_{k=1}^n a_{sk}b_{kj}, \text{ and } c_{tj} = \sum_{k=1}^n a_{tk}b_{kj}. \text{ Because } a_{sk} = a_{tk} \text{ for } k = 1, \dots, n, \text{ rows } s \text{ and } t \text{ of } AB$$

are the same.

78. (a) No, the matrices have different sizes.
 (b) No, the matrices have different sizes.
 (c) Yes; No, BA is undefined.

80. $1.2 \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix} = \begin{bmatrix} 84 & 60 & 30 \\ 42 & 120 & 84 \end{bmatrix}$

82. (a) Multiply the matrix for 2010 by $\frac{1}{3090}$. This produces a matrix giving the information as percents of the total population.

$$A = \frac{1}{3090} \begin{bmatrix} 12,306 & 35,240 & 7830 \\ 16,095 & 41,830 & 9051 \\ 27,799 & 72,075 & 14,985 \\ 5698 & 13,717 & 2710 \\ 12,222 & 31,867 & 5901 \end{bmatrix} \approx \begin{bmatrix} 3.98 & 11.40 & 2.53 \\ 5.21 & 13.54 & 2.93 \\ 9.00 & 23.33 & 4.85 \\ 1.84 & 4.44 & 0.88 \\ 3.96 & 10.31 & 1.91 \end{bmatrix}$$

Multiply the matrix for 2013 by $\frac{1}{3160}$. This produces a matrix giving the information as percents of the total population.

$$B = \frac{1}{3160} \begin{bmatrix} 12,026 & 35,471 & 8446 \\ 15,772 & 41,985 & 9791 \\ 27,954 & 73,703 & 16,727 \\ 5710 & 14,067 & 3104 \\ 12,124 & 32,614 & 6636 \end{bmatrix} \approx \begin{bmatrix} 3.81 & 11.23 & 2.67 \\ 4.99 & 13.29 & 3.10 \\ 8.85 & 23.32 & 5.29 \\ 1.81 & 4.45 & 0.98 \\ 3.84 & 10.32 & 2.10 \end{bmatrix}$$

$$(b) \quad B - A = \begin{bmatrix} 3.81 & 11.23 & 2.67 \\ 4.99 & 13.29 & 3.10 \\ 8.85 & 23.32 & 5.29 \\ 1.81 & 4.45 & 0.98 \\ 3.84 & 10.32 & 2.10 \end{bmatrix} - \begin{bmatrix} 3.98 & 11.40 & 2.53 \\ 5.21 & 13.54 & 2.93 \\ 9.00 & 23.33 & 4.85 \\ 1.84 & 4.44 & 0.88 \\ 3.96 & 10.31 & 1.91 \end{bmatrix} = \begin{bmatrix} -0.18 & -0.18 & 0.14 \\ -0.22 & -0.25 & 0.17 \\ -0.15 & -0.001 & 0.44 \\ -0.04 & 0.01 & 0.11 \\ -0.12 & 0.01 & 0.19 \end{bmatrix}$$

- (c) The 65+ age group is projected to show relative growth from 2010 to 2013 over all regions because its column in $B - A$ contains all positive percents.

$$84. AB = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right] \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \end{array} \right]$$

86. (a) True. The number of elements in a row of the first matrix must be equal to the number of elements in a column of the second matrix. See page 43 of the text.

(b) True. See page 45 of the text.

Section 2.2 Properties of Matrix Operations

$$2. \begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6+0+(-11) & 8+5+(-7) \\ -1+(-3)+2 & 0+(-1)+(-1) \end{bmatrix} = \begin{bmatrix} -5 & 6 \\ -2 & -2 \end{bmatrix}$$

$$4. \frac{1}{2}([5 \ -2 \ 4 \ 0] + [14 \ 6 \ -18 \ 9]) = \frac{1}{2}[5+14 \ -2+6 \ 4+(-18) \ 0+9] = \frac{1}{2}[19 \ 4 \ -14 \ 9] = \left[\frac{19}{2} \ 2 \ -7 \ \frac{9}{2}\right]$$

$$\begin{aligned} 6. -1 \begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6} \left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix} \right) &= \begin{bmatrix} -4 & -11 \\ 2 & 1 \\ -9 & -3 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -5+7 & -1+5 \\ 3+(-9) & 4+(-1) \\ 0+6 & 13+(-1) \end{bmatrix} \\ &= \begin{bmatrix} -4 & -11 \\ 2 & 1 \\ -9 & -3 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 2 & 4 \\ -6 & 3 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} -4 & -11 \\ 2 & 1 \\ -9 & -3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -1 & \frac{1}{2} \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -4 + \frac{1}{3} & -11 + \frac{2}{3} \\ 2 + (-1) & 1 + \frac{1}{2} \\ -9 + 1 & -3 + 2 \end{bmatrix} = \begin{bmatrix} -\frac{11}{3} & -\frac{31}{3} \\ 1 & \frac{3}{2} \\ -8 & -1 \end{bmatrix} \end{aligned}$$

$$8. A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

$$10. (a+b)B = (3+(-4)) \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = (-1) \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

$$12. (ab)O = (3)(-4) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (-12) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$14. (a) X = 3A - 2B$$

$$\begin{aligned} &= \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ 4 & 0 \\ -8 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix} \end{aligned}$$

$$(b) 2X = 2A - B$$

$$\begin{aligned} 2X &= \begin{bmatrix} -4 & -2 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix} \\ 2X &= \begin{bmatrix} -4 & -5 \\ 0 & 0 \\ 10 & -7 \end{bmatrix} \\ X &= \begin{bmatrix} -2 & -\frac{5}{2} \\ 0 & 0 \\ 5 & -\frac{7}{2} \end{bmatrix} \end{aligned}$$

(c) $2X + 3A = B$

$$2X + \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$

$$2X = \begin{bmatrix} 6 & 6 \\ -1 & 0 \\ -13 & 11 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix}$$

(d) $2A + 4B = -2X$

$$\begin{bmatrix} -4 & -2 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 12 \\ 8 & 0 \\ -16 & -4 \end{bmatrix} = -2X$$

$$\begin{bmatrix} -4 & 10 \\ 10 & 0 \\ -10 & -12 \end{bmatrix} = -2X$$

$$\begin{bmatrix} 2 & -5 \\ -5 & 0 \\ 5 & 6 \end{bmatrix} = X$$

$$16. \ c(CB) = (-2) \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \right)$$

$$= (-2) \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 \\ 2 & 6 \end{bmatrix}$$

$$18. \ C(BC) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & -1 \end{bmatrix}$$

$$20. \ B(C + O) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix}$$

$$22. \ B(cA) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \left((-2) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -10 & 0 \\ 2 & 0 & 10 \end{bmatrix}$$

$$24. \ (a) \ (AB)C = \left(\begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -5 & 0 \\ -2 & 3 & 3 \end{bmatrix} \right) \begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 26 & 6 \\ 7 & -14 & -9 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 0 \\ -12 & 5 \end{bmatrix}$$

$$(b) \ A(BC) = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} \left(\begin{bmatrix} 1 & -5 & 0 \\ -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 0 \\ -12 & 5 \end{bmatrix}$$

$$26. \ AB = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix}$$

$$BA = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \neq AB$$

$$28. \ AC = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 9 \\ 16 & -8 & 12 \\ 4 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix} = BC$$

But $A \neq B$.

$$30. \ AB = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

But $A \neq O$ and $B \neq O$.

$$32. \quad AT = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$34. \quad A + IA = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\ = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$36. \quad A^2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\text{So, } A^4 = (A^2)^2 = I_2^2 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

38. In general, $AB \neq BA$ for matrices.

$$40. \quad D^T = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}^T = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}$$

$$42. \quad (AB)^T = \left(\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \right)^T = \begin{bmatrix} 1 & 1 \\ -4 & -2 \end{bmatrix}^T = \begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix}$$

$$44. \quad (AB)^T = \left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \right)^T = \begin{bmatrix} 4 & 0 & -7 \\ 2 & 4 & 7 \\ 4 & 2 & 2 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 4 & 2 \\ -7 & 7 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 4 & 2 \\ -7 & 7 & 2 \end{bmatrix}$$

$$46. \quad (a) \quad A^T A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 11 & 21 \end{bmatrix}$$

$$(b) \quad AA^T = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 25 & -8 \\ 2 & -8 & 4 \end{bmatrix}$$

$$48. \quad (a) \quad A^T A = \begin{bmatrix} 4 & 2 & 14 & 6 \\ -3 & 0 & -2 & 8 \\ 2 & 11 & 12 & -5 \\ 0 & -1 & -9 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 252 & 8 & 168 & -104 \\ 8 & 77 & -70 & 50 \\ 168 & -70 & 294 & -139 \\ -104 & 50 & -139 & 98 \end{bmatrix}$$

$$(b) \quad AA^T = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 & 14 & 6 \\ -3 & 0 & -2 & 8 \\ 2 & 11 & 12 & -5 \\ 0 & -1 & -9 & 4 \end{bmatrix} = \begin{bmatrix} 29 & 30 & 86 & -10 \\ 30 & 126 & 169 & -47 \\ 86 & 169 & 425 & -28 \\ -10 & -47 & -28 & 141 \end{bmatrix}$$

$$50. \quad A^{17} = \begin{bmatrix} (1)^{17} & 0 & 0 & 0 & 0 \\ 0 & (-1)^{17} & 0 & 0 & 0 \\ 0 & 0 & (1)^{17} & 0 & 0 \\ 0 & 0 & 0 & (-1)^{17} & 0 \\ 0 & 0 & 0 & 0 & (1)^{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$52. \quad A^{20} = \begin{bmatrix} (1)^{20} & 0 & 0 & 0 & 0 \\ 0 & (-1)^{20} & 0 & 0 & 0 \\ 0 & 0 & (1)^{20} & 0 & 0 \\ 0 & 0 & 0 & (-1)^{20} & 0 \\ 0 & 0 & 0 & 0 & (1)^{20} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$54. \text{ Because } A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (-1)^3 & 0 \\ 0 & 0 & (3)^3 \end{bmatrix}, \text{ you have } A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

56. (a) False. In general, for $n \times n$ matrices A and B it is *not* true that $AB = BA$. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

$$\text{Then } AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = BA.$$

(b) False. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = AC$, but $B \neq C$.

(c) True. See Theorem 2.6, part 2 on page 57.

$$\begin{aligned} 58. \quad & aX + A(bB) = b(AB + IB) && \text{Original equation} \\ & aX + (Ab)B = b(AB + B) && \text{Associative property; property of the identity matrix} \\ & aX + bAB = bAB + bB && \text{Property of scalar multiplication; distributive property} \\ & aX + bAB + (-bAB) = bAB + bB + (-bAB) && \text{Add } -bAB \text{ to both sides.} \\ & aX = bAB + bB + (-bAB) && \text{Additive inverse} \\ & aX = bAB + (-bAB) + bB && \text{Commutative property} \\ & aX = bB && \text{Additive inverse} \\ & X = \frac{b}{a}B && \text{Divide by } a. \end{aligned}$$

$$\begin{aligned} 60. \quad f(A) &= -10 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5 \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} - 2 \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^2 + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^3 \\ &= - \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} + \begin{bmatrix} 10 & 5 & -5 \\ 5 & 0 & 10 \\ -5 & 5 & 15 \end{bmatrix} - 2 \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0 & 5 & -5 \\ 5 & -10 & 10 \\ -5 & 5 & 5 \end{bmatrix} - 2 \begin{bmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 5 & -5 \\ 5 & -10 & 10 \\ -5 & 5 & 5 \end{bmatrix} - \begin{bmatrix} 12 & 2 & -6 \\ 0 & 6 & 10 \\ -8 & 4 & 24 \end{bmatrix} + \begin{bmatrix} 16 & 3 & -13 \\ -2 & 5 & 21 \\ -18 & 8 & 44 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 & -12 \\ 3 & -11 & 21 \\ -15 & 9 & 25 \end{bmatrix} \end{aligned}$$

$$62. (cd)A = (cd)[a_{ij}] = [(cd)a_{ij}] = [c(da_{ij})] = c[da_{ij}] = c(dA)$$

$$64. (c + d)A = (c + d)[a_{ij}] = [(c + d)a_{ij}] = [ca_{ij} + da_{ij}] = [ca_{ij}] + [da_{ij}] = c[a_{ij}] + d[a_{ij}] = cA + dA$$

66. (a) To show that $A(BC) = (AB)C$, compare the ij th entries in the matrices on both sides of this equality. Assume that A has size $n \times p$, B has size $p \times r$, and C has size $r \times m$. Then the entry in the k th row and the j th column of BC is

$\sum_{l=1}^r b_{kl}c_{lj}$. Therefore, the entry in i th row and j th column of $A(BC)$ is

$$\sum_{k=1}^p a_{ik} \sum_{l=1}^r b_{kl}c_{lj} = \sum_{k,l} a_{ik}b_{kl}c_{lj}.$$

The entry in the i th row and j th column of $(AB)C$ is $\sum_{l=1}^r d_{il}c_{lj}$, where d_{il} is the entry of AB in the i th row and the l th column.

So, $d_{il} = \sum_{k=1}^p a_{ik}b_{kl}$ for each $l = 1, \dots, r$. So, the ij th entry of $(AB)C$ is

$$\sum_{i=1}^r \sum_{k=1}^p a_{ik}b_{kl}c_{lj} = \sum_{k,l} a_{ik}b_{kl}c_{lj}.$$

Because all corresponding entries of $A(BC)$ and $(AB)C$ are equal and both matrices are of the same size ($n \times m$), you conclude that $A(BC) = (AB)C$.

(b) The entry in the i th row and j th column of $(A + B)C$ is $(a_{il} + b_{il})c_{1j} + (a_{i2} + b_{i2})c_{2j} + \dots + (a_{in} + b_{in})c_{nj}$, whereas the entry in the i th row and j th column of $AC + BC$ is $(a_{i1}c_{1j} + \dots + a_{in}c_{nj}) + (b_{i1}c_{1j} + \dots + b_{in}c_{nj})$, which are equal by the distributive law for real numbers.

(c) The entry in the i th row and j th column of $c(AB)$ is $c[a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}]$. The corresponding entry for $(cA)B$ is $(ca_{i1})b_{1j} + (ca_{i2})b_{2j} + \dots + (ca_{in})b_{nj}$ and the corresponding entry for $A(cB)$ is $a_{i1}(cb_{1j}) + a_{i2}(cb_{2j}) + \dots + a_{in}(cb_{nj})$.

Because these three expressions are equal, you have shown that $c(AB) = (cA)B = A(cB)$.

$$68. (2) (A + B)^T = ([a_{ij}] + [b_{ij}])^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^T + B^T$$

$$(3) (cA)^T = (c[a_{ij}])^T = [ca_{ij}]^T = [ca_{ji}] = c[a_{ji}] = c(A^T)$$

(4) The entry in the i th row and j th column of $(AB)^T$ is $a_{j1}b_{1i} + a_{j2}b_{2i} + \dots + a_{jn}b_{ni}$. On the other hand, the entry in the i th row and j th column of $B^T A^T$ is $b_{1i}a_{j1} + b_{2i}a_{j2} + \dots + b_{ni}a_{jn}$, which is the same.

$$70. (a) \text{ Answers will vary. Sample answer: } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

(b) Let A and B be symmetric.

If $AB = BA$, then $(AB)^T = B^T A^T = BA = AB$ and AB is symmetric.

If $(AB)^T = AB$, then $AB = (AB)^T = B^T A^T = BA$ and $AB = BA$.

$$72. \text{ Because } A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A^T, \text{ the matrix is symmetric.}$$

$$74. \text{ Because } -A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = A^T, \text{ the matrix is skew-symmetric.}$$

76. If $A^T = -A$ and $B^T = -B$, then $(A + B)^T = A^T + B^T = -A - B = -(A + B)$, which implies that $A + B$ is skew-symmetric.

78. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$.

$$A - A^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} - a_{21} & a_{13} - a_{31} & \cdots & a_{1n} - a_{n1} \\ a_{21} - a_{12} & 0 & a_{23} - a_{32} & \cdots & a_{2n} - a_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} - a_{1n} & a_{n2} - a_{2n} & a_{n3} - a_{3n} & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} - a_{21} & a_{13} - a_{31} & \cdots & a_{1n} - a_{n1} \\ -(a_{12} - a_{21}) & 0 & a_{23} - a_{32} & \cdots & a_{2n} - a_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ -(a_{1n} - a_{n1}) & -(a_{2n} - a_{n2}) & -(a_{3n} - a_{n3}) & \cdots & 0 \end{bmatrix}$$

So, $A - A^T$ is skew-symmetric.

Section 2.3 The Inverse of a Matrix

2. $AB = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2-1 & 1-1 \\ -2+2 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. $AB = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 3 & 6 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Use the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}.$$

So, the inverse is

$$A^{-1} = \frac{1}{2(2) - (-2)(2)} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

10. Use the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}.$$

So, the inverse is

$$A^{-1} = \frac{1}{(1)(-3) - (-2)(2)} \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}.$$

12. Using the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$$

you see that $ad - bc = (-1)(-3) - (1)(3) = 0$. So, the matrix has no inverse.

14. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \ A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{bmatrix}$$

16. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 10 & 5 & -7 & 1 & 0 & 0 \\ -5 & 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \ A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & -10 & -4 & 27 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} -10 & -4 & 27 \\ 2 & 1 & -5 \\ -13 & -5 & 35 \end{bmatrix}.$$

18. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 3 & 2 & 5 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

20. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 2 & 0 & 1 & 0 \\ 1 & -\frac{1}{2} & -\frac{5}{2} & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

22. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 1 & 0 & 0 \\ -0.3 & 0.2 & 0.2 & 0 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \ A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0.8 \\ 0 & 1 & 0 & -10 & 4 & 4.4 \\ 0 & 0 & 1 & 10 & -2 & -3.2 \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 0 & -2 & 0.8 \\ -10 & 4 & 4.4 \\ 10 & -2 & -3.2 \end{bmatrix}.$$

24. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

26. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \ A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

28. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 4 & 8 & -7 & 14 & 1 & 0 & 0 & 0 \\ 2 & 5 & -4 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -7 & 0 & 0 & 1 & 0 \\ 3 & 6 & -5 & 10 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using elementary row operations, reduce the matrix as follows.

$$[A \ I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 27 & -10 & 4 & -29 \\ 0 & 1 & 0 & 0 & -16 & 5 & -2 & 18 \\ 0 & 0 & 1 & 0 & -17 & 4 & -2 & 20 \\ 0 & 0 & 0 & 1 & -7 & 2 & -1 & 8 \end{bmatrix}$$

Therefore the inverse is

$$A^{-1} = \begin{bmatrix} 27 & -10 & 4 & -29 \\ -16 & 5 & -2 & 18 \\ -17 & 4 & -2 & 20 \\ -7 & 2 & -1 & 8 \end{bmatrix}$$

30. Adjoin the identity matrix to form

$$[A \ I] = \begin{bmatrix} 1 & 3 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using elementary row operations, reduce the matrix as follows.

$$[I \ A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1.5 & -4 & 2.6 \\ 0 & 1 & 0 & 0 & 0 & 0.5 & 1 & -0.8 \\ 0 & 0 & 1 & 0 & 0 & 0 & -0.5 & 0.1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.2 \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 1 & -1.5 & -4 & 2.6 \\ 0 & 0.5 & 1 & -0.8 \\ 0 & 0 & -0.5 & 0.1 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$40. A^{-2} = (A^{-1})^2 = \left(\frac{1}{2} \begin{bmatrix} -15 & -4 & 28 \\ -1 & 0 & 2 \\ 23 & 6 & -42 \end{bmatrix} \right)^2 = \frac{1}{4} \begin{bmatrix} 873 & 228 & -1604 \\ 61 & 16 & -112 \\ -1317 & -344 & 2420 \end{bmatrix}$$

$$A^{-2} = (A^2)^{-1} = \begin{bmatrix} 48 & 4 & 32 \\ -29 & 48 & -17 \\ 22 & 9 & 15 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 873 & 228 & -1604 \\ 61 & 16 & -112 \\ -1317 & -344 & 2420 \end{bmatrix}$$

The results are equal.

$$32. A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$$

$$ad - bc = (1)(2) - (-2)(-3) = -4$$

$$A^{-1} = -\frac{1}{4} \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

$$34. A = \begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$$

$$ad - bc = (-12)(-2) - 3(5) = 24 - 15 = 9$$

$$A^{-1} = \frac{1}{9} \begin{bmatrix} -2 & -3 \\ -5 & -12 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & -\frac{1}{3} \\ -\frac{5}{9} & -\frac{4}{3} \end{bmatrix}$$

$$36. A = \begin{bmatrix} -\frac{1}{4} & \frac{9}{4} \\ \frac{5}{3} & \frac{8}{9} \end{bmatrix}$$

$$ad - bc = \left(-\frac{1}{4}\right)\left(\frac{8}{9}\right) - \left(\frac{9}{4}\right)\left(\frac{5}{3}\right) = -\frac{143}{36}$$

$$A^{-1} = -\frac{36}{143} \begin{bmatrix} \frac{8}{9} & -\frac{9}{4} \\ -\frac{5}{3} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{32}{143} & \frac{81}{143} \\ \frac{60}{143} & \frac{9}{143} \end{bmatrix}$$

$$38. A^{-2} = (A^{-1})^2 = \left(\frac{1}{47} \begin{bmatrix} 6 & -7 \\ 5 & 2 \end{bmatrix} \right)^2 = \frac{1}{2209} \begin{bmatrix} 1 & -56 \\ 40 & -31 \end{bmatrix}$$

$$A^{-2} = (A^2)^{-1} = \begin{bmatrix} -31 & 56 \\ -40 & 1 \end{bmatrix}^{-1} = \frac{1}{2209} \begin{bmatrix} 1 & -56 \\ 40 & -31 \end{bmatrix}$$

The results are equal.

42. (a) $(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix} \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

$$= \frac{1}{77} \begin{bmatrix} -4 & 9 \\ -9 & 1 \end{bmatrix}$$

(b) $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}^T = \begin{bmatrix} -\frac{2}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$

(c) $(2A)^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2} \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{1}{14} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$

44. (a) $(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} 6 & 5 & -3 \\ -2 & 4 & -1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -25 & 24 \\ -6 & 10 & 7 \\ 17 & 7 & 15 \end{bmatrix}$$

(b) $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 4 \\ -4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

(c) $(2A)^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -2 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 2 & 1 & \frac{1}{2} \end{bmatrix}$

46. The coefficient matrix for each system is

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

and the formula for the inverse of a 2×2 matrix produces

$$A^{-1} = \frac{1}{2+2} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

The solution is: $x = 1$ and $y = 5$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

The solution is: $x = -1$ and $y = -1$.

48. The coefficient matrix for each system is

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Using the algorithm to invert a matrix, you find that the inverse is

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}$$

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The solution is: $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The solution is: $x_1 = 1$, $x_2 = 0$, and $x_3 = 1$.

50. Using a graphing utility or software program, you have

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & -1 & 3 & -1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 & -1 \\ 2 & 1 & 4 & 1 & -1 \\ 3 & 1 & 1 & -2 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 3 \\ -1 \\ 5 \end{bmatrix}.$$

The solution is: $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = 0$, and $x_5 = 1$.

52. Using a graphing utility or software program, you have
 $A\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 & -5 & -1 \\ 3 & 6 & -5 & -6 & 3 & 3 \\ 2 & -3 & 1 & 3 & -1 & -2 \\ -1 & 4 & -4 & -6 & 2 & 4 \\ 3 & -1 & 5 & 2 & -3 & -5 \\ -2 & 3 & -4 & -6 & 1 & 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \text{ and}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ -11 \\ 0 \\ -9 \\ 1 \\ -12 \end{bmatrix}.$$

The solution is: $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$, $x_5 = 0$, and $x_6 = 1$.

54. The inverse of A is given by

$$A^{-1} = \frac{1}{x-4} \begin{bmatrix} -2 & -x \\ 1 & 2 \end{bmatrix}.$$

Letting $A^{-1} = A$, you find that $\frac{1}{x-4} = -1$.

So, $x = 3$.

56. The matrix $\begin{bmatrix} x & 2 \\ -3 & 4 \end{bmatrix}$ will be singular if

$$ad - bc = (x)(4) - (-3)(2) = 0, \text{ which implies that}$$

$$4x = -6 \text{ or } x = -\frac{3}{2}.$$

58. First, find $4A$.

$$4A = \left[(4A)^{-1} \right]^{-1} = \frac{1}{4+12} \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}$$

Then, multiply by $\frac{1}{4}$ to obtain

$$A = \frac{1}{4}(4A) = \frac{1}{4} \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{32} & -\frac{1}{16} \\ \frac{3}{64} & \frac{1}{32} \end{bmatrix}.$$

60. Using the formula for the inverse of a 2×2 matrix, you have

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{\sec^2 \theta - \tan^2 \theta} \begin{bmatrix} \sec \theta & -\tan \theta \\ -\tan \theta & \sec \theta \end{bmatrix} \\ &= \begin{bmatrix} \sec \theta & -\tan \theta \\ -\tan \theta & \sec \theta \end{bmatrix}. \end{aligned}$$

62. Adjoin the identity matrix to form

$$[F \quad I] = \begin{bmatrix} 0.017 & 0.010 & 0.008 & 1 & 0 & 0 \\ 0.010 & 0.012 & 0.010 & 0 & 1 & 0 \\ 0.008 & 0.010 & 0.017 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \quad F^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 115.56 & -100 & 4.44 \\ 0 & 1 & 0 & -100 & 250 & -100 \\ 0 & 0 & 1 & 4.44 & -100 & 115.56 \end{bmatrix}$$

$$\text{So, } F^{-1} = \begin{bmatrix} 115.56 & -100 & 4.44 \\ -100 & 250 & -100 \\ 4.44 & -100 & 115.56 \end{bmatrix} \text{ and}$$

$$\mathbf{w} = F^{-1} \mathbf{d} = \begin{bmatrix} 115.56 & -100 & 4.44 \\ -100 & 250 & -100 \\ 4.44 & -100 & 115.56 \end{bmatrix} \begin{bmatrix} 0 \\ 0.15 \\ 0 \end{bmatrix} = \begin{bmatrix} -15 \\ 37.5 \\ -15 \end{bmatrix}.$$

64. $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$ and

$$(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$$

$$\text{So, } (A^{-1})^T = (A^T)^{-1}.$$

66. $(I - 2A)(I - 2A) = I^2 - 2IA - 2AI + 4A^2$
- $$= I - 4A + 4A^2$$
- $$= I - 4A + 4A \quad (\text{because } A = A^2)$$
- $$= I$$

$$\text{So, } (I - 2A)^{-1} = I - 2A.$$

68. Because $ABC = I$, A is invertible and $A^{-1} = BC$.

$$\text{So, } ABCA = A \text{ and } BC A = I.$$

$$\text{So, } B^{-1} = CA.$$

70. Let $A^2 = A$ and suppose A is nonsingular. Then, A^{-1} exists, and you have the following.

$$A^{-1}(A^2) = A^{-1}A$$

$$(A^{-1}A)A = I$$

$$A = I$$

72. (a) True. See Theorem 2.8, part 1 on page 67.

(b) False. For example, consider the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,
which is not invertible, but $1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$.

(c) False. If A is a square matrix then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is a nonsingular matrix.

74. A has an inverse if $a_{ii} \neq 0$ for all $i = 1 \dots n$ and

$$A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{a_{nn}} \end{bmatrix}.$$

76. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(a) $A^2 - 2A + 5I = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $A\left(\frac{1}{5}(2I - A)\right) = \frac{1}{5}(2A - A^2) = \frac{1}{5}(5I) = I$

Similarly, $\left(\frac{1}{5}(2I - A)\right)A = I$. Or, $\frac{1}{5}(2I - A) = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = A^{-1}$ directly.

(c) The calculation in part (b) did not depend on the entries of A .

78. Let C be the inverse of $(I - AB)$, that is $C = (I - AB)^{-1}$. Then $C(I - AB) = (I - AB)C = I$.

Consider the matrix $I + BCA$. Claim that this matrix is the inverse of $I - BA$. To check this claim, show that $(I + BCA)(I - BA) = (I - BA)(I + BCA) = I$.

$$\begin{aligned} \text{First, show } (I - BA)(I + BCA) &= I - BA + BCA - BABCA \\ &= I - BA + B(C - ABC)A \\ &= I - BA + B\underbrace{((I - AB)C)}_I A \\ &= I - BA + BA = I \end{aligned}$$

Similarly, show $(I + BCA)(I - BA) = I$.

80. Answers will vary. Sample answer:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \text{ or } A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

82. $AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{1}{ad-bc} \\ \frac{-b}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^{-1}A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Section 2.4 Elementary Matrices

2. This matrix is *not* elementary, because it is not square.

4. This matrix is elementary. It can be obtained by interchanging the two rows of I_2 .

6. This matrix is elementary. It can be obtained by multiplying the first row of I_3 by 2, and adding the result to the third row.

8. This matrix is *not* elementary, because two elementary row operations are required to obtain it from I_4 .

10. C is obtained by adding the third row of A to the first row. So,

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

12. A is obtained by adding -1 times the third row of C to the first row. So,

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14. Answers will vary. Sample answer:

| Matrix | Elementary Row Operation | Elementary Matrix |
|--|------------------------------------|---|
| $\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & -3 & 6 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ | $R_1 \leftrightarrow R_2$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ | $(\frac{1}{3})R_2 \rightarrow R_2$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ | $(\frac{1}{2})R_3 \rightarrow R_3$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ |

$$\text{So, } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

16. Answers will vary. Sample answer:

| Matrix | Elementary Row Operation | Elementary Matrix |
|--|------------------------------------|---|
| $\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 3 & -2 & -4 \end{bmatrix}$ | $(-2)R_1 + R_2 \rightarrow R_2$ | $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 0 & -11 & -4 \end{bmatrix}$ | $(-3)R_1 + R_3 \rightarrow R_3$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & -11 & -4 \end{bmatrix}$ | $(-1)R_2 \rightarrow R_2$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$ | $(11)R_2 + R_3 \rightarrow R_3$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ | $(\frac{1}{7})R_3 \rightarrow R_3$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$ |

$$\text{So, } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & -1 \\ 3 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

18. Matrix Elementary Row Operations Elementary Matrix

| | | |
|---|-------------------------------------|---|
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & -3 & 3 & 9 \\ 0 & 17 & -1 & -3 \\ 4 & 8 & -5 & 1 \end{bmatrix}$ | $R_3 + (-2)R_1 \rightarrow R_3$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & -3 & 3 & 9 \\ 0 & 17 & -1 & -3 \\ 0 & 32 & -5 & -7 \end{bmatrix}$ | $R_4 + (-4)R_2 \rightarrow R_4$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 17 & -1 & -3 \\ 0 & 32 & -5 & -7 \end{bmatrix}$ | $(-\frac{1}{3})R_2 \rightarrow R_2$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 16 & 48 \\ 0 & 32 & -5 & -7 \end{bmatrix}$ | $R_3 + (-17)R_2 \rightarrow R_3$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -17 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 16 & 48 \\ 0 & 0 & 27 & 89 \end{bmatrix}$ | $R_4 + (-32)R_2 \rightarrow R_2$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -32 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 27 & 89 \end{bmatrix}$ | $(\frac{1}{16})R_3 \rightarrow R_3$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ | $R_4 + (-27)R_3 \rightarrow R_4$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -27 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | $(\frac{1}{8})R_4 \rightarrow R_4$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}$ |

So,
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -27 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -32 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -17 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & -3 & 3 & 9 \\ 2 & 5 & -1 & 1 \\ 4 & 8 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

20. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, multiply the first row by $\frac{1}{25}$ to obtain

$$E^{-1} = \begin{bmatrix} \frac{1}{25} & 0 \\ 0 & 1 \end{bmatrix}.$$

22. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, add 3 times the second row to the third row to obtain

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

24. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, add $-k$ times the third row to the second row to obtain

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

26. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\frac{1}{2} R_1 \rightarrow R_1 \right) \quad E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_2 - R_1 \rightarrow R_2 \quad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Use the elementary matrices to find the inverse.

$$A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

28. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \left(\frac{1}{2} R_2 \rightarrow R_2 \right) \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} R_1 + 2R_3 \rightarrow R_1 \quad E_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} R_2 - \left(\frac{1}{2} \right) R_3 \rightarrow R_2 \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Use the elementary matrices to find the inverse.

$$\begin{aligned} A^{-1} &= E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

For Exercises 30–36, answers will vary. Sample answers are shown below.

30. The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is itself an elementary matrix, so the factorization is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

32. Reduce the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ as follows.

| <u>Matrix</u> | <u>Elementary Row Operation</u> | <u>Elementary Matrix</u> |
|---|------------------------------------|---|
| $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ | Add -2 times row one to row two. | $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ | Multiply row two by -1 . | $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | Add -1 times row two to row one. | $E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ |

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

34. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$ as follows.

| <u>Matrix</u> | <u>Elementary Row Operation</u> | <u>Elementary Matrix</u> |
|---|--------------------------------------|--|
| $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ | Add -2 times row one to row two. | $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ | Add -1 times row one to row three. | $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | Add -1 times row two to row three. | $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | Add -3 times row three to row one. | $E_4 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | Add -2 times row two to row one. | $E_5 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

36. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{bmatrix} \quad \left(\frac{1}{4}\right)R_1 \rightarrow R_1$$

$$E_1 = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix} \quad R_4 - R_1 \rightarrow R_4$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \left(-\frac{2}{5}\right)R_4 \rightarrow R_4$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{2}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -R_3 \rightarrow R_3$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 - \left(\frac{1}{2}\right)R_4 \rightarrow R_1$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 - R_4 \rightarrow R_2$$

$$E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 + 2R_4 \rightarrow R_3$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, one way to factor A is

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1} \\ &= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

38. (a) EA has the same rows as A except the two rows that are interchanged in E will be interchanged in EA .
 (b) Multiplying a matrix on the left by E interchanges the same two rows that are interchanged from I_n in E .
 So, multiplying E by itself interchanges the rows twice and $E^2 = I_n$.

$$40. A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & 0 \\ -b & 1+ab & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}.$$

42. (a) False. It is impossible to obtain the zero matrix by applying any elementary row operation to the identity matrix.
- (b) True. If $A = E_1 E_2 \dots E_k$, where each E_i is an elementary matrix, then A is invertible (because every elementary matrix is) and $A^{-1} = E_k^{-1} \dots E_2^{-1} E_1^{-1}$.
- (c) True. See equivalent conditions (2) and (3) of Theorem 2.15.

44. MatrixElementary Matrix

$$\begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = A$$

$$\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = U$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$E_1 A = U \Rightarrow A = E_1^{-1} U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = LU$$

46. MatrixElementary Matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 10 & 12 & 3 \end{bmatrix} = A$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 12 & 3 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 7 \end{bmatrix} = U \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E_2 E_1 A = U \Rightarrow A = E_1^{-1} E_2^{-1} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 7 \end{bmatrix} = LU$$

48. MatrixElementary Matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = A$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = U \Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

$$Ly = \mathbf{b}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 15 \\ -1 \end{bmatrix}$$

$$y_1 = 4, -y_1 + y_2 = -4 \Rightarrow y_2 = 0,$$

$$3y_1 + 2y_2 + y_3 = 15 \Rightarrow y_3 = 3, \text{ and } y_4 = -1.$$

$$U\mathbf{x} = \mathbf{y}: \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ -1 \end{bmatrix}$$

$$x_4 = 1, x_3 = 1, x_2 - x_3 = 0 \Rightarrow x_2 = 1, \text{ and } x_1 = 2.$$

So, the solution to the system $A\mathbf{x} = \mathbf{b}$ is: $x_1 = 2$,

$$x_2 = x_3 = x_4 = 1.$$

$$50. A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq A.$$

Because $A^2 \neq A$, A is *not* idempotent.

$$52. A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because $A^2 \neq A$, A is *not* idempotent.

54. Assume A is idempotent. Then

$$A^2 = A$$

$$(A^2)^T = A^T$$

$$(A^T A^T) = A^T$$

which means that A^T is idempotent.

Now assume A^T is idempotent. Then

$$A^T A^T = A^T$$

$$(A^T A^T)^T = (A^T)^T$$

$$AA = A$$

which means that A is idempotent.

$$\begin{aligned} 56. (AB)^2 &= (AB)(AB) \\ &= A(BA)B \\ &= A(AB)B \\ &= (AA)(BB) \\ &= AB \end{aligned}$$

So, $(AB)^2 = AB$, and AB is idempotent.

58. If A is row-equivalent to B , then

$$A = E_k \cdots E_2 E_1 B,$$

where E_1, \dots, E_k are elementary matrices.

So,

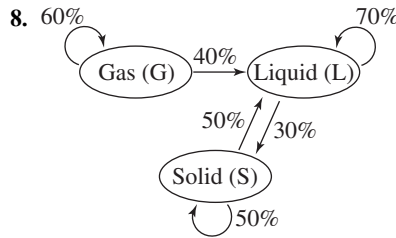
$$B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A,$$

which shows that B is row equivalent to A .

60. (a) When an elementary row operation is performed on a matrix A , perform the same operation on I to obtain the matrix E .
- (b) Keep track of the row operations used to reduce A to an upper triangular matrix U . If a row reduces to U using only the row operation of adding a multiple of one row to another row below it, then the inverse of the product of the elementary matrices is the matrix L , and $A = LU$.
- (c) For the system $A\mathbf{x} = \mathbf{b}$, find an LU factorization of A . Then solve the system $L\mathbf{y} = \mathbf{b}$ for \mathbf{y} and $U\mathbf{x} = \mathbf{y}$ for \mathbf{x} .

Section 2.5 Markov Chains

- The matrix is *not* stochastic because every entry of a stochastic matrix satisfies the inequality $0 \leq a_{ij} \leq 1$.
- The matrix is *not* stochastic because the sum of entries in a column of a stochastic matrix is 1.
- The matrix is stochastic because each entry is between 0 and 1, and each column adds up to 1.



The matrix of transition probabilities is shown.

$$P = \begin{matrix} & \begin{matrix} \text{From} \\ \text{G} & \text{L} & \text{S} \end{matrix} \\ \begin{matrix} \text{G} \\ \text{L} \\ \text{S} \end{matrix} & \begin{bmatrix} 0.60 & 0 & 0 \\ 0.40 & 0.70 & 0.50 \\ 0 & 0.30 & 0.50 \end{bmatrix} \end{matrix} \left. \begin{matrix} \\ \\ \end{matrix} \right\} \begin{matrix} \text{To} \\ \\ \end{matrix}$$

The initial state matrix represents the amounts of the physical states is shown.

$$X_0 = \begin{bmatrix} 0.20(10,000) \\ 0.60(10,000) \\ 0.20(10,000) \end{bmatrix} = \begin{bmatrix} 2000 \\ 6000 \\ 2000 \end{bmatrix}$$

To represent the amount of each physical state after the catalyst is added, multiply P by X_0 to obtain

$$PX_0 = \begin{bmatrix} 0.60 & 0 & 0 \\ 0.40 & 0.70 & 0.50 \\ 0 & 0.30 & 0.50 \end{bmatrix} \begin{bmatrix} 2000 \\ 6000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 1200 \\ 6000 \\ 2800 \end{bmatrix}.$$

So, after the catalyst is added there are 1200 molecules in a gas state, 6000 molecules in a liquid state, and 2800 molecules in a solid state.

$$\begin{aligned} 10. \quad X_1 &= PX_0 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{15} \\ \frac{1}{3} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0.2\bar{6} \\ 0.\bar{3} \\ 0.4 \end{bmatrix} \\ X_2 &= PX_1 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} \frac{4}{15} \\ \frac{1}{3} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{17}{75} \\ \frac{49}{150} \\ \frac{67}{150} \end{bmatrix} = \begin{bmatrix} 0.22\bar{6} \\ 0.32\bar{6} \\ 0.44\bar{6} \end{bmatrix} \\ X_3 &= PX_2 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} \frac{17}{75} \\ \frac{49}{150} \\ \frac{67}{150} \end{bmatrix} = \begin{bmatrix} \frac{151}{750} \\ \frac{239}{750} \\ \frac{12}{25} \end{bmatrix} = \begin{bmatrix} 0.201\bar{3} \\ 0.318\bar{6} \\ 0.48 \end{bmatrix} \end{aligned}$$

12. Form the matrix representing the given transition probabilities. A represents infected mice and B noninfected.

$$P = \begin{matrix} & \begin{matrix} \text{From} \\ A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \end{matrix} \left. \begin{matrix} \\ \end{matrix} \right\} \begin{matrix} A \\ B \end{matrix} \text{ To}$$

The state matrix representing the current population is

$$X_0 = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}.$$

- (a) The state matrix for next week is

$$X_1 = PX_0 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.87 \end{bmatrix}.$$

So, next week $0.13(1000) = 130$ mice will be infected.

$$(b) \quad X_2 = PX_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 0.13 \\ 0.87 \end{bmatrix} = \begin{bmatrix} 0.113 \\ 0.887 \end{bmatrix}$$

$$X_3 = PX_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 0.113 \\ 0.887 \end{bmatrix} = \begin{bmatrix} 0.1113 \\ 0.8887 \end{bmatrix}$$

In 3 weeks, $0.1113(1000) \approx 111$ mice will be infected.

14. Form the matrix representing the given transition probabilities. Let S represent those who swim and B represent those who play basketball.

$$P = \begin{array}{c} \text{From} \\ \begin{matrix} S & B \end{matrix} \\ \begin{bmatrix} 0.30 & 0.40 \\ 0.70 & 0.60 \end{bmatrix} \end{array} \begin{array}{c} S \\ B \end{array} \text{ To}$$

The state matrix representing the students is

$$X_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{array}{c} S \\ B \end{array}$$

- (a) The state matrix for tomorrow is

$$X_1 = PX_0 = \begin{bmatrix} 0.30 & 0.40 \\ 0.70 & 0.60 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.36 \\ 0.64 \end{bmatrix}$$

So, tomorrow $0.36(250) = 90$ students will swim and $0.64(250) = 160$ students will play basketball.

- (b) The state matrix for two days from now is

$$X_2 = P^2X_0 = \begin{bmatrix} 0.37 & 0.36 \\ 0.63 & 0.64 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.364 \\ 0.636 \end{bmatrix}$$

So, two days from now $0.364(250) = 91$ students will swim and $0.636(250) = 159$ students will play basketball.

- (c) The state matrix for four days from now is

$$X_4 = P^4X_0 = \begin{bmatrix} 0.363637 & 0.363637 \\ 0.636363 & 0.636363 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.36364 \\ 0.63636 \end{bmatrix}$$

So, four days from now, $0.36364(250) \approx 91$ students will swim and $0.63636(250) \approx 159$ students will play basketball.

16. Form the matrix representing the given transition probabilities. Let A represent users of Brand A, B users of Brand B, and N users of neither brands.

$$P = \begin{array}{c} \text{From} \\ \begin{matrix} A & B & N \end{matrix} \\ \begin{bmatrix} 0.75 & 0.15 & 0.10 \\ 0.20 & 0.75 & 0.15 \\ 0.05 & 0.10 & 0.75 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ N \end{array} \text{ To}$$

The state matrix representing the current product usage is

$$X_0 = \begin{bmatrix} \frac{2}{11} \\ \frac{3}{11} \\ \frac{5}{11} \end{bmatrix} \begin{array}{c} A \\ B \\ N \end{array}$$

- (a) The state matrix for next month is

$$X_1 = P^1X_0 = \begin{bmatrix} 0.75 & 0.15 & 0.10 \\ 0.20 & 0.75 & 0.15 \\ 0.05 & 0.10 & 0.75 \end{bmatrix} \begin{bmatrix} \frac{2}{11} \\ \frac{3}{11} \\ \frac{5}{11} \end{bmatrix} = \begin{bmatrix} 0.222\bar{7} \\ 0.30\bar{9} \\ 0.37\bar{2} \end{bmatrix}$$

So, next month the distribution of users will be

$$0.222\bar{7} \cdot 110,000 = 24,500 \text{ for Brand A,}$$

$$0.30\bar{9} \cdot 110,000 = 34,000 \text{ for Brand B, and}$$

$$0.37\bar{2} \cdot 110,000 = 41,500 \text{ for neither.}$$

$$(b) X_2 = P^2X_0 \approx \begin{bmatrix} 0.2511 \\ 0.3330 \\ 0.325 \end{bmatrix}$$

In 2 months, the distribution of users will be
 $0.2511 \cdot 110,000 = 27,625$ for Brand A,
 $0.3330 \cdot 110,000 = 36,625$ for Brand B, and
 $0.325 \cdot 110,000 = 35,750$ for neither.

$$(c) X_{18} = P^{18}X_0 \approx \begin{bmatrix} 0.3139 \\ 0.3801 \\ 0.2151 \end{bmatrix}$$

In 18 months, the distribution of users will be
 $0.3139 \cdot 110,000 \approx 34,530$ for Brand A,
 $0.3801 \cdot 110,000 \approx 41,808$ for Brand B, and
 $0.2151 \cdot 110,000 \approx 23,662$ for neither.

18. The stochastic matrix

$$P = \begin{bmatrix} 0 & 0.3 \\ 1 & 0.7 \end{bmatrix}$$

is regular because P^2 has only positive entries.

$$\begin{aligned} P\bar{X} = \bar{X} &\Rightarrow \begin{bmatrix} 0 & 0.3 \\ 1 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\Rightarrow \begin{aligned} 0.3x_2 &= x_1 \\ x_1 + 0.7x_2 &= x_2 \end{aligned} \end{aligned}$$

Because $x_1 + x_2 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -x_1 + 0.3x_2 &= 0 \\ x_1 - 0.3x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The solution to the system is $x_2 = \frac{10}{13}$ and

$$x_1 = 1 - \frac{10}{13} = \frac{3}{13}.$$

$$\text{So, } \bar{X} = \begin{bmatrix} \frac{3}{13} \\ \frac{10}{13} \end{bmatrix}.$$

20. The stochastic matrix

$$P = \begin{bmatrix} 0.2 & 0 \\ 0.8 & 1 \end{bmatrix}$$

is not regular because every power of P has a zero in the second column.

$$\begin{aligned} P\bar{X} = \bar{X} &\Rightarrow \begin{bmatrix} 0.2 & 0 \\ 0.8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\Rightarrow \begin{aligned} 0.2x_1 &= x_1 \\ 0.8x_1 + x_2 &= x_2 \end{aligned} \end{aligned}$$

Because $x_1 + x_2 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -0.8x_1 &= 0 \\ 0.8x_1 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The solution of the system is $x_1 = 0$ and $x_2 = 1$.

$$\text{So, } \bar{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

22. The stochastic matrix

$$P = \begin{bmatrix} \frac{2}{5} & \frac{7}{10} \\ \frac{3}{5} & \frac{3}{10} \end{bmatrix}$$

is regular because P^1 has only positive entries.

$$\begin{aligned} P\bar{X} = \bar{X} &\Rightarrow \begin{bmatrix} \frac{2}{5} & \frac{7}{10} \\ \frac{3}{5} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\Rightarrow \begin{aligned} \frac{2}{5}x_1 + \frac{7}{10}x_2 &= x_1 \\ \frac{3}{5}x_1 + \frac{3}{10}x_2 &= x_2 \end{aligned} \end{aligned}$$

Because $x_1 + x_2 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -\frac{3}{5}x_1 + \frac{7}{10}x_2 &= 0 \\ \frac{3}{5}x_1 - \frac{7}{10}x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The solution of the system is $x_2 = \frac{6}{13}$ and

$$x_1 = 1 - \frac{6}{13} = \frac{7}{13}.$$

$$\text{So, } \bar{X} = \begin{bmatrix} \frac{7}{13} \\ \frac{6}{13} \end{bmatrix}.$$

24. The stochastic matrix

$$P = \begin{bmatrix} \frac{2}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{4}{9} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

is regular because P^1 has only positive entries.

$$\begin{aligned} P\bar{X} = \bar{X} &\Rightarrow \begin{bmatrix} \frac{2}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{4}{9} & \frac{1}{4} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &\Rightarrow \begin{aligned} \frac{2}{9}x_1 + \frac{1}{4}x_2 + \frac{1}{3}x_3 &= x_1 \\ \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &= x_2 \\ \frac{4}{9}x_1 + \frac{1}{4}x_2 + \frac{1}{3}x_3 &= x_3 \end{aligned} \end{aligned}$$

Because $x_1 + x_2 + x_3 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -\frac{7}{9}x_1 + \frac{1}{4}x_2 + \frac{1}{3}x_3 &= 0 \\ \frac{1}{3}x_1 - \frac{1}{2}x_2 + \frac{1}{3}x_3 &= 0 \\ \frac{4}{9}x_1 + \frac{1}{4}x_2 + \frac{2}{3}x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is $x_3 = 0.33$, $x_2 = 0.4$, and $x_1 = 1 - 0.4 - 0.33 = 0.27$.

$$\text{So, } \bar{X} = \begin{bmatrix} 0.27 \\ 0.4 \\ 0.33 \end{bmatrix}.$$

26. The stochastic matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{6} & \frac{3}{5} & 0 \end{bmatrix}$$

is regular because P^2 has only positive entries.

$$P\bar{X} = \bar{X} \Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{6} & \frac{3}{5} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} \frac{1}{2}x_1 + \frac{1}{5}x_2 + x_3 &= x_1 \\ \frac{1}{3}x_1 + \frac{1}{5}x_2 &= x_2 \\ \frac{1}{6}x_1 + \frac{3}{5}x_2 &= x_3 \end{aligned}$$

Because $x_1 + x_2 + x_3 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -\frac{1}{2}x_1 + \frac{1}{5}x_2 + x_3 &= 0 \\ \frac{1}{3}x_1 - \frac{4}{5}x_2 &= 0 \\ \frac{1}{6}x_1 + \frac{3}{5}x_2 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is

$$x_3 = \frac{5}{22}, x_2 = \frac{5}{17} - \frac{5}{17}\left(\frac{5}{22}\right) = \frac{5}{22}, \text{ and}$$

$$x_1 = 1 - \frac{5}{22} - \frac{5}{22} = \frac{6}{11}.$$

$$\text{So, } \bar{X} = \begin{bmatrix} \frac{6}{11} \\ \frac{5}{22} \\ \frac{5}{22} \end{bmatrix} \approx \begin{bmatrix} 0.54 \\ 0.22\bar{7} \\ 0.22\bar{7} \end{bmatrix}.$$

28. The stochastic matrix

$$P = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.7 & 1 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$$

is not regular because every power of P has two zeros in the second column.

$$P\bar{X} = \bar{X} \Rightarrow \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.7 & 1 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} 0.1x_1 + 0.3x_3 &= x_1 \\ 0.7x_1 + x_2 + 0.3x_3 &= x_2 \\ 0.2x_1 + 0.4x_3 &= x_3 \end{aligned}$$

Because $x_1 + x_2 + x_3 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -0.9x_1 + 0.3x_3 &= 0 \\ 0.7x_1 + 0.3x_3 &= 0 \\ 0.2x_1 - 0.6x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is $x_3 = 0$, $x_2 = 1 - 0 = 1$, and $x_1 = 1 - 1 - 0 = 0$.

$$\text{So, } \bar{X} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

30. The stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not regular because every power of P has three zeros in the first column.

$$P\bar{X} = \bar{X} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$x_1 = x_1$$

$$x_3 = x_2$$

$$x_2 = x_3$$

$$x_4 = x_4$$

Because $x_1 + x_2 + x_3 + x_4 = 1$, the system of linear equations is as follows.

$$\begin{aligned} 0 &= 0 \\ -x_2 + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

Let $x_3 = s$ and $x_4 = t$. The solution of the system is $x_4 = t$, $x_3 = s$, $x_2 = s$, and $x_1 = 1 - 2s - t$, where $0 \leq s \leq 1$, $0 \leq t \leq 1$, and $2s + t \leq 1$.

32. Exercise 3: To find \bar{X} , let $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then use the

matrix equation $P\bar{X} = \bar{X}$ to obtain

$$\begin{bmatrix} 0.\bar{3} & 0.1\bar{6} & 0.25 \\ 0.\bar{3} & 0.\bar{6} & 0.25 \\ 0.\bar{3} & 0.1\bar{6} & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$0.\bar{3}x_1 + 0.1\bar{6}x_2 + 0.25x_3 = x_1$$

$$0.\bar{3}x_1 + 0.\bar{6}x_2 + 0.25x_3 = x_2$$

$$0.\bar{3}x_1 + 0.1\bar{6}x_2 + 0.5x_3 = x_3$$

Use these equations and the fact that $x_1 + x_2 + x_3 = 1$ to write the system of linear equations shown.

$$-0.\bar{6}x_1 + 0.1\bar{6}x_2 + 0.25x_3 = 0$$

$$0.\bar{3}x_1 - 0.\bar{3}x_2 + 0.25x_3 = 0$$

$$0.\bar{3}x_1 + 0.1\bar{6}x_2 + 0.5x_3 = 0$$

$$x_1 + x_2 + x_3 = 1$$

The solution of the system is

$$x_1 = \frac{3}{13}, x_2 = \frac{6}{13}, \text{ and } x_3 = \frac{4}{13}.$$

So, the steady state matrix is

$$\bar{X} = \begin{bmatrix} \frac{3}{13} \\ \frac{6}{13} \\ \frac{4}{13} \end{bmatrix}.$$

Exercise 5: To find \bar{X} , let $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then use the

matrix equation $P\bar{X} = \bar{X}$ to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or

$$x_1 = x_1$$

$$x_2 = x_2$$

$$x_3 = x_3$$

$$x_4 = x_4$$

Use these equations and the fact that

$x_1 + x_2 + x_3 + x_4 = 1$ to write the system of linear equations shown.

$$x_1 + x_2 + x_3 + x_4 = 1$$

Let $x_2 = r$, $x_3 = s$, and $x_4 = t$, where r , s , and t are real numbers between 0 and 1.

The solution of the system is

$x_1 = 1 - r - s - t$, $x_2 = r$, $x_3 = s$, and $x_4 = t$, where

r , s , and t are real numbers such that

$0 \leq r \leq 1$, $0 \leq s \leq 1$, $0 \leq t \leq 1$, and $r + s + t \leq 1$.

So, the steady state matrix is

$$\bar{X} = \begin{bmatrix} 1 - r - s - t \\ r \\ s \\ t \end{bmatrix}.$$

Exercise 6: To find \bar{X} , let $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then use the

matrix equation $P\bar{X} = \bar{X}$ to obtain

$$\begin{bmatrix} \frac{1}{2} & \frac{2}{9} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or

$$\frac{1}{2}x_1 + \frac{2}{9}x_2 + \frac{1}{4}x_3 + \frac{4}{15}x_4 = x_1$$

$$\frac{1}{6}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 + \frac{4}{15}x_4 = x_2$$

$$\frac{1}{6}x_1 + \frac{2}{9}x_2 + \frac{1}{4}x_3 + \frac{4}{15}x_4 = x_3$$

$$\frac{1}{6}x_1 + \frac{2}{9}x_2 + \frac{1}{4}x_3 + \frac{1}{5}x_4 = x_4$$

Use these equations and the fact that

$x_1 + x_2 + x_3 + x_4 = 1$ to write the system of equations shown.

$$-\frac{1}{2}x_1 + \frac{2}{9}x_2 + \frac{1}{4}x_3 + \frac{4}{15}x_4 = 0$$

$$\frac{1}{6}x_1 - \frac{2}{3}x_2 + \frac{1}{4}x_3 + \frac{4}{15}x_4 = 0$$

$$\frac{1}{6}x_1 - \frac{2}{9}x_2 - \frac{3}{4}x_3 + \frac{4}{15}x_4 = 0$$

$$\frac{1}{6}x_1 + \frac{2}{9}x_2 + \frac{1}{4}x_3 - \frac{4}{5}x_4 = 0$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

The solution of the system is

$$x_1 = \frac{24}{73}, x_2 = \frac{18}{73}, x_3 = \frac{16}{73}, \text{ and } x_4 = \frac{15}{73}.$$

So, the steady state matrix is

$$\bar{X} = \begin{bmatrix} \frac{24}{73} \\ \frac{18}{73} \\ \frac{16}{73} \\ \frac{15}{73} \end{bmatrix} \approx \begin{bmatrix} 0.3288 \\ 0.2466 \\ 0.2192 \\ 0.2055 \end{bmatrix}.$$

34. Form the matrix representing the given transition probabilities. Let A represent those who received an “A” and let N represent those who did not.

$$P = \begin{array}{c} \begin{array}{cc} \text{From} \\ A & N \end{array} \\ \left[\begin{array}{cc} 0.70 & 0.10 \\ 0.30 & 0.90 \end{array} \right] \begin{array}{c} A \\ N \end{array} \end{array} \left\{ \begin{array}{c} A \\ N \end{array} \right. \text{To}$$

To find the steady state matrix, solve the equation $P\bar{X} = \bar{X}$, where $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and use the fact that $x_1 + x_2 = 1$

to write a system of equations.

$$\begin{aligned} 0.70x_1 + 0.10x_2 &= x_1 & -0.3x_1 + 0.1x_2 &= 0 \\ 0.30x_1 + 0.90x_2 &= x_2 & \Rightarrow 0.3x_1 - 0.1x_2 &= 0 \\ x_1 + x_2 &= 1 & x_1 + x_2 &= 1 \end{aligned}$$

The solution of the system is $x_1 = \frac{1}{4}$ and $x_2 = \frac{3}{4}$. So, the steady state matrix is $\bar{X} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$. This indicates that eventually $\frac{1}{4}$

of the students will receive assignment grades of “A” and $\frac{3}{4}$ of the students will not.

36. Form the matrix representing transition probabilities. Let A represent Theatre A, let B represent Theatre B, and let N represent neither theatre.

$$P = \begin{array}{c} \begin{array}{ccc} \text{From} \\ A & B & N \end{array} \\ \left[\begin{array}{ccc} 0.10 & 0.06 & 0.03 \\ 0.05 & 0.08 & 0.04 \\ 0.85 & 0.86 & 0.97 \end{array} \right] \begin{array}{c} A \\ B \\ N \end{array} \end{array} \left\{ \begin{array}{c} A \\ B \\ N \end{array} \right. \text{To}$$

To find the steady state matrix, solve the equation $P\bar{X} = \bar{X}$ where $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and use the fact that $x_1 + x_2 + x_3 = 1$

to write a system of equations.

$$\begin{aligned} 0.10x_1 + 0.06x_2 + 0.03x_3 &= x_1 & -0.90x_1 + 0.06x_2 + 0.03x_3 &= 0 \\ 0.05x_1 + 0.08x_2 + 0.04x_3 &= x_2 & \Rightarrow 0.05x_1 - 0.92x_2 + 0.04x_3 &= 0 \\ 0.85x_1 + 0.86x_2 + 0.97x_3 &= x_3 & 0.85x_1 + 0.86x_2 - 0.03x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 & x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is $x_1 = \frac{4}{119}$, $x_2 = \frac{5}{119}$, and $x_3 = \frac{110}{119}$. So, the steady state matrix is $\bar{X} = \begin{bmatrix} \frac{4}{119} \\ \frac{5}{119} \\ \frac{110}{119} \end{bmatrix}$. This indicates

that eventually $\frac{4}{119} \approx 3.4\%$ of the people will attend Theatre A, $\frac{5}{119} \approx 4.2\%$ of the people will attend Theatre B, and $\frac{110}{119} \approx 92.4\%$ of the people will attend neither theatre on any given night.

38. The matrix is not absorbing; The first state S_1 is absorbing, however the corresponding Markov chain is not absorbing because there is no way to move from S_2 or S_3 to S_1 .
40. The matrix is absorbing; The fourth state S_4 is absorbing and it is possible to move from any of the states to S_4 in one transition.

42. Use the matrix equation $P\bar{X} = \bar{X}$, or

$$\begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.7 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 = 1$ to write the linear system

$$\begin{aligned} -0.9x_1 &= 0 \\ 0.2x_1 &= 0 \\ 0.7x_1 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of this system is $x_1 = 0$, $x_2 = 1 - t$, and $x_3 = t$, where t is a real number such that $0 \leq t \leq 1$.

So, the steady state matrix is $\bar{X} = \begin{bmatrix} 0 \\ 1-t \\ t \end{bmatrix}$, where

$$0 \leq t \leq 1.$$

44. Use the matrix equation $P\bar{X} = \bar{X}$ or

$$\begin{bmatrix} 0.7 & 0 & 0.2 & 0.1 \\ 0.1 & 1 & 0.5 & 0.6 \\ 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 + x_4 = 1$ to write the linear system

$$\begin{aligned} -0.3x_1 &+ 0.2x_3 + 0.1x_4 = 0 \\ 0.1x_1 &+ 0.5x_3 + 0.6x_4 = 0 \\ &- 0.8x_3 + 0.2x_4 = 0 \\ 0.2x_1 &+ 0.1x_3 - 0.9x_4 = 0 \\ x_1 + x_2 &+ x_3 + x_4 = 1 \end{aligned}$$

The solution of this system is $x_1 = 0$, $x_2 = 1$, $x_3 = 0$,

and $x_4 = 0$. So, the steady state matrix is $\bar{X} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

46. Let S_n be the state that Player 1 has n chips.

$$P = \begin{array}{ccccc} & \text{From} & & & \\ & S_0 & S_1 & S_2 & S_3 & S_4 \\ \begin{bmatrix} 1 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \end{bmatrix} & \begin{matrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{matrix} & \text{To and } X_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

So,

$$P^n X_0 \rightarrow \bar{P} X_0 = \begin{bmatrix} \frac{49}{58} \\ 0 \\ 0 \\ 0 \\ \frac{9}{58} \end{bmatrix}$$

So, the probability that Player 1 reaches S_4 and wins the tournament is $\frac{9}{58} \approx 0.155$.

48. (a) To find the n th state matrix of a Markov chain, compute $X_n = P^n X_0$, where X_0 is the initial state matrix.
- (b) To find the steady state matrix of a Markov chain, determine the limit of $P^n X_0$, as $n \rightarrow \infty$, where X_0 is the initial state matrix.
- (c) The regular Markov chain is $PX_0, P^2X_0, P^3X_0, \dots$, where P is a regular stochastic matrix and X_0 is the initial state matrix.
- (d) An absorbing Markov chain is a Markov chain with at least one absorbing state and it is possible for a member of the population to move from any nonabsorbing state to an absorbing state in a finite number of transitions.
- (e) An absorbing Markov chain is concerned with having an entry of 1 and the rest 0 in a column, whereas a regular Markov chain is concerned with the repeated multiplication of the regular stochastic matrix.

50. (a) When the chain reaches S_1 or S_4 , it is certain in the next step to transition to an adjacent state, S_2 or S_3 , respectively, so S_1 and S_4 reflect to S_2 or S_3 .

$$(b) \quad P = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 1 & 0 & 0.3 & 0 \\ 0 & 0.6 & 0 & 1 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}$$

$$(c) \quad P^{30} \approx \begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & \frac{5}{12} & 0 & \frac{5}{12} \\ \frac{5}{6} & 0 & \frac{5}{6} & 0 \\ 0 & \frac{7}{12} & 0 & \frac{7}{12} \end{bmatrix}$$

$$P^{30} \approx \begin{bmatrix} 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{5}{6} & 0 & \frac{5}{12} & 0 \\ 0 & \frac{5}{6} & 0 & \frac{5}{6} \\ \frac{7}{12} & 0 & \frac{7}{12} & 0 \end{bmatrix}$$

Other high even or odd powers of P give similar results where the columns alternate.

$$(d) \quad \bar{X} = \begin{bmatrix} \frac{1}{12} \\ \frac{5}{24} \\ \frac{5}{12} \\ \frac{7}{24} \end{bmatrix}$$

Half the sum entries in the corresponding columns of P^n and P^{n+1} approach the corresponding entries in \bar{X} .

52. (a) Yes, it is possible.
(b) Yes, it is possible.

Both matrices X satisfy $P^1 X = X$. The steady state matrix depends on the initial state matrix. In general,

$$\text{the steady state matrix is } \bar{X} = \begin{bmatrix} \frac{6}{11} - t \\ \frac{5}{11} - \frac{5}{6}t \\ \frac{5}{6}t \\ t \end{bmatrix},$$

where t is any real number such that $0 \leq t \leq \frac{6}{11}$. In

part (a) $t = 0$ and in part (b), $t = \frac{6}{11}$.

54. Let

$$P = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$$

be a 2×2 stochastic matrix, and consider the system of equations $PX = X$.

$$\begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

You have

$$ax_1 + bx_2 = x_1$$

$$(1-a)x_1 + (1-b)x_2 = x_2$$

or

$$(a-1)x_1 + bx_2 = 0$$

$$(1-a)x_1 - bx_2 = 0.$$

Letting $x_1 = b$ and $x_2 = 1-a$, you have the 2×1 state matrix X satisfying $PX = X$

$$X = \begin{bmatrix} b \\ 1-a \end{bmatrix}.$$

56. Let P be a regular stochastic matrix and X_0 be the initial state matrix.

$$\begin{aligned} \lim_{n \rightarrow \infty} P^n X_0 &= \lim_{n \rightarrow \infty} P^n (x_1 + x_2 + \cdots + x_k) \\ &= \lim_{n \rightarrow \infty} P^n \cdot x_1 + \lim_{n \rightarrow \infty} P^n \cdot x_2 + \cdots + \lim_{n \rightarrow \infty} P^n \cdot x_k \\ &= \bar{P}x_1 + \bar{P}x_2 + \cdots + \bar{P}x_k \\ &= \bar{P}(x_1 + x_2 + \cdots + x_k) \\ &= \bar{P}X_0 \\ &= \bar{X}, \text{ where } \bar{X} \text{ is a unique steady state matrix.} \end{aligned}$$

Section 2.6 More Applications of Matrix Operations

2. Divide the message into groups of four and form the uncoded matrices.

$$\begin{array}{cccccccccccccc} \text{H} & \text{E} & \text{L} & \text{P} & _ & \text{I} & \text{S} & _ & \text{C} & \text{O} & \text{M} & \text{I} & \text{N} & \text{G} & _ & _ \\ [8 & 5 & 12 & 16] & [0 & 9 & 19 & 0] & [3 & 15 & 13 & 9] & [14 & 7 & 0 & 0] \end{array}$$

Multiplying each uncoded row matrix on the right by A yields the coded row matrices

$$\begin{aligned} [8 \ 5 \ 12 \ 16]A &= [8 \ 5 \ 12 \ 16] \begin{bmatrix} -2 & 3 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ 3 & 1 & -2 & -4 \end{bmatrix} \\ &= [15 \ 33 \ -23 \ -43] \end{aligned}$$

$$[0 \ 9 \ 19 \ 0]A = [-28 \ -10 \ 28 \ 47]$$

$$[3 \ 15 \ 13 \ 9]A = [-7 \ 20 \ 7 \ 2]$$

$$[14 \ 7 \ 0 \ 0]A = [-35 \ 49 \ -7 \ -7].$$

So, the coded message is 15, 33, -23, -43, -28, -10, 28, 47, -7, 20, 7, 2, -35, 49, -7, -7.

4. Find $A^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$, and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[85 \ 120] \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = [20 \ 15] \Rightarrow \text{T, O}$$

$$[6 \ 8]A^{-1} = [0 \ 2] \Rightarrow _, \text{B}$$

$$[10 \ 15]A^{-1} = [5 \ 0] \Rightarrow \text{E, } _$$

$$[84 \ 117]A^{-1} = [15 \ 18] \Rightarrow \text{O, R}$$

$$[42 \ 56]A^{-1} = [0 \ 14] \Rightarrow _, \text{N}$$

$$[90 \ 125]A^{-1} = [15 \ 20] \Rightarrow \text{O, T}$$

$$[60 \ 80]A^{-1} = [0 \ 20] \Rightarrow _, \text{T}$$

$$[30 \ 45]A^{-1} = [15 \ 0] \Rightarrow \text{O, } _$$

$$[19 \ 26]A^{-1} = [2 \ 5] \Rightarrow \text{B, E}$$

So, the message is TO_BE_OR_NOT_TO_BE.

6. Find $A^{-1} = \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix}$, and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[112 \ -140 \ 83]A^{-1} = [112 \ -140 \ 83] \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix} = [8 \ 1 \ 22] \Rightarrow \text{H, A, V}$$

$$[19 \ -25 \ 13]A^{-1} = [5 \ 0 \ 1] \Rightarrow \text{E, } _, \text{A}$$

$$[72 \ -76 \ 61]A^{-1} = [0 \ 7 \ 18] \Rightarrow _, \text{G, R}$$

$$[95 \ -118 \ 71]A^{-1} = [5 \ 1 \ 20] \Rightarrow \text{E, A, T}$$

$$[20 \ 21 \ 38]A^{-1} = [0 \ 23 \ 5] \Rightarrow _, \text{W, E}$$

$$[35 \ -23 \ 36]A^{-1} = [5 \ 11 \ 5] \Rightarrow \text{E, K, E}$$

$$[42 \ -48 \ 32]A^{-1} = [14 \ 4 \ 0] \Rightarrow \text{N, D, } _$$

The message is HAVE_A_GREAT_WEEKEND_.

8. Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and find that

$$\begin{array}{rcl} & & \text{-- S} \\ [-19 & -19] \begin{bmatrix} a & b \\ c & d \end{bmatrix} & = & [0 \quad 19] \\ & & \text{U E} \\ [37 & 16] \begin{bmatrix} a & b \\ c & d \end{bmatrix} & = & [21 \quad 5]. \end{array}$$

This produces a system of 4 equations.

$$\begin{array}{rcl} -19a & -19c & = 0 \\ -19b & -19d & = 19 \\ 37a & +16c & = 21 \\ 37b & +16d & = 5. \end{array}$$

Solving this system, you find $a = 1$, $b = 1$, $c = -1$, and $d = -2$. So,

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}.$$

Multiply each coded row matrix on the right by A^{-1} to yield the uncoded row matrices.

$$\begin{array}{l} [3 \quad 1], [14 \quad 3], [5 \quad 12], [0 \quad 15], [18 \quad 4], \\ [5 \quad 18], [19 \quad 0], [0 \quad 19], [21 \quad 5]. \end{array}$$

This corresponds to the message
CANCEL_ORDERS_SUE.

10. You have

$$\begin{array}{l} [45 \quad -35] \begin{bmatrix} w & x \\ y & z \end{bmatrix} = [10 \quad 15] \text{ and} \\ [38 \quad -30] \begin{bmatrix} w & x \\ y & z \end{bmatrix} = [8 \quad 14]. \end{array}$$

$$\text{So, } 45w - 35y = 10 \quad \text{and} \quad 45x - 35z = 15$$

$$38w - 30y = 8 \quad \quad 38x - 30z = 14.$$

Solving these two systems gives $w = y = 1$, $x = -2$, and $z = -3$. So,

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}.$$

(b) Decoding, you have:

$$[45 \quad -35]A^{-1} = [10 \quad 15] \Rightarrow \text{J, O}$$

$$[38 \quad -30]A^{-1} = [8 \quad 14] \Rightarrow \text{H, N}$$

$$[18 \quad -18]A^{-1} = [0 \quad 18] \Rightarrow \text{_, R}$$

$$[35 \quad -30]A^{-1} = [5 \quad 20] \Rightarrow \text{E, T}$$

$$[81 \quad -60]A^{-1} = [21 \quad 18] \Rightarrow \text{U, R}$$

$$[42 \quad -28]A^{-1} = [14 \quad 0] \Rightarrow \text{N, _}$$

$$[75 \quad -55]A^{-1} = [20 \quad 15] \Rightarrow \text{T, O}$$

$$[2 \quad -2]A^{-1} = [0 \quad 2] \Rightarrow \text{_, B}$$

$$[22 \quad -21]A^{-1} = [1 \quad 19] \Rightarrow \text{A, S}$$

$$[15 \quad -10]A^{-1} = [5 \quad 0] \Rightarrow \text{E, _}$$

The message is JOHN_RETURN_TO_BASE_.

12. Use the given information to find D .

$$D = \begin{bmatrix} 0.30 & 0.20 \\ 0.40 & 0.40 \end{bmatrix} \begin{array}{l} \text{User} \\ \text{A} \quad \text{B} \end{array} \left. \begin{array}{l} \text{A} \\ \text{B} \end{array} \right\} \text{Supplier}$$

The equation $X = DX + E$ may be rewritten in the form $(I - D)X = E$, that is

$$\begin{bmatrix} 0.7 & -0.2 \\ -0.4 & 0.6 \end{bmatrix} X = \begin{bmatrix} 10,000 \\ 20,000 \end{bmatrix}.$$

Solve this system by using Gauss-Jordan elimination to obtain

$$x \approx \begin{bmatrix} 29,412 \\ 52,941 \end{bmatrix}.$$

14. From the given matrix
- D
- , form the linear system

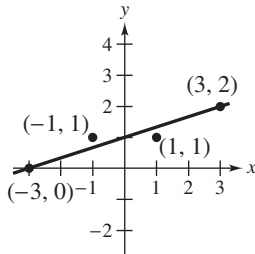
$$X = DX + E, \text{ which can be written as } (I - D)X = E,$$

that is

$$\begin{bmatrix} 0.8 & -0.4 & -0.4 \\ -0.4 & 0.8 & -0.2 \\ 0 & -0.2 & 0.8 \end{bmatrix} X = \begin{bmatrix} 5000 \\ 2000 \\ 8000 \end{bmatrix}.$$

$$\text{Solving this system, } X = \begin{bmatrix} 21,875 \\ 17,000 \\ 14,250 \end{bmatrix}.$$

16. (a) The line that best fits the given points is shown in the graph.



- (b) Using the matrices

$$X = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$\text{you have } X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}, X^T Y = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \text{ and}$$

$$A = (X^T X)^{-1} X^T Y = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{10} \end{bmatrix}.$$

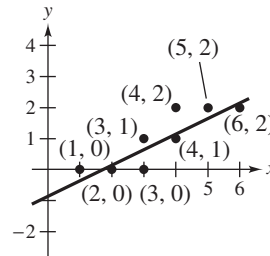
So, the least squares regression line is $y = \frac{3}{10}x + 1$.

- (c) Solving
- $Y = XA + E$
- for
- E
- , you have

$$E = Y - XA = \begin{bmatrix} -0.1 \\ 0.3 \\ -0.3 \\ 0.1 \end{bmatrix}.$$

So, the sum of the squares error is $E^T E = 0.2$.

18. (a) The line that best fits the given points is shown in the graph.



- (b) Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 8 & 28 \\ 28 & 116 \end{bmatrix}, X^T Y = \begin{bmatrix} 8 \\ 37 \end{bmatrix}, \text{ and}$$

$$A = (X^T X)^{-1} (X^T Y) = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}.$$

So, the least squares regression line is $y = \frac{1}{2}x - \frac{3}{4}$.

- (c) Solving
- $Y = XA + E$
- for
- E
- , you have

$$E = Y - XA = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}^T$$

and the sum of the squares error is $E^T E = 1.5$.

20. Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix},$$

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 39 \end{bmatrix}, \text{ and}$$

$$A = (X^T X)^{-1} (X^T Y) = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}.$$

So, the least squares regression line is $y = \frac{3}{2}x - \frac{3}{2}$.

22. Using matrices

$$X = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 5 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 40 \end{bmatrix}, \quad X^T Y = \begin{bmatrix} 8 \\ 32 \end{bmatrix}, \text{ and}$$

$$A = (X^T X)^{-1} (X^T Y) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{40} \end{bmatrix} \begin{bmatrix} 8 \\ 32 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.8 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.8x + 2$.

24. Using matrices

$$X = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}, \quad X^T Y = \begin{bmatrix} 7 \\ -13 \end{bmatrix}, \text{ and}$$

$$A = (X^T X)^{-1} (X^T Y) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix} \begin{bmatrix} 7 \\ -13 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{13}{20} \end{bmatrix}.$$

So, the least squares regression line is
 $y = -0.65x + 1.75$.

26. Using matrices

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 5 \\ 1 & 8 \\ 1 & 10 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -4 \\ -5 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 8 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 5 \\ 1 & 8 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 27 \\ 27 & 205 \end{bmatrix},$$

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 8 & 10 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 0 \\ -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -70 \end{bmatrix}, \text{ and}$$

$$A = (X^T X)^{-1} (X^T Y) = \frac{1}{296} \begin{bmatrix} 205 & -27 \\ -27 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ -70 \end{bmatrix} = \frac{1}{296} \begin{bmatrix} 1890 \\ -350 \end{bmatrix}.$$

So, the least squares regression line is

$$y = -\frac{175}{148}x + \frac{945}{148}.$$

28. Using matrices

$$X = \begin{bmatrix} 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0.72 \\ 0.92 \\ 1.17 \\ 1.34 \\ 1.60 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 5 & 55 \\ 55 & 615 \end{bmatrix} \text{ and } X^T Y = \begin{bmatrix} 5.75 \\ 65.43 \end{bmatrix}.$$

$$A = (X^T X)^{-1} X^T Y = \begin{bmatrix} -1.248 \\ 0.218 \end{bmatrix}$$

So, the least squares regression line is

$$y = 0.218x - 1.248.$$

30. (a) To encode a message, convert the message to numbers and partition it into uncoded row matrices of size $1 \times n$.

Then multiply on the right by an invertible $n \times n$ matrix A to obtain coded row matrices. To decode a message, multiply the coded row matrices on the right by A^{-1} and convert the numbers back to letters.

(b) A Leontief input-output model uses an $n \times n$ matrix to represent the input needs of an economic system, and an $n \times 1$ matrix to represent any external demands on the system.

(c) The coefficients of the least squares regression line are given by $A = (X^T X)^{-1} X^T Y$.

Review Exercises for Chapter 2

$$2. -2 \begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8 \begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -10 & 8 \\ -12 & 0 \end{bmatrix} + \begin{bmatrix} 56 & 8 \\ 8 & 16 \\ 8 & 32 \end{bmatrix} = \begin{bmatrix} 54 & 4 \\ -2 & 24 \\ -4 & 32 \end{bmatrix}$$

$$4. \begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1(6) + 5(4) & 1(-2) + 5(0) & 1(8) + 5(0) \\ 2(6) - 4(4) & 2(-2) - 4(0) & 2(8) - 4(0) \end{bmatrix} = \begin{bmatrix} 26 & -2 & 8 \\ -4 & -4 & 16 \end{bmatrix}$$

$$6. \begin{bmatrix} 2 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 24 & 12 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 24 & 16 \end{bmatrix}$$

8. Letting $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, the

system can be written as

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}.$$

Using Gaussian elimination, the solution of the system is

$$\mathbf{x} = \begin{bmatrix} \frac{6}{7} \\ -\frac{23}{7} \end{bmatrix}.$$

10. Letting $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -3 & -3 \\ 4 & -2 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 10 \\ 22 \\ -2 \end{bmatrix}$,

the system can be written as

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & -3 & -3 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \\ -2 \end{bmatrix}.$$

Using Gaussian elimination, the solution of the system is

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix}.$$

12. $A^T = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix}$$

14. $A^T = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 6 \\ -3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} = [14]$$

16. From the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

you see that $ad - bc = 4(2) - (-1)(-8) = 0$, and so the matrix has no inverse.

18. Begin by adjoining the identity matrix to the given matrix.

$$[A \ I] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

This matrix reduces to

$$[I \ A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

So, the inverse matrix is

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

20. $A \quad \mathbf{x} \quad \mathbf{b}$

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Because $A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$, solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

22. $A \quad \mathbf{x} \quad \mathbf{b}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Using Gauss-Jordan elimination, you find that

$$A^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}.$$

Solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

24. $A \quad \mathbf{x} \quad \mathbf{b}$

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Because $A^{-1} = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix}$, solve the equation

$A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{18}{11} \\ -\frac{19}{11} \end{bmatrix}$$

26. $A \quad \mathbf{x} \quad \mathbf{b}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -7 \end{bmatrix}$$

Using Gauss-Jordan elimination, you find that

$$A^{-1} = \begin{bmatrix} \frac{5}{18} & \frac{1}{9} & \frac{1}{6} \\ -\frac{8}{9} & \frac{4}{9} & -\frac{1}{3} \\ \frac{17}{18} & -\frac{2}{9} & \frac{1}{6} \end{bmatrix}$$

Solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{5}{18} & \frac{1}{9} & \frac{1}{6} \\ -\frac{8}{9} & \frac{4}{9} & -\frac{1}{3} \\ \frac{17}{18} & -\frac{2}{9} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} -\frac{23}{18} \\ \frac{17}{9} \\ -\frac{17}{18} \end{bmatrix}$$

28. Because $(2A)^{-1} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$, you can use the formula for

the inverse of a 2×2 matrix to obtain

$$2A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2-0} \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}.$$

$$\text{So, } A = \frac{1}{4} \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -1 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

30. The matrix $\begin{bmatrix} 2 & x \\ 1 & 4 \end{bmatrix}$ will be nonsingular if

$ad - bc = (2)(4) - (1)(x) \neq 0$, which implies that $x \neq 8$.

32. Because the given matrix represents 6 times the second row, the inverse will be $\frac{1}{6}$ times the second row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For Exercises 34 and 36, answers will vary. Sample answers are shown below.

34. Begin by finding a sequence of elementary row operations to write A in reduced row-echelon form.

| Matrix | Elementary Row Operation | Elementary Matrix |
|---|-----------------------------|--|
| $\begin{bmatrix} 1 & -4 \\ -3 & 13 \end{bmatrix}$ | Interchange the rows. | $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ | Add 3 times row 1 to row 2. | $E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | Add 4 times row 2 to row 1. | $E_3 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ |

Then, you can factor A as follows.

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

36. Begin by finding a sequence of elementary row operations to write A in reduced row-echelon form.

| Matrix | Elementary Row Operation | Elementary Matrix |
|---|--------------------------------------|---|
| $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$ | Multiply row one by $\frac{1}{3}$. | $E_1 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | Add -1 times row one to row three. | $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | Add -2 times row three to row one. | $E_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | Multiply row two by $\frac{1}{2}$. | $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

So, you can factor A as follows.

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

38. Letting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, you have

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, many answers are possible.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ etc.}$$

40. There are many possible answers.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\text{But, } BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0.$$

42. Because $(A^{-1} + B^{-1})(A^{-1} + B^{-1}) = I$, if $(A^{-1} + B^{-1})^{-1}$ exists, it is sufficient to show that $(A^{-1} + B^{-1})(A(A + B)^{-1}B) = I$ for equality of the second factors in each equation.

$$\begin{aligned}
 (A^{-1} + B^{-1})(A(A + B)^{-1}B) &= A^{-1}(A(A + B)^{-1}B) + B^{-1}(A(A + B)^{-1}B) \\
 &= A^{-1}A(A + B)^{-1}B + B^{-1}A(A + B)^{-1}B \\
 &= I(A + B)^{-1}B + B^{-1}A(A + B)^{-1}B \\
 &= (I + B^{-1}A)((A + B)^{-1}B) \\
 &= (B^{-1}B + B^{-1}A)((A + B)^{-1}B) \\
 &= B^{-1}(B + A)(A + B)^{-1}B \\
 &= B^{-1}(A + B)(A + B)^{-1}B \\
 &= B^{-1}IB \\
 &= B^{-1}B \\
 &= I
 \end{aligned}$$

Therefore, $(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B$.

44. Answers will vary. Sample answer:

Matrix

Elementary Matrix

$$\begin{bmatrix} -3 & 1 \\ 12 & 0 \end{bmatrix} = A$$

$$\begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix} = U$$

$$E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$EA = U$$

$$A = E^{-1}U = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix} = LU$$

46. Matrix

Elementary Matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = A$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_3E_2E_1A = U \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

48. MatrixElementary Matrix

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 2 & 1 & 1 & -2 \end{bmatrix} = A$$

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$EA = U \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

$$Ly = \mathbf{b}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 8 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{y}: \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$50. \quad 1.1 \quad \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix} = \begin{bmatrix} 110 & 99 & 77 & 33 \\ 44 & 22 & 66 & 66 \end{bmatrix}$$

52. (a) In matrix B , grading system 1 counts each midterm as 25% of the grade and the final exam as 50% of the grade.

Grading system 2 counts each midterm as 20% of the grade and the final exam as 60% of the grade.

$$(b) \quad AB = \begin{bmatrix} 78 & 82 & 80 \\ 84 & 88 & 85 \\ 92 & 93 & 90 \\ 88 & 86 & 90 \\ 74 & 78 & 80 \\ 96 & 95 & 98 \end{bmatrix} \begin{bmatrix} 0.25 & 0.20 \\ 0.25 & 0.20 \\ 0.50 & 0.60 \end{bmatrix} = \begin{bmatrix} 80 & 80 \\ 85.5 & 85.4 \\ 91.25 & 91 \\ 88.5 & 88.8 \\ 78 & 78.4 \\ 96.75 & 97 \end{bmatrix}$$

- (c) Two students received an "A" in each grading system.

$$\begin{aligned} 54. \quad f(A) &= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^3 - 3 \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

56. The matrix is not stochastic because the sum of entries in columns 1 and 2 do not add up to 1.

58. This matrix is stochastic because each entry is between 0 and 1, and each column adds up to 1.

$$60. \quad X_1 = PX_0 = \begin{bmatrix} 0.307 \\ 0.693 \end{bmatrix}$$

$$X_2 = PX_1 = \begin{bmatrix} 0.38246 \\ 0.61754 \end{bmatrix}$$

$$X_3 = PX_2 = \begin{bmatrix} 0.3659 \\ 0.6341 \end{bmatrix}$$

$$62. \quad X_1 = PX_0 = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{27} \\ \frac{10}{27} \end{bmatrix} \approx \begin{bmatrix} 0.4 \\ 0.185 \\ 0.370 \end{bmatrix}$$

$$X_2 = PX_1 = \begin{bmatrix} \frac{37}{81} \\ \frac{22}{81} \\ \frac{22}{81} \end{bmatrix} \approx \begin{bmatrix} 0.4568 \\ 0.2716 \\ 0.2716 \end{bmatrix}$$

$$X_3 = PX_2 = \begin{bmatrix} \frac{103}{243} \\ \frac{59}{243} \\ \frac{1}{3} \end{bmatrix} \approx \begin{bmatrix} 0.4239 \\ 0.2428 \\ 0.3 \end{bmatrix}$$

64. Begin by forming the matrix of transition probabilities.

$$P = \begin{array}{c} \begin{array}{ccc} \text{From Region} \\ \hline 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.85 & 0.15 & 0.10 \\ 0.10 & 0.80 & 0.10 \\ 0.05 & 0.05 & 0.80 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \end{array} \left. \vphantom{\begin{array}{ccc} 1 & 2 & 3 \end{array}} \right\} \begin{array}{l} \text{To Region} \\ 1 \\ 2 \\ 3 \end{array}$$

(a) The population in each region after 1 year is given by

$$PX = \begin{bmatrix} 0.85 & 0.15 & 0.10 \\ 0.10 & 0.80 & 0.10 \\ 0.05 & 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0.3\bar{6} \\ 0.\bar{3} \\ 0.3 \end{bmatrix}.$$

$$\text{So, } 300,000 \begin{bmatrix} 0.3\bar{6} \\ 0.\bar{3} \\ 0.3 \end{bmatrix} = \begin{bmatrix} 110,000 \\ 100,000 \\ 90,000 \end{bmatrix} \begin{array}{l} \text{Region 1} \\ \text{Region 2} \\ \text{Region 3} \end{array}$$

(b) The population in each region after 3 years is given by

$$P^3X = \begin{bmatrix} 0.665375 & 0.322375 & 0.2435 \\ 0.219 & 0.562 & 0.219 \\ 0.115625 & 0.115625 & 0.5375 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0.4104 \\ 0.\bar{3} \\ 0.25625 \end{bmatrix}.$$

$$\text{So, } 300,000 \begin{bmatrix} 0.4104 \\ 0.\bar{3} \\ 0.25625 \end{bmatrix} = \begin{bmatrix} 123,125 \\ 100,000 \\ 76,875 \end{bmatrix} \begin{array}{l} \text{Region 1} \\ \text{Region 2} \\ \text{Region 3} \end{array}$$

66. The stochastic matrix

$$P = \begin{bmatrix} 1 & \frac{4}{7} \\ 0 & \frac{3}{7} \end{bmatrix}$$

is not regular because P^n has a zero in the first column for all powers.

To find \bar{X} , begin by letting $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then use the

matrix equation $P\bar{X} = \bar{X}$ to obtain

$$\begin{bmatrix} 1 & \frac{4}{7} \\ 0 & \frac{3}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Use these matrices and the fact that $x_1 + x_2 = 1$ to write the system of linear equations shown.

$$\frac{4}{7}x_2 = 0$$

$$-\frac{4}{7}x_2 = 0$$

$$x_1 + x_2 = 1$$

The solution of the system is $x_1 = 1$ and $x_2 = 0$

So, the steady state matrix is $\bar{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

68. The stochastic matrix

$$P = \begin{bmatrix} 0 & 0 & 0.2 \\ 0.5 & 0.9 & 0 \\ 0.5 & 0.1 & 0.8 \end{bmatrix}$$

is regular because P^2 has only positive entries.

To find \bar{X} , let $\bar{X} = [x_1 \ x_2 \ x_3]^T$. Then use the matrix equation $P\bar{X} = \bar{X}$ to obtain.

$$\begin{bmatrix} 0 & 0 & 0.2 \\ 0.5 & 0.9 & 0 \\ 0.5 & 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Use these matrices and the fact that $x_1 + x_2 + x_3 = 1$ to write the system of linear equations shown.

$$-x_1 + 0.2x_3 = 0$$

$$0.5x_1 - 0.1x_2 = 0$$

$$0.5x_1 + 0.1x_2 - 0.2x_3 = 0$$

$$x_1 + x_2 + x_3 = 1$$

The solution of the system is $x_1 = \frac{1}{11}$, $x_2 = \frac{5}{11}$, and

$$x_3 = \frac{5}{11}.$$

So, the steady state matrix is $\bar{X} = \begin{bmatrix} \frac{1}{11} \\ \frac{5}{11} \\ \frac{5}{11} \end{bmatrix}$.

70. Form the matrix representing the given probabilities. Let C represent the classified documents, D represent the declassified documents, and S represent the shredded documents.

$$P = \begin{array}{c} \begin{array}{ccc} \text{From} \\ C & D & S \end{array} \\ \begin{bmatrix} 0.70 & 0.20 & 0 \\ 0.10 & 0.75 & 0 \\ 0.20 & 0.05 & 1 \end{bmatrix} \begin{array}{l} C \\ D \\ S \end{array} \end{array} \left. \vphantom{\begin{array}{c} \begin{array}{ccc} \text{From} \\ C & D & S \end{array} \\ \begin{bmatrix} 0.70 & 0.20 & 0 \\ 0.10 & 0.75 & 0 \\ 0.20 & 0.05 & 1 \end{bmatrix} \begin{array}{l} C \\ D \\ S \end{array} \end{array} \right\} \text{To}$$

Solve the equation $P\bar{X} = \bar{X}$, where $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and use the fact that $x_1 + x_2 + x_3 = 1$ to write a system of equations.

$$\begin{aligned} 0.70x_1 + 0.20x_2 &= x_1 & -0.3x_1 + 0.2x_2 &= 0 \\ 0.10x_1 + 0.75x_2 &= x_2 & \Rightarrow 0.1x_1 - 0.25x_2 &= 0 \\ 0.20x_1 + 0.05x_2 + x_3 &= x_3 & 0.2x_1 + 0.05x_2 &= 0 \\ x_1 + x_2 + x_3 &= 1 & x_1 + x_2 + x_3 &= 0 \end{aligned}$$

So, the steady state matrix is $\bar{X} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

This indicates that eventually all of the documents will be shredded.

72. The matrix

$$P = \begin{bmatrix} 1 & 0 & 0.38 \\ 0 & 0.30 & 0 \\ 0 & 0.70 & 0.62 \end{bmatrix}$$

is absorbing. The first state S_1 is absorbing and it is possible to move from S_2 to S_1 in two transitions and to move from S_3 to S_1 in one transition.

74. (a) False. See Exercise 65, page 61.

(b) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$A + B$ is a *singular* matrix, while both A and B are *nonsingular* matrices.

76. (a) True. See Section 2.5, Example 4(b).

(b) False. See Section 2.5, Example 7(a).

78. The uncoded row matrices are

$$\begin{array}{cccccccccccccccc} B & E & A & M & _ & M & E & _ & U & P & _ & S & C & O & T & T & Y & _ \\ [2 & 5 & 1] & [13 & 0 & 13] & [5 & 0 & 21] & [16 & 0 & 19] & [3 & 15 & 20] & [20 & 25 & 0] \end{array}$$

Multiplying each 1×3 matrix on the right by A yields the coded row matrices.

$$[17 \ 6 \ 20] \ [0 \ 0 \ 13] \ [-32 \ -16 \ -43] \ [-6 \ -3 \ 7] \ [11 \ -2 \ -3] \ [115 \ 45 \ 155]$$

So, the coded message is

17, 6, 20, 0, 0, 13, -32, -16, -43, -6, -3, 7, 11, -2, -3, 115, 45, 155.

80. Find
- A^{-1}
- to be

$$A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

and the coded row matrices are

$$[11 \ 52], [-8 \ -9], [-13 \ -39], [5 \ 20], [12 \ 56], [5 \ 20], [-2 \ 7], [9 \ 41], [25 \ 100].$$

Multiplying each coded row matrix on the right by A^{-1} yields the uncoded row matrices.

$$\begin{array}{cccccccccccccccc} \text{S} & \text{H} & & \text{O} & \text{W} & & \text{M} & \text{E} & & \text{T} & \text{H} & \text{E} & & \text{M} & \text{O} & \text{N} & \text{E} & \text{Y} & \\ [19 & 8] & [15 & 23] & [0 & 13] & [5 & 0] & [20 & 8] & [5 & 0] & [13 & 15] & [14 & 5] & [25 & 0] \end{array}$$

So, the message is SHOW_ME_THE_MONEY_.

82. Find
- A^{-1}
- to be

$$A^{-1} = \begin{bmatrix} \frac{4}{13} & \frac{2}{13} & \frac{1}{13} \\ \frac{8}{13} & -\frac{9}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{4}{13} & -\frac{2}{13} \end{bmatrix},$$

and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[66 \ 27 \ -31]A^{-1} = [66 \ 27 \ -31] \begin{bmatrix} \frac{4}{13} & \frac{2}{13} & \frac{1}{13} \\ \frac{8}{13} & -\frac{9}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{4}{13} & -\frac{2}{13} \end{bmatrix} = [25 \ 1 \ 14] \Rightarrow \text{Y, A, N}$$

$$[37 \ 5 \ -9]A^{-1} = [11 \ 5 \ 5] \Rightarrow \text{K, E, E}$$

$$[61 \ 46 \ -73]A^{-1} = [19 \ 0 \ 23] \Rightarrow \text{S, }, \text{W}$$

$$[46 \ -14 \ 9]A^{-1} = [9 \ 14 \ 0] \Rightarrow \text{I, N, }$$

$$[94 \ 21 \ -49]A^{-1} = [23 \ 15 \ 18] \Rightarrow \text{W, O, R}$$

$$[32 \ -4 \ 12]A^{-1} = [12 \ 4 \ 0] \Rightarrow \text{L, D, }$$

$$[66 \ 31 \ -53]A^{-1} = [19 \ 5 \ 18] \Rightarrow \text{S, E, R}$$

$$[47 \ 33 \ -67]A^{-1} = [9 \ 5 \ 19] \Rightarrow \text{I, E, S}$$

$$[32 \ 19 \ -56]A^{-1} = [0 \ 9 \ 14] \Rightarrow \text{, I, N}$$

$$[43 \ -9 \ -20]A^{-1} = [0 \ 19 \ 5] \Rightarrow \text{, S, E}$$

$$[68 \ 23 \ -34]A^{-1} = [22 \ 5 \ 14] \Rightarrow \text{V, E, N}$$

The message is YANKEES_WIN_WORLD_SERIES_IN_SEVEN.

84. Solve the equation
- $X = DX + E$
- for
- X
- to obtain
- $(I - D)X = E$
- , which corresponds to solving the augmented matrix.

$$\left[\begin{array}{ccc|c} 0.9 & -0.3 & -0.2 & 3000 \\ 0 & 0.8 & -0.3 & 3500 \\ -0.4 & -0.1 & 0.9 & 8500 \end{array} \right]$$

The solution to this system is

$$X = \begin{bmatrix} 10,000 \\ 10,000 \\ 15,000 \end{bmatrix}.$$

86. Using the matrices

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 4 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 5 & 20 \\ 20 & 90 \end{bmatrix}, \quad X^T Y = \begin{bmatrix} 14 \\ 63 \end{bmatrix}, \text{ and}$$

$$A = (X^T X)^{-1} X^T Y = \begin{bmatrix} 1.8 & -0.4 \\ -0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 14 \\ 63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.7x$.

88. Using the matrices

$$X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \\ -3 \end{bmatrix}, \text{ you have}$$

$$X^T X = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}, X^T Y = \begin{bmatrix} 2 \\ -18 \end{bmatrix}, \text{ and } A = (X^T X)^{-1} X^T Y = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 2 \\ -18 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -1.8 \end{bmatrix}.$$

So, the least squares regression line is $y = -1.8x + 0.4$, or $y = -\frac{9}{5}x + \frac{2}{5}$.

90. (a) Using the matrices $X = \begin{bmatrix} 1 & 8 \\ 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \end{bmatrix}$ and $Y = \begin{bmatrix} 2.93 \\ 3.00 \\ 3.01 \\ 3.10 \\ 3.21 \\ 3.39 \end{bmatrix}$, you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 9 & 10 & 11 & 12 & 13 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \end{bmatrix} = \begin{bmatrix} 6 & 63 \\ 63 & 679 \end{bmatrix}$$

and

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 9 & 10 & 11 & 12 & 13 \end{bmatrix} \begin{bmatrix} 2.93 \\ 3.00 \\ 3.01 \\ 3.10 \\ 3.21 \\ 3.39 \end{bmatrix} = \begin{bmatrix} 18.64 \\ 197.23 \end{bmatrix}.$$

Now, using $(X^T X)^{-1}$ to find the coefficient matrix A , you have

$$A = (X^T X)^{-1} X^T Y = \begin{bmatrix} \frac{97}{15} & \frac{-3}{5} \\ \frac{-3}{5} & \frac{2}{35} \end{bmatrix} \begin{bmatrix} 18.64 \\ 197.23 \end{bmatrix} \approx \begin{bmatrix} 2.2007 \\ 0.0863 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.0863x + 2.2007$.

(b) Using a graphing utility, the regression line is $y = 0.0863x + 2.2007$.

(c)

| Year | 2008 | 2009 | 2009 | 2010 | 2011 | 2012 | 2013 |
|-----------|------|------|------|------|------|------|------|
| Actual | 2.93 | 2.93 | 3.00 | 3.01 | 3.10 | 3.21 | 3.39 |
| Estimated | 2.89 | 2.89 | 2.98 | 3.06 | 3.15 | 3.24 | 3.32 |

The estimated values are close to the actual values.

Project Solutions for Chapter 2

1 Exploring Matrix Multiplication

1. Test 1 seems to be the more difficult test. The averages were:

$$\text{Test 1 average} = 75$$

$$\text{Test 2 average} = 85.5$$

2. Anna, David, Chris, Bruce

3. Answers will vary. Sample answer:

$M \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents scores on the first test.

$M \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represents scores on the second test.

4. Answers will vary. Sample answer:

$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} M$ represents Anna's scores.

$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} M$ represents Chris's scores.

5. Answers will vary. Sample answer:

$M \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ represents the sum of the test scores for each

student, and $\frac{1}{2} M \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ represents each student's average.

6. $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} M$ represents the sum of scores on each test;

$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} M$ represents the average on each test.

7. $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} M \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ represents the overall points total for all students on all tests.

8. $\frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 80.25$

9. $M \begin{bmatrix} 1.1 \\ 1.0 \end{bmatrix}$

2 Nilpotent Matrices

1. $A^2 \neq 0$ and $A^3 = 0$, so the index is 3.

2. (a) Nilpotent of index 2

(b) Not nilpotent

(c) Nilpotent of index 2

(d) Not nilpotent

(e) Nilpotent of index 2

(f) Nilpotent of index 3

3. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ index 2; $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ index 3

4. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ index 2; $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ index 3;

$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ index 4

5. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

6. No. If A is nilpotent and invertible, then $A^k = O$ for some k and $A^{k-1} \neq O$. So,

$$A^{-1}A = I \Rightarrow O = A^{-1}A^k = (A^{-1}A)A^{k-1} = IA^{k-1} \neq O,$$

which is impossible.

7. If A is nilpotent, then $(A^k)^T = (A^T)^k = O$. But

$(A^T)^{k-1} = (A^{k-1})^T \neq O$, which shows that A^T is nilpotent with the same index.

8. Let A be nilpotent of index k . Then

$$(I - A)(A^{k-1} + A^{k-2} + \cdots + A^2 + A + I) = I - A^k = I,$$

which shows that

$$(A^{k-1} + A^{k-2} + \cdots + A^2 + A + I)$$

is the inverse of $I - A$.