CHAPTER TWO

Solutions for Section 2.1

Exercises

1. For t between 2 and 5, we have

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{400-135}{5-2} = \frac{265}{3} \text{ km/hr.}$$

The average velocity on this part of the trip was $265/3$ km/hr.

2. The average velocity over a time period is the change in position divided by the change in time. Since the function $x(t)$ gives the position of the particle, we find the values of $x(0) = -2$ and $x(4) = -6$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta x(t)}{\Delta t} = \frac{x(4)-x(0)}{4-0} = \frac{-6-(-2)}{4} = -1 \text{ meters/sec.}$$

3. The average velocity over a time period is the change in position divided by the change in time. Since the function $x(t)$ gives the position of the particle, we find the values of $x(2) = 14$ and $x(8) = -4$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta x(t)}{\Delta t} = \frac{x(8)-x(2)}{8-2} = \frac{-4-14}{6} = -3 \text{ angstroms/sec.}$$

4. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we read off the graph that $s(0) = 1$ and $s(3) = 4$. Thus,

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3)-s(0)}{3-0} = \frac{4-1}{3} = 1 \text{ meter/sec.}$$

5. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we read off the graph that $s(1) = 2$ and $s(3) = 6$. Thus,

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(3)-s(1)}{3-1} = \frac{6-2}{2} = 2 \text{ meters/sec.}$$

6. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we find the values of $s(2) = e^2 - 1 = 6.389$ and $s(4) = e^4 - 1 = 53.598$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(4)-s(2)}{4-2} = \frac{53.598-6.389}{2} = 23.605\ \mu\text{m/sec.}$$

7. The average velocity over a time period is the change in the distance divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we find the values of $s(\pi/3) = 4 + 3\sqrt{3}/2$ and $s(7\pi/3) = 4 + 3\sqrt{3}/2$. Using these values, we find

$$\text{Average velocity } = \frac{\Delta s(t)}{\Delta t} = \frac{s(7\pi/3)-s(\pi/3)}{7\pi/3-\pi/3} = \frac{4+3\sqrt{3}/2-(4+3\sqrt{3}/2)}{2\pi} = 0 \text{ cm/sec.}$$

Though the particle moves, its average velocity is zero, since it is at the same position at $t = \pi/3$ and $t = 7\pi/3$.

8. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{3.63 - 3}{0.1} = 6.3 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{3.0603 - 3}{0.01} = 6.03 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{3.006003 - 3}{0.001} = 6.003 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 6, so we estimate the instantaneous velocity at $t = 1$ to be 6 m/sec.

9. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 0$ and $t = 0.1$. We have

$$\text{Average velocity} = \frac{f(0.1) - f(0)}{0.1 - 0} = \frac{0.004 - 0}{0.1} = 0.04 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(0.01) - f(0)}{0.01 - 0} = \frac{0.000004}{0.01} = 0.0004 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(0.001) - f(0)}{1.001 - 1} = \frac{4 \times 10^{-9} - 0}{0.001} = 4 \times 10^{-6} \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 0$ the average velocity appears to be getting closer and closer to 0, so we estimate the instantaneous velocity at $t = 0$ to be 0 m/sec.

Looking at a graph of $s = f(t)$ we see that a line tangent to the graph at $t = 0$ is horizontal, confirming our result.

10. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{0.808496 - 0.909297}{0.1} = -1.00801 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{0.900793 - 0.909297}{0.01} = -0.8504 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{0.908463 - 0.909297}{0.001} = -0.834 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to -0.83, so we estimate the instantaneous velocity at $t = 1$ to be -0.83 m/sec. In this case, more estimates with smaller values of h would be very helpful in making a better estimate.

11. See Figure 2.1.

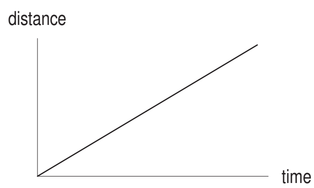


Figure 2.1

12. See Figure 2.2.

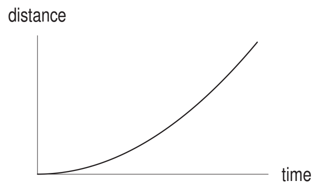


Figure 2.2

13. See Figure 2.3.

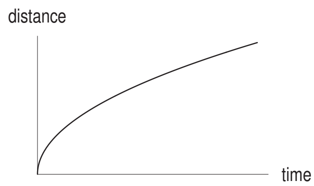


Figure 2.3

Problems

14. Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{(3+0.1)^3-27}{0.1} = 27.91$$
$$\frac{(3+0.01)^3-27}{0.01} = 27.09$$
$$\frac{(3+0.001)^3-27}{0.001} = 27.009.$$

These calculations suggest that $\lim_{h\to 0} \frac{(3+h)^3-27}{h} = 27.$

15. Using radians,

h	$(\cos h - 1)/h$
0.01	-0.005
0.001	-0.0005
0.0001	-0.00005

These values suggest that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$

16. Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{7^{0.1}-1}{0.1} = 2.148$$
$$\frac{7^{0.01}-1}{0.01} = 1.965$$
$$\frac{7^{0.001}-1}{0.001} = 1.948$$
$$\frac{7^{0.0001}-1}{0.0001} = 1.946.$$

This suggests that $\lim_{h\to 0} \frac{7^h-1}{h} \approx 1.9.$

17. Using $h = 0.1, 0.01, 0.001$, we see

h	$(e^{1+h} - e)/h$
0.01	2.7319
0.001	2.7196
0.0001	2.7184

These values suggest that $\lim_{h\to 0} \dfrac{e^{1+h} - e}{h} = 2.7$. In fact, this limit is e.

18.

Slope	-3	-1	0	$1/2$	1	2
Point	F	C	E	A	B	D

19. The slope is positive at A and D; negative at C and F. The slope is most positive at A; most negative at F.

20. $0 <$ slope at $C <$ slope at $B <$ slope of $AB < 1 <$ slope at A. (Note that the line $y = x$, has slope 1.)

21. Since $f(t)$ is concave down between $t = 1$ and $t = 3$, the average velocity between the two times should be less than the instantaneous velocity at $t = 1$ but greater than the instantaneous velocity at time $t = 3$, so $D < A < C$. For analogous reasons, $F < B < E$. Finally, note that f is decreasing at $t = 5$ so $E < 0$, but increasing at $t = 0$, so $D > 0$. Therefore, the ordering from smallest to greatest of the given quantities is

$$F < B < E < 0 < D < A < C.$$

22.

$$\begin{pmatrix}\text{Average velocity}\\ 0 < t < 0.2\end{pmatrix} = \frac{s(0.2) - s(0)}{0.2 - 0} = \frac{0.5}{0.2} = 2.5 \text{ ft/sec.}$$

$$\begin{pmatrix}\text{Average velocity}\\ 0.2 < t < 0.4\end{pmatrix} = \frac{s(0.4) - s(0.2)}{0.4 - 0.2} = \frac{1.3}{0.2} = 6.5 \text{ ft/sec.}$$

A reasonable estimate of the velocity at $t = 0.2$ is the average: $\frac{1}{2}(6.5 + 2.5) = 4.5$ ft/sec.

23. One possibility is shown in Figure 2.4.

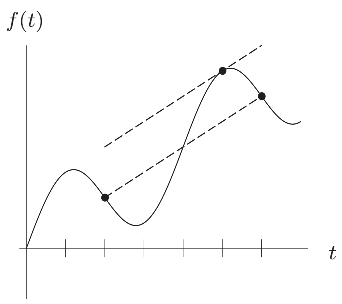


Figure 2.4

24. (a) When $t = 0$, the ball is on the bridge and its height is $f(0) = 36$, so the bridge is 36 feet above the ground.

(b) After 1 second, the ball's height is $f(1) = -16 + 50 + 36 = 70$ feet, so it traveled $70 - 36 = 34$ feet in 1 second, and its average velocity was 34 ft/sec.

(c) At $t = 1.001$, the ball's height is $f(1.001) = 70.017984$ feet, and its velocity about $\frac{70.017984-70}{1.001-1} = 17.984 \approx 18$ ft/sec.

(d) We complete the square:

$$\begin{aligned} f(t) &= -16t^2 + 50t + 36 \\ &= -16\left(t^2 - \frac{25}{8}t\right) + 36 \\ &= -16\left(t^2 - \frac{25}{8}t + \frac{625}{256}\right) + 36 + 16\left(\frac{625}{256}\right) \\ &= -16(t - \tfrac{25}{16})^2 + \tfrac{1201}{16} \end{aligned}$$

so the graph of f is a downward parabola with vertex at the point $(25/16, 1201/16) = (1.6, 75.1)$. We see from Figure 2.5 that the ball reaches a maximum height of about 75 feet. The velocity of the ball is zero when it is at the peak, since the tangent is horizontal there.

(e) The ball reaches its maximum height when $t = \frac{25}{16} = 1.6$.

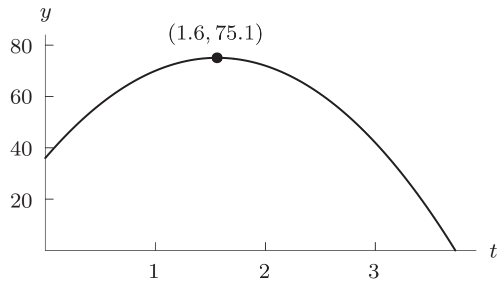


Figure 2.5

25. $\lim\limits_{h\to 0} \dfrac{(2+h)^2 - 4}{h} = \lim\limits_{h\to 0} \dfrac{4 + 4h + h^2 - 4}{h} = \lim\limits_{h\to 0}(4+h) = 4$

26. $\lim\limits_{h\to 0} \dfrac{(1+h)^3 - 1}{h} = \lim\limits_{h\to 0} \dfrac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim\limits_{h\to 0} \dfrac{h(3 + 3h + h^2)}{h} = \lim\limits_{h\to 0} 3 + 3h + h^2 = 3.$

27. $\lim\limits_{h\to 0} \dfrac{3(2+h)^2 - 12}{h} = \lim\limits_{h\to 0} \dfrac{12 + 12h + 3h^2 - 12}{h} = \lim\limits_{h\to 0} \dfrac{h(12+3h)}{h} = \lim\limits_{h\to 0} 12 + 3h = 12.$

28. $\lim\limits_{h\to 0} \dfrac{(3+h)^2 - (3-h)^2}{2h} = \lim\limits_{h\to 0} \dfrac{9 + 6h + h^2 - 9 + 6h - h^2}{2h} = \lim\limits_{h\to 0} \dfrac{12h}{2h} = \lim\limits_{h\to 0} 6 = 6.$

Strengthen Your Understanding

29. Speed is the magnitude of velocity, so it is always positive or zero; velocity has both magnitude and direction.

30. We expand and simplify first

$$\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{(4 + 4h + h^2) - 4}{h} = \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0}(4+h) = 4.$$

31. Since the tangent line to the curve at $t = 4$ is almost horizontal, the instantaneous velocity is almost zero. At $t = 2$ the slope of the tangent line, and hence the instantaneous velocity, is relatively large and positive.

32. $f(t) = t^2$. The slope of the graph of $y = f(t)$ is negative for $t < 0$ and positive for $t > 0$.
Many other answers are possible.

33. One possibility is the position function $s(t) = t^2$. Any function that is symmetric about the line $t = 0$ works.
For $s(t) = t^2$, the slope of a tangent line (representing the velocity) is negative at $t = -1$ and positive at $t = 1$, and that the magnitude of the slopes (the speeds) are the same.

34. False. For example, the car could slow down or even stop at one minute after 2 pm, and then speed back up to 60 mph at one minute before 3 pm. In this case the car would travel only a few miles during the hour, much less than 50 miles.

35. False. Its average velocity for the time between 2 pm and 4 pm is 40 mph, but the car could change its speed a lot during that time period. For example, the car might be motionless for an hour then go 80 mph for the second hour. In that case the velocity at 2 pm would be 0 mph.

36. True. During a short enough time interval the car can not change its velocity very much, and so it velocity will be nearly constant. It will be nearly equal to the average velocity over the interval.

37. True. The instantaneous velocity is a limit of the average velocities. The limit of a constant equals that constant.

38. True. By definition, Average velocity = Distance traveled/Time.

39. False. Instantaneous velocity equals a *limit* of difference quotients.

Solutions for Section 2.2

Exercises

1. The derivative, $f'(2)$, is the rate of change of x^3 at $x = 2$. Notice that each time x changes by 0.001 in the table, the value of x^3 changes by 0.012. Therefore, we estimate

$$f'(2) = \begin{array}{c}\text{Rate of change}\\ \text{of } f \text{ at } x = 2\end{array} \approx \frac{0.012}{0.001} = 12.$$

The function values in the table look exactly linear because they have been rounded. For example, the exact value of x^3 when $x = 2.001$ is 8.012006001, not 8.012. Thus, the table can tell us only that the derivative is approximately 12. Example 5 on page 95 shows how to compute the derivative of $f(x)$ exactly.

2. With $h = 0.01$ and $h = -0.01$, we have the difference quotients

$$\frac{f(1.01) - f(1)}{0.01} = 3.0301 \quad \text{and} \quad \frac{f(0.99) - f(1)}{-0.01} = 2.9701.$$

With $h = 0.001$ and $h = -0.001$,

$$\frac{f(1.001) - f(1)}{0.001} = 3.003001 \quad \text{and} \quad \frac{f(0.999) - f(1)}{-0.001} = 2.997001.$$

The values of these difference quotients suggest that the limit is about 3.0. We say

$$f'(1) = \begin{array}{c}\text{Instantaneous rate of change of } f(x) = x^3\\ \text{with respect to } x \text{ at } x = 1\end{array} \approx 3.0.$$

3. (a) Using the formula for the average rate of change gives

$$\begin{array}{c}\text{Average rate of change}\\ \text{of revenue for } 1 \le q \le 2\end{array} = \frac{R(2) - R(1)}{1} = \frac{160 - 90}{1} = 70 \text{ dollars/kg}.$$

$$\begin{array}{c}\text{Average rate of change}\\ \text{of revenue for } 2 \le q \le 3\end{array} = \frac{R(3) - R(2)}{1} = \frac{210 - 160}{1} = 50 \text{ dollars/kg}.$$

So we see that the average rate decreases as the quantity sold in kilograms increases.

(b) With $h = 0.01$ and $h = -0.01$, we have the difference quotients

$$\frac{R(2.01) - R(2)}{0.01} = 59.9 \text{ dollars/kg} \quad \text{and} \quad \frac{R(1.99) - R(2)}{-0.01} = 60.1 \text{ dollars/kg}.$$

With $h = 0.001$ and $h = -0.001$,

$$\frac{R(2.001) - R(2)}{0.001} = 59.99 \text{ dollars/kg} \quad \text{and} \quad \frac{R(1.999) - R(2)}{-0.001} = 60.01 \text{ dollars/kg}.$$

The values of these difference quotients suggest that the instantaneous rate of change is about 60 dollars/kg. To confirm that the value is exactly 60, that is, that $R'(2) = 60$, we would need to take the limit as $h \to 0$.

4. (a) Using a calculator we obtain the values found in the table below:

x	1	1.5	2	2.5	3
e^x	2.72	4.48	7.39	12.18	20.09

(b) The average rate of change of $f(x) = e^x$ between $x = 1$ and $x = 3$ is

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3 - 1} = \frac{e^3 - e}{3 - 1} \approx \frac{20.09 - 2.72}{2} = 8.69.$$

(c) First we find the average rates of change of $f(x) = e^x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$:

$$\text{Average rate of change} = \frac{f(2) - f(1.5)}{2 - 1.5} = \frac{e^2 - e^{1.5}}{2 - 1.5} \approx \frac{7.39 - 4.48}{0.5} = 5.82$$
$$\text{Average rate of change} = \frac{f(2.5) - f(2)}{2.5 - 2} = \frac{e^{2.5} - e^2}{2.5 - 2} \approx \frac{12.18 - 7.39}{0.5} = 9.58.$$

Now we approximate the instantaneous rate of change at $x = 2$ by averaging these two rates:

$$\text{Instantaneous rate of change} \approx \frac{5.82 + 9.58}{2} = 7.7.$$

5. (a)

Table 2.1

x	1	1.5	2	2.5	3
$\log x$	0	0.18	0.30	0.40	0.48

(b) The average rate of change of $f(x) = \log x$ between $x = 1$ and $x = 3$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{\log 3 - \log 1}{3 - 1} \approx \frac{0.48 - 0}{2} = 0.24$$

(c) First we find the average rates of change of $f(x) = \log x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$.

$$\frac{\log 2 - \log 1.5}{2 - 1.5} = \frac{0.30 - 0.18}{0.5} \approx 0.24$$
$$\frac{\log 2.5 - \log 2}{2.5 - 2} = \frac{0.40 - 0.30}{0.5} \approx 0.20$$

Now we approximate the instantaneous rate of change at $x = 2$ by finding the average of the above rates, i.e.

$$\left(\begin{array}{c}\text{the instantaneous rate of change} \\ \text{of } f(x) = \log x \text{ at } x = 2\end{array}\right) \approx \frac{0.24 + 0.20}{2} = 0.22.$$

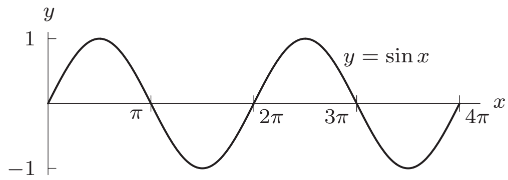
6. In Table 2.2, each x increase of 0.001 leads to an increase in $f(x)$ by about 0.031, so

$$f'(3) \approx \frac{0.031}{0.001} = 31.$$

Table 2.2

x	2.998	2.999	3.000	3.001	3.002
$x^3 + 4x$	38.938	38.969	39.000	39.031	39.062

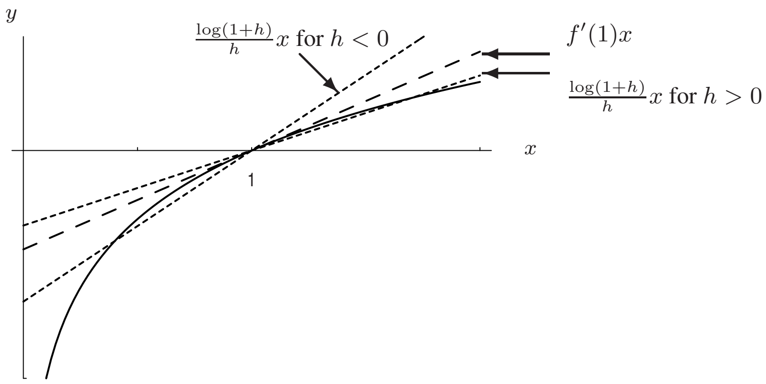
7.



Since $\sin x$ is decreasing for values near $x = 3\pi$, its derivative at $x = 3\pi$ is negative.

8. $f'(1) = \lim_{h\to 0} \dfrac{\log(1+h) - \log 1}{h} = \lim_{h\to 0} \dfrac{\log(1+h)}{h}$

Evaluating $\frac{\log(1+h)}{h}$ for $h = 0.01, 0.001$, and 0.0001, we get $0.43214, 0.43408, 0.43427$, so $f'(1) \approx 0.43427$. The corresponding secant lines are getting steeper, because the graph of $\log x$ is concave down. We thus expect the limit to be more than 0.43427 . If we consider negative values of h, the estimates are too large. We can also see this from the graph below:



9. We estimate $f'(2)$ using the average rate of change formula on a small interval around 2. We use the interval $x = 2$ to $x = 2.001$. (Any small interval around 2 gives a reasonable answer.) We have

$$f'(2) \approx \frac{f(2.001) - f(2)}{2.001 - 2} = \frac{3^{2.001} - 3^2}{2.001 - 2} = \frac{9.00989 - 9}{0.001} = 9.89.$$

10. (a) The average rate of change from $x = a$ to $x = b$ is the slope of the line between the points on the curve with $x = a$ and $x = b$. Since the curve is concave down, the line from $x = 1$ to $x = 3$ has a greater slope than the line from $x = 3$ to $x = 5$, and so the average rate of change between $x = 1$ and $x = 3$ is greater than that between $x = 3$ and $x = 5$.

(b) Since f is increasing, $f(5)$ is the greater.

(c) As in part (a), f is concave down and f' is decreasing throughout so $f'(1)$ is the greater.

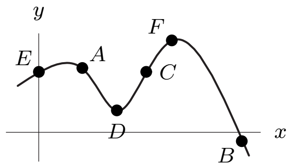
11. Since $f'(x) = 0$ where the graph is horizontal, $f'(x) = 0$ at $x = d$. The derivative is positive at points b and c, but the graph is steeper at $x = c$. Thus $f'(x) = 0.5$ at $x = b$ and $f'(x) = 2$ at $x = c$. Finally, the derivative is negative at points a and e but the graph is steeper at $x = e$. Thus, $f'(x) = -0.5$ at $x = a$ and $f'(x) = -2$ at $x = e$. See Table 2.3.

Thus, we have $f'(d) = 0$, $f'(b) = 0.5$, $f'(c) = 2$, $f'(a) = -0.5$, $f'(e) = -2$.

Table 2.3

x	$f'(x)$
d	0
b	0.5
c	2
a	-0.5
e	-2

12. One possible choice of points is shown below.



Problems

13. The statements $f(100) = 35$ and $f'(100) = 3$ tell us that at $x = 100$, the value of the function is 35 and the function is increasing at a rate of 3 units for a unit increase in x. Since we increase x by 2 units in going from 100 to 102, the value of the function goes up by approximately $2 \cdot 3 = 6$ units, so

$$f(102) \approx 35 + 2 \cdot 3 = 35 + 6 = 41.$$

14. The answers to parts (a)–(d) are shown in Figure 2.6.

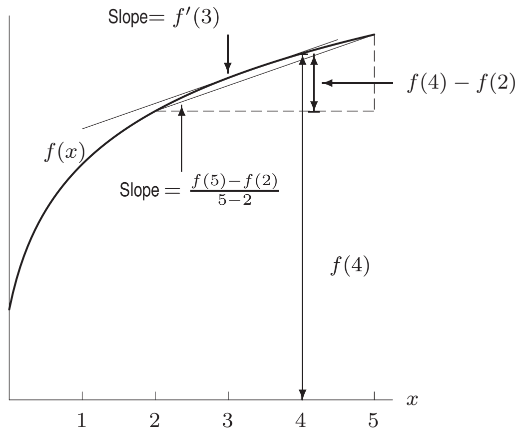


Figure 2.6

15. (a) Since f is increasing, $f(4) > f(3)$.
(b) From Figure 2.7, it appears that $f(2) - f(1) > f(3) - f(2)$.
(c) The quantity $\dfrac{f(2) - f(1)}{2 - 1}$ represents the slope of the secant line connecting the points on the graph at $x = 1$ and $x = 2$. This is greater than the slope of the secant line connecting the points at $x = 1$ and $x = 3$ which is $\dfrac{f(3) - f(1)}{3 - 1}$.
(d) The function is steeper at $x = 1$ than at $x = 4$ so $f'(1) > f'(4)$.

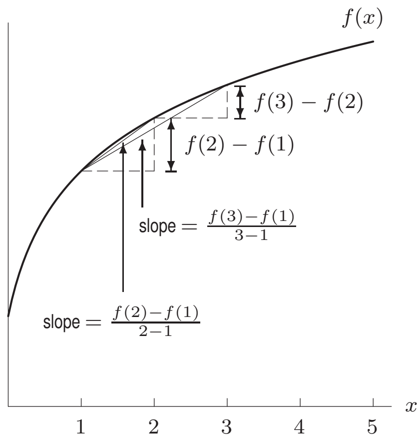


Figure 2.7

16. Figure 2.8 shows the quantities in which we are interested.

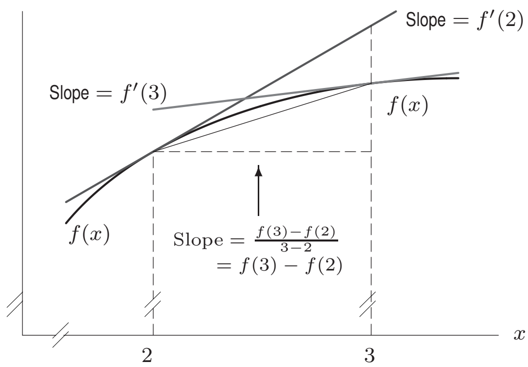


Figure 2.8

The quantities $f'(2)$, $f'(3)$ and $f(3) - f(2)$ have the following interpretations:

- $f'(2) =$ slope of the tangent line at $x = 2$
- $f'(3) =$ slope of the tangent line at $x = 3$
- $f(3) - f(2) = \frac{f(3)-f(2)}{3-2} =$ slope of the secant line from $f(2)$ to $f(3)$.

From Figure 2.8, it is clear that $0 < f(3) - f(2) < f'(2)$. By extending the secant line past the point $(3, f(3))$, we can see that it lies above the tangent line at $x = 3$.

Thus

$$0 < f'(3) < f(3) - f(2) < f'(2).$$

17. The coordinates of A are $(4, 25)$. See Figure 2.9. The coordinates of B and C are obtained using the slope of the tangent line. Since $f'(4) = 1.5$, the slope is 1.5

From A to B, $\Delta x = 0.2$, so $\Delta y = 1.5(0.2) = 0.3$. Thus, at C we have $y = 25 + 0.3 = 25.3$. The coordinates of B are $(4.2, 25.3)$.

From A to C, $\Delta x = -0.1$, so $\Delta y = 1.5(-0.1) = -0.15$. Thus, at C we have $y = 25 - 0.15 = 24.85$. The coordinates of C are $(3.9, 24.85)$.

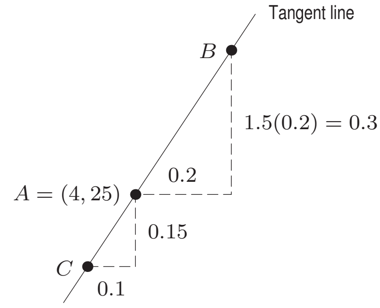


Figure 2.9

18. (a) Since the point $B = (2, 5)$ is on the graph of g, we have $g(2) = 5$.
(b) The slope of the tangent line touching the graph at $x = 2$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{5 - 5.02}{2 - 1.95} = \frac{-0.02}{0.05} = -0.4.$$

Thus, $g'(2) = -0.4$.

19. See Figure 2.10.

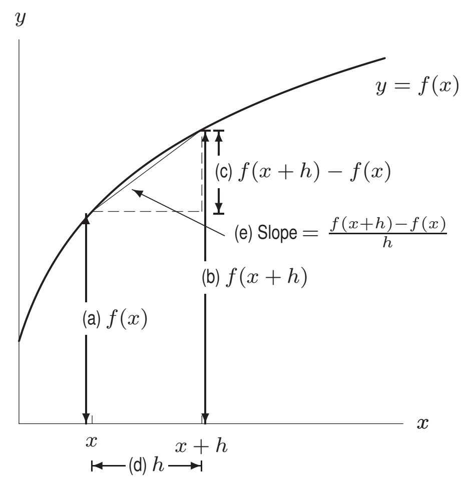


Figure 2.10

20. See Figure 2.11.

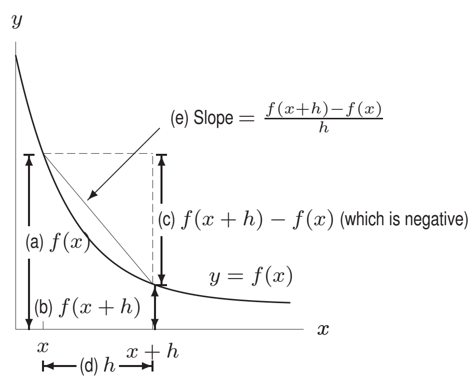


Figure 2.11

21. (a) For the line from A to B,

$$\text{Slope} = \frac{f(b) - f(a)}{b - a}.$$

(b) The tangent line at point C appears to be parallel to the line from A to B. Assuming this to be the case, the lines have the same slope.

(c) There is only one other point, labeled D in Figure 2.12, at which the tangent line is parallel to the line joining A and B.

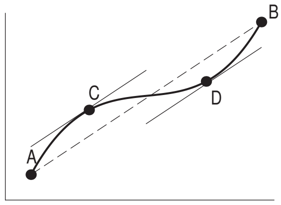


Figure 2.12

22. (a) Figure 2.13 shows the graph of an even function. We see that since f is symmetric about the y-axis, the tangent line at $x = -10$ is just the tangent line at $x = 10$ flipped about the y-axis, so the slope of one tangent is the negative of that of the other. Therefore, $f'(-10) = -f'(10) = -6$.

(b) From part (a) we can see that if f is even, then for any x, we have $f'(-x) = -f'(x)$. Thus $f'(-0) = -f'(0)$, so $f'(0) = 0$.

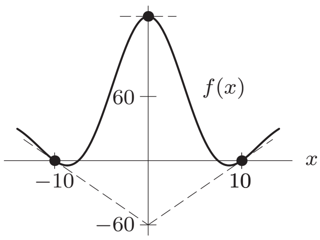


Figure 2.13

23. Figure 2.14 shows the graph of an odd function. We see that since g is symmetric about the origin, its tangent line at $x = -4$ is just the tangent line at $x = 4$ flipped about the origin, so they have the same slope. Thus, $g'(-4) = 5$.

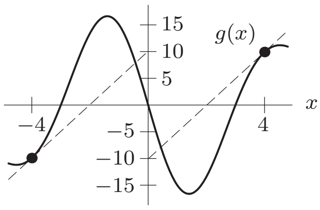


Figure 2.14

24. (a)

$$f'(0) = \lim_{h \to 0} \frac{\overbrace{\sin h}^{h \text{ in degrees}} - \overbrace{\sin 0}^{0}}{h} = \frac{\sin h}{h}.$$

To four decimal places,

$$\frac{\sin 0.2}{0.2} \approx \frac{\sin 0.1}{0.1} \approx \frac{\sin 0.01}{0.01} \approx \frac{\sin 0.001}{0.001} \approx 0.01745$$

so $f'(0) \approx 0.01745$.

(b) Consider the ratio $\frac{\sin h}{h}$. As we approach 0, the numerator, $\sin h$, will be much smaller in magnitude if h is in degrees than it would be if h were in radians. For example, if $h = 1^\circ$ radian, $\sin h = 0.8415$, but if $h = 1$ degree, $\sin h = 0.01745$. Thus, since the numerator is smaller for h measured in degrees while the denominator is the same, we expect the ratio $\frac{\sin h}{h}$ to be smaller.

25. We find the derivative using a difference quotient:

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2+3+h-(3^2+3)}{h}$$
$$= \lim_{h\to 0} \frac{9+6h+h^2+3+h-9-3}{h} = \lim_{h\to 0} \frac{7h+h^2}{h} = \lim_{h\to 0}(7+h) = 7.$$

Thus at $x = 3$, the slope of the tangent line is 7. Since $f(3) = 3^2 + 3 = 12$, the line goes through the point $(3, 12)$, and therefore its equation is

$$y - 12 = 7(x-3) \quad \text{or} \quad y = 7x - 9.$$

The graph is in Figure 2.15.

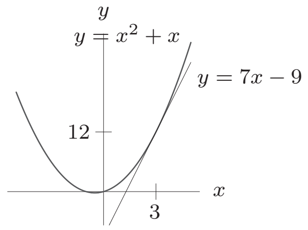


Figure 2.15

26. Using a difference quotient with $h = 0.001$, say, we find

$$f'(1) \approx \frac{1.001\ln(1.001) - 1\ln(1)}{1.001-1} = 1.0005$$
$$f'(2) \approx \frac{2.001\ln(2.001) - 2\ln(2)}{2.001-2} = 1.6934$$

The fact that f' is larger at $x = 2$ than at $x = 1$ suggests that f is concave up between $x = 1$ and $x = 2$.

27. We want $f'(2)$. The exact answer is

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^{2+h}-4}{h},$$

but we can approximate this. If $h = 0.001$, then

$$\frac{(2.001)^{2.001}-4}{0.001} \approx 6.779$$

and if $h = 0.0001$ then

$$\frac{(2.0001)^{2.0001}-4}{0.0001} \approx 6.773,$$

so $f'(2) \approx 6.77$.

28. Notice that we can't get all the information we want just from the graph of f for $0 \le x \le 2$, shown on the left in Figure 2.16. Looking at this graph, it looks as if the slope at $x = 0$ is 0. But if we zoom in on the graph near $x = 0$, we get the graph of f for $0 \le x \le 0.05$, shown on the right in Figure 2.16. We see that f does dip down quite a bit between $x = 0$ and $x \approx 0.11$. In fact, it now looks like $f'(0)$ is around -1. Note that since $f(x)$ is undefined for $x < 0$, this derivative only makes sense as we approach zero from the right.

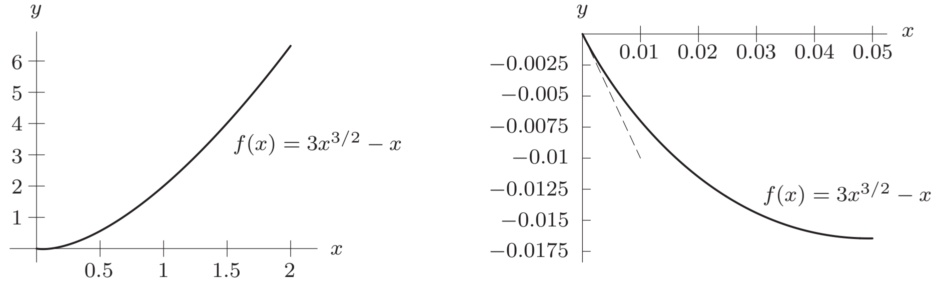


Figure 2.16

We zoom in on the graph of f near $x = 1$ to get a more accurate picture from which to estimate $f'(1)$. A graph of f for $0.7 \le x \le 1.3$ is shown in Figure 2.17. [Keep in mind that the axes shown in this graph don't cross at the origin!] Here we see that $f'(1) \approx 3.5$.

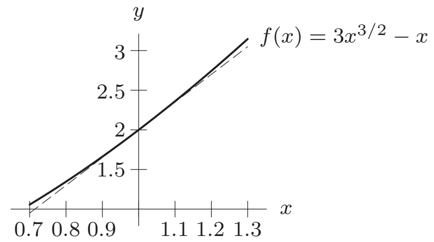


Figure 2.17

29.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\ln(\cos(1+h)) - \ln(\cos 1)}{h}$$

For $h = 0.001$, the difference quotient $= -1.55912$; for $h = 0.0001$, the difference quotient $= -1.55758$. The instantaneous rate of change of f therefore appears to be about -1.558 at $x = 1$.
At $x = \frac{\pi}{4}$, if we try $h = 0.0001$, then

$$\text{difference quotient} = \frac{\ln[\cos(\frac{\pi}{4} + 0.0001)] - \ln(\cos\frac{\pi}{4})}{0.0001} \approx -1.0001.$$

The instantaneous rate of change of f appears to be about -1 at $x = \frac{\pi}{4}$.

30. The quantity $f(0)$ represents the population on October 17, 2006, so $f(0) = 300$ million.
The quantity $f'(0)$ represents the rate of change of the population (in millions per year). Since

$$\frac{1 \text{ person}}{11 \text{ seconds}} = \frac{1/10^6 \text{ million people}}{11/(60\cdot 60\cdot 24\cdot 365)\text{ years}} = 2.867 \text{ million people/year},$$

so we have $f'(0) = 2.867$.

31. We want to approximate $P'(0)$ and $P'(7)$. Since for small h

$$P'(0) \approx \frac{P(h) - P(0)}{h},$$

if we take $h = 0.01$, we get

$$\begin{aligned} P'(0) &\approx \frac{1.267(1.007)^{0.01} - 1.267}{0.01} = 0.00884 \text{ billion/year} \\ &= 8.84 \text{ million people/year in 2000}, \\ P'(7) &\approx \frac{1.267(1.007)^{7.01} - 1.267(1.007)^{7}}{0.01} = 0.00928 \text{ billion/year} \\ &= 9.28 \text{ million people/year in 2007} \end{aligned}$$

32. (a) From Figure 2.18, it appears that the slopes of the tangent lines to the two graphs are the same at each x. For $x = 0$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ at 0 are

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h)-0}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \frac{1}{2}h \\ &= 0, \end{aligned}$$

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} \\ &= \lim_{h\to 0} \frac{g(h)-g(0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}h^2+3-3}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \frac{1}{2}h \\ &= 0. \end{aligned}$$

For $x = 2$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ are

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2-\frac{1}{2}(2)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2)-2}{h} \\ &= \lim_{h\to 0} \frac{2+2h+\frac{1}{2}h^2-2}{h} \\ &= \lim_{h\to 0} \frac{2h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(2+\frac{1}{2}h\right) \\ &= 2, \end{aligned}$$

$$\begin{aligned} g'(2) &= \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2+3-(\frac{1}{2}(2)^2+3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2-\frac{1}{2}(2)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2)-2}{h} \\ &= \lim_{h\to 0} \frac{2+2h+\frac{1}{2}(h^2)-2}{h} \\ &= \lim_{h\to 0} \frac{2h+\frac{1}{2}(h^2)}{h} \\ &= \lim_{h\to 0} \left(2+\frac{1}{2}h\right) \\ &= 2. \end{aligned}$$

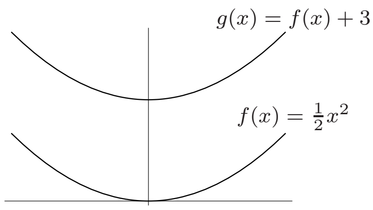


Figure 2.18

For $x = x_0$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ are

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2+2x_0h+h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0+\frac{1}{2}h\right) \\ &= x_0, \end{aligned}$$

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2+3-(\frac{1}{2}(x_0)^2+3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 - \frac{1}{2}(x_0)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2+2x_0h+h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h+\frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0+\frac{1}{2}h\right) \\ &= x_0. \end{aligned}$$

(b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)+C-(f(x)+C)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= f'(x). \end{aligned}$$

33. As h gets smaller, round-off error becomes important. When $h = 10^{-12}$, the quantity $2^h - 1$ is so close to 0 that the calculator rounds off the difference to 0, making the difference quotient 0. The same thing will happen when $h = 10^{-20}$.

34. (a) Table 2.4 shows that near $x = 1$, every time the value of x increases by 0.001, the value of x^2 increases by approximately 0.002. This suggests that

$$f'(1) \approx \frac{0.002}{0.001} = 2.$$

Table 2.4 *Values of $f(x) = x^2$ near $x = 1$*

x	x^2	Difference in successive x^2 values
0.998	0.996004	
		0.001997
0.999	0.998001	
		0.001999
1.000	1.000000	
		0.002001
1.001	1.002001	
		0.002003
1.002	1.004004	
↑ x increments of 0.001		↑ All approximately 0.002

(b) The derivative is the limit of the difference quotient, so we look at

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}.$$

Using the formula for f, we have

$$f'(1) = \lim_{h\to 0} \frac{(1+h)^2-1^2}{h} = \lim_{h\to 0} \frac{(1+2h+h^2)-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h}.$$

Since the limit only examines values of h close to, but not equal to zero, we can cancel h in the expression $(2h+h^2)/h$. We get

$$f'(1) = \lim_{h\to 0}\frac{h(2+h)}{h} = \lim_{h\to 0}(2+h).$$

This limit is 2, so $f'(1)=2$. At $x=1$ the rate of change of x^2 is 2.

(c) Since the derivative is the rate of change, $f'(1)=2$ means that for small changes in x near $x=1$, the change in $f(x)=x^2$ is about twice as big as the change in x. As an example, if x changes from 1 to 1.1, a net change of 0.1, then $f(x)$ changes by about 0.2. Figure 2.19 shows this geometrically. Near $x=1$ the function is approximately linear with slope of 2.

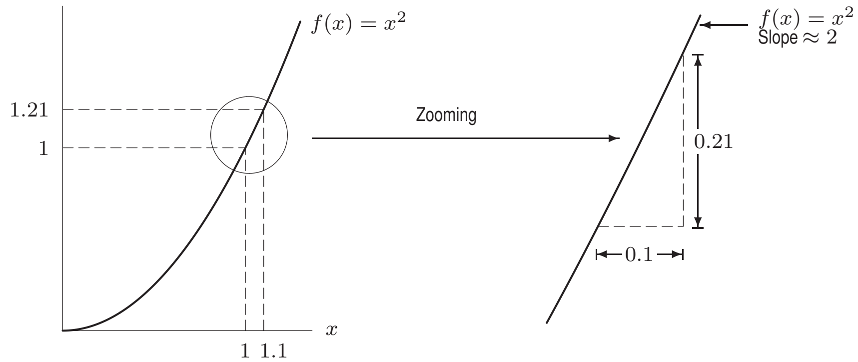


Figure 2.19: Graph of $f(x)=x^2$ near $x=1$ has slope ≈ 2

35. $\displaystyle\lim_{h\to 0}\frac{(-3+h)^2-9}{h} = \lim_{h\to 0}\frac{9-6h+h^2-9}{h} = \lim_{h\to 0}\frac{h(-6+h)}{h} = \lim_{h\to 0}-6+h = -6.$

36. $\displaystyle\lim_{h\to 0}\frac{(2-h)^3-8}{h} = \lim_{h\to 0}\frac{8-12h+6h^2-h^3-8}{h} = \lim_{h\to 0}\frac{h(-12+6h-h^2)}{h} = \lim_{h\to 0}-12+6h-h^2 = -12.$

37. $\displaystyle\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{1+h}-1\right) = \lim_{h\to 0}\frac{1-(1+h)}{(1+h)h} = \lim_{h\to 0}\frac{-1}{1+h} = -1$

38. $\displaystyle\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(1+h)^2}-1\right) = \lim_{h\to 0}\frac{1-(1+2h+h^2)}{h(1+h)^2} = \lim_{h\to 0}\frac{-2-h}{(1+h)^2} = -2$

39. $\displaystyle\sqrt{4+h}-2 = \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{\sqrt{4+h}+2} = \frac{4+h-4}{\sqrt{4+h}+2} = \frac{h}{\sqrt{4+h}+2}.$

Therefore $\displaystyle\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$

40. $\displaystyle\frac{1}{\sqrt{4+h}}-\frac{1}{2} = \frac{2-\sqrt{4+h}}{2\sqrt{4+h}} = \frac{(2-\sqrt{4+h})(2+\sqrt{4+h})}{2\sqrt{4+h}(2+\sqrt{4+h})} = \frac{4-(4+h)}{2\sqrt{4+h}(2+\sqrt{4+h})}.$

Therefore $\displaystyle\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{4+h}}-\frac{1}{2}\right) = \lim_{h\to 0}\frac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})} = -\frac{1}{16}$

41. Using the definition of the derivative, we have

$$\begin{aligned} f'(10) &= \lim_{h\to 0}\frac{f(10+h)-f(10)}{h} \\ &= \lim_{h\to 0}\frac{5(10+h)^2-5(10)^2}{h} \\ &= \lim_{h\to 0}\frac{500+100h+5h^2-500}{h} \\ &= \lim_{h\to 0}\frac{100h+5h^2}{h} \\ &= \lim_{h\to 0}\frac{h(100+5h)}{h} \\ &= \lim_{h\to 0}100+5h \\ &= 100. \end{aligned}$$

42. Using the definition of the derivative, we have

$$\begin{aligned} f'(-2) &= \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} \\ &= \lim_{h\to 0}\frac{(-2+h)^3-(-2)^3}{h} \\ &= \lim_{h\to 0}\frac{(-8+12h-6h^2+h^3)-(-8)}{h} \\ &= \lim_{h\to 0}\frac{12h-6h^2+h^3}{h} \\ &= \lim_{h\to 0}\frac{h(12-6h+h^2)}{h} \\ &= \lim_{h\to 0}(12-6h+h^2), \end{aligned}$$

which goes to 12 as $h \to 0$. So $f'(-2) = 12$.

43. Using the definition of the derivative

$$\begin{aligned} g'(-1) &= \lim_{h\to 0}\frac{g(-1+h)-g(-1)}{h} \\ &= \lim_{h\to 0}\frac{((-1+h)^2+(-1+h))-((-1)^2+(-1))}{h} \\ &= \lim_{h\to 0}\frac{(1-2h+h^2-1+h)-(0)}{h} \\ &= \lim_{h\to 0}\frac{-h+h^2}{h} = \lim_{h\to 0}(-1+h) = -1. \end{aligned}$$

44.

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{((1+h)^3+5)-(1^3+5)}{h} \\ &= \lim_{h\to 0}\frac{1+3h+3h^2+h^3+5-1-5}{h} = \lim_{h\to 0}\frac{3h+3h^2+h^3}{h} \\ &= \lim_{h\to 0}(3+3h+h^2) = 3. \end{aligned}$$

45.

$$\begin{aligned} g'(2) &= \lim_{h\to 0}\frac{g(2+h)-g(2)}{h} = \lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h} \\ &= \lim_{h\to 0}\frac{2-(2+h)}{h(2+h)2} = \lim_{h\to 0}\frac{-h}{h(2+h)2} \\ &= \lim_{h\to 0}\frac{-1}{(2+h)2} = -\frac{1}{4} \end{aligned}$$

46.

$$\begin{aligned} g'(2) &= \lim_{h\to 0}\frac{g(2+h)-g(2)}{h} = \lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{2^2}}{h} \\ &= \lim_{h\to 0}\frac{2^2-(2+h)^2}{2^2(2+h)^2h} = \lim_{h\to 0}\frac{4-4-4h-h^2}{4h(2+h)^2} \\ &= \lim_{h\to 0}\frac{-4h-h^2}{4h(2+h)^2} = \lim_{h\to 0}\frac{-4-h}{4(2+h)^2} \\ &= \frac{-4}{4(2)^2} = -\frac{1}{4}. \end{aligned}$$

47. As we saw in the answer to Problem 41, the slope of the tangent line to $f(x) = 5x^2$ at $x = 10$ is 100. When $x = 10$, $f(x) = 500$ so $(10, 500)$ is a point on the tangent line. Thus $y = 100(x - 10) + 500 = 100x - 500$.

48. As we saw in the answer to Problem 42, the slope of the tangent line to $f(x) = x^3$ at $x = -2$ is 12. When $x = -2$, $f(x) = -8$ so we know the point $(-2, -8)$ is on the tangent line. Thus the equation of the tangent line is $y = 12(x + 2) - 8 = 12x + 16$.

49. We know that the slope of the tangent line to $f(x) = x$ when $x = 20$ is 1. When $x = 20$, $f(x) = 20$ so $(20, 20)$ is on the tangent line. Thus the equation of the tangent line is $y = 1(x - 20) + 20 = x$.

50. First find the derivative of $f(x) = 1/x^2$ at $x = 1$.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1^2}}{h} \\ &= \lim_{h\to 0} \frac{1^2 - (1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{1 - (1 + 2h + h^2)}{h(1+h)^2} \\ &= \lim_{h\to 0} \frac{-2h - h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-2-h}{(1+h)^2} = -2 \end{aligned}$$

Thus the tangent line has a slope of -2 and goes through the point $(1, 1)$, and so its equation is

$$y - 1 = -2(x - 1) \quad \text{or} \quad y = -2x + 3.$$

Strengthen Your Understanding

51. The graph of $f(x) = \log x$ is increasing, so $f'(0.5) > 0$.

52. The derivative of a function at a point is the slope of the tangent line, not the tangent line itself.

53. $f(x) = e^x$.
Many other answers are possible.

54. A linear function is of the form $f(x) = ax + b$. The derivative of this function is the slope of the line $y = ax + b$, so $f'(x) = a$, so $a = 2$. One such function is $f(x) = 2x + 1$.

55. True. The derivative of a function is the limit of difference quotients. A few difference quotients can be computed from the table, but the limit can not be computed from the table.

56. True. The derivative $f'(10)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point where $x = 10$. When you zoom in on $y = f(x)$ close enough it is not possible to see the difference between the tangent line and the graph of f on the calculator screen. The line you see on the calculator is a little piece of the tangent line, so its slope is the derivative $f'(10)$.

57. True. This is seen graphically. The derivative $f'(a)$ is the slope of the line tangent to the graph of f at the point P where $x = a$. The difference quotient $(f(b) - f(a))/(b - a)$ is the slope of the secant line with endpoints on the graph of f at the points where $x = a$ and $x = b$. The tangent and secant lines cross at the point P. The secant line goes above the tangent line for $x > a$ because f is concave up, and so the secant line has higher slope.

58. (a). This is best observed graphically.

Solutions for Section 2.3

Exercises

1. (a) We use the interval to the right of $x = 2$ to estimate the derivative. (Alternately, we could use the interval to the left of 2, or we could use both and average the results.) We have

$$f'(2) \approx \frac{f(4) - f(2)}{4 - 2} = \frac{24 - 18}{4 - 2} = \frac{6}{2} = 3.$$

We estimate $f'(2) \approx 3$.

(b) We know that $f'(x)$ is positive when $f(x)$ is increasing and negative when $f(x)$ is decreasing, so it appears that $f'(x)$ is positive for $0 < x < 4$ and is negative for $4 < x < 12$.

2. For $x = 0$, 5, 10, and 15, we use the interval to the right to estimate the derivative. For $x = 20$, we use the interval to the left. For $x = 0$, we have

$$f'(0) \approx \frac{f(5) - f(0)}{5 - 0} = \frac{70 - 100}{5 - 0} = \frac{-30}{5} = -6.$$

Similarly, we find the other estimates in Table 2.5.

Table 2.5

x	0	5	10	15	20
$f'(x)$	-6	-3	-1.8	-1.2	-1.2

3. The graph is that of the line $y = -2x + 2$. The slope, and hence the derivative, is -2. See Figure 2.20.

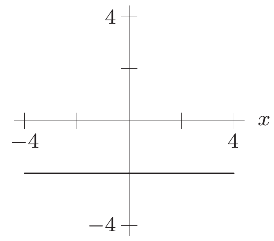


Figure 2.20

4. See Figure 2.21.

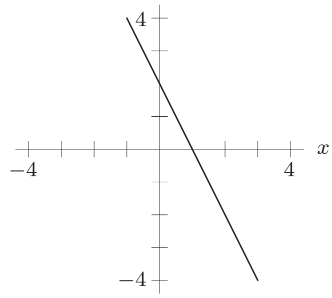


Figure 2.21

5. See Figure 2.22.

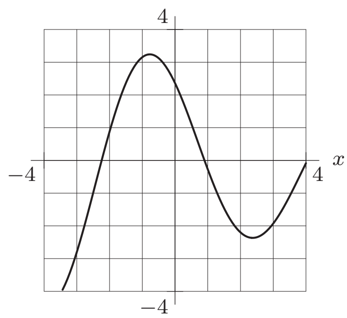


Figure 2.22

6. See Figure 2.23.

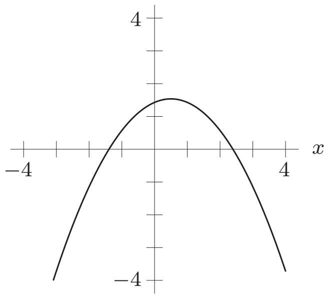


Figure 2.23

7. The slope of this curve is approximately -1 at $x = -4$ and at $x = 4$, approximately 0 at $x = -2.5$ and $x = 1.5$, and approximately 1 at $x = 0$. See Figure 2.24.

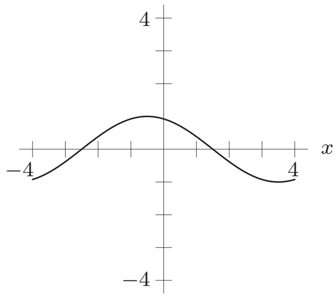


Figure 2.24

8. See Figure 2.25.

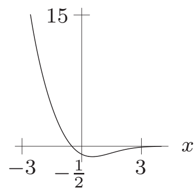


Figure 2.25

9. See Figure 2.26.

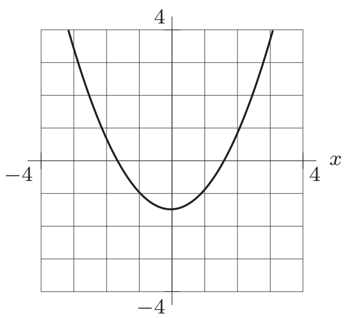


Figure 2.26

10. See Figure 2.27.

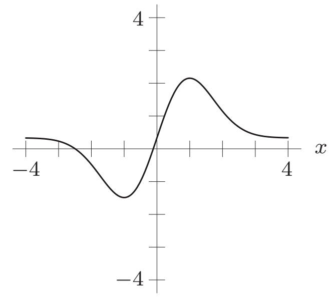


Figure 2.27

11. See Figure 2.28.

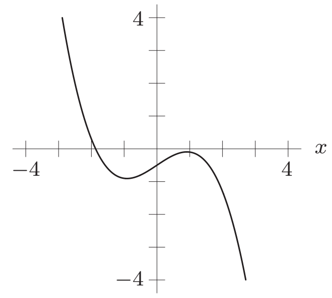


Figure 2.28

12. See Figure 2.29.

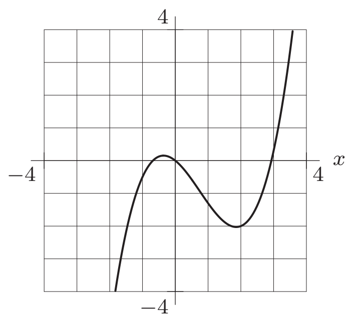


Figure 2.29

13. See Figures 2.30 and 2.31.

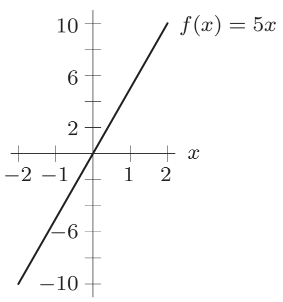


Figure 2.30

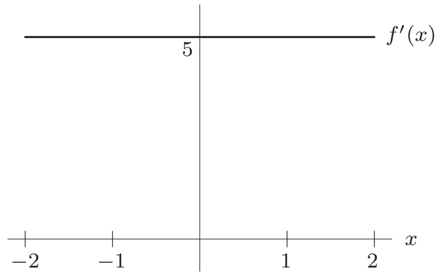


Figure 2.31

14. See Figures 2.32 and 2.33.

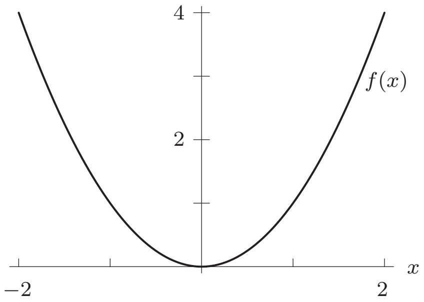


Figure 2.32

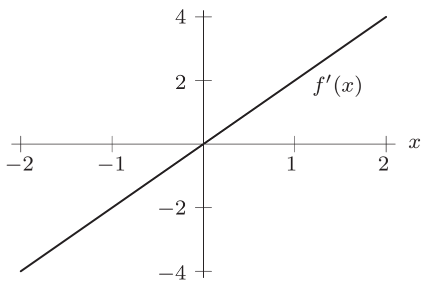


Figure 2.33

15. See Figures 2.34 and 2.35.

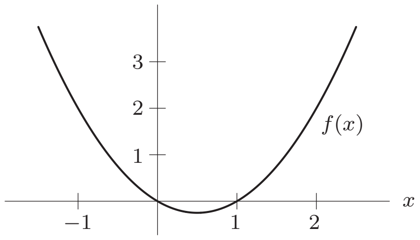


Figure 2.34

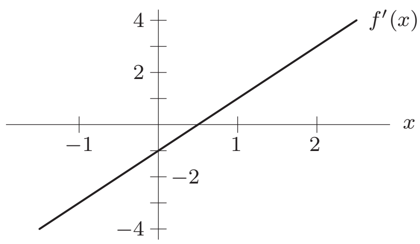


Figure 2.35

16. The graph of $f(x)$ and its derivative look the same, as in Figures 2.36 and 2.37.

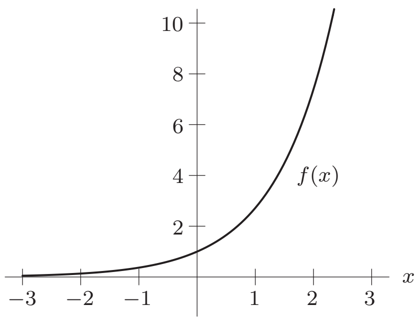


Figure 2.36

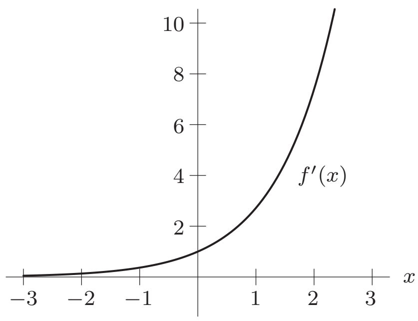


Figure 2.37

17. See Figures 2.38 and 2.39.

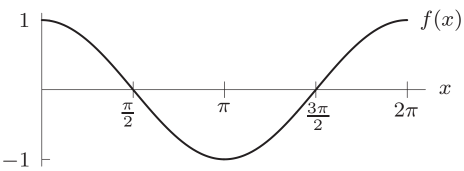


Figure 2.38

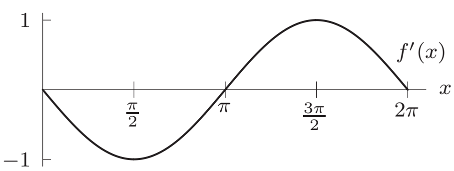


Figure 2.39

18. See Figures 2.40 and 2.41.

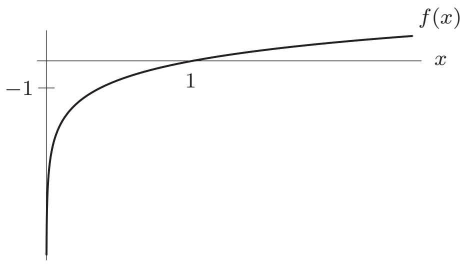


Figure 2.40

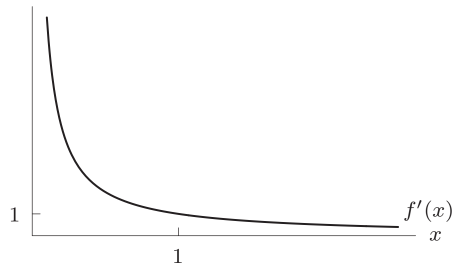


Figure 2.41

19. Since $1/x = x^{-1}$, using the power rule gives

$$k'(x) = (-1)x^{-2} = -\frac{1}{x^2}.$$

Using the definition of the derivative, we have

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x} \\ &= \lim_{h\to 0} \frac{-h}{h(x+h)x} = \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}. \end{aligned}$$

20. Since $1/x^2 = x^{-2}$, using the power rule gives

$$l'(x) = -2x^{-3} = -\frac{2}{x^3}.$$

Using the definition of the derivative, we have

$$\begin{aligned} l'(x) &= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2xh-h^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^2x^2} = -\frac{2}{x^3}. \end{aligned}$$

21. Using the definition of the derivative,

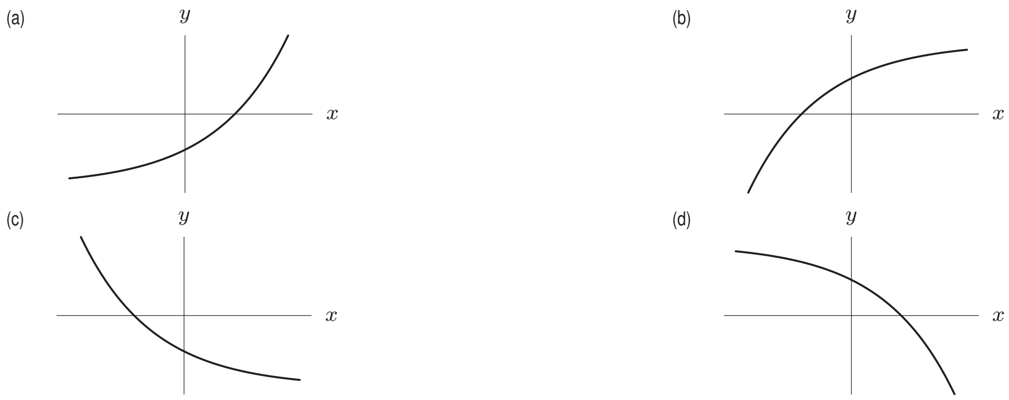
$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2-3-(2x^2-3)}{h} \\ &= \lim_{h\to 0} \frac{2(x^2+2xh+h^2)-3-2x^2+3}{h} = \lim_{h\to 0} \frac{4xh+2h^2}{h} \\ &= \lim_{h\to 0} (4x+2h) = 4x. \end{aligned}$$

22. Using the definition of the derivative, we have

$$\begin{aligned} m'(x) &= \lim_{h\to 0} \frac{m(x+h)-m(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+1}-\frac{1}{x+1}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+1-x-h-1}{(x+1)(x+h+1)}\right) = \lim_{h\to 0} \frac{-h}{h(x+1)(x+h+1)} \\ &= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)^2}. \end{aligned}$$

Problems

23.



24. Since $f'(x) > 0$ for $x < -1$, $f(x)$ is increasing on this interval.
Since $f'(x) < 0$ for $x > -1$, $f(x)$ is decreasing on this interval.
Since $f'(x) = 0$ at $x = -1$, the tangent to $f(x)$ is horizontal at $x = -1$.
One possible shape for $y = f(x)$ is shown in Figure 2.42.

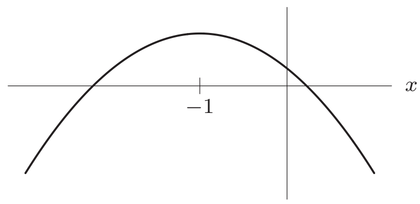


Figure 2.42

25.

x	$\ln x$
0.998	−0.0020
0.999	−0.0010
1.000	0.0000
1.001	0.0010
1.002	0.0020

x	$\ln x$
1.998	0.6921
1.999	0.6926
2.000	0.6931
2.001	0.6936
2.002	0.6941

x	$\ln x$
4.998	1.6090
4.999	1.6092
5.000	1.6094
5.001	1.6096
5.002	1.6098

x	$\ln x$
9.998	2.3024
9.999	2.3025
10.000	2.3026
10.001	2.3027
10.002	2.3028

At $x = 1$, the values of $\ln x$ are increasing by 0.001 for each increase in x of 0.001, so the derivative appears to be 1. At $x = 2$, the increase is 0.0005 for each increase of 0.001, so the derivative appears to be 0.5. At $x = 5$, $\ln x$ increases by 0.0002 for each increase of 0.001 in x, so the derivative appears to be 0.2. And at $x = 10$, the increase is 0.0001 over intervals of 0.001, so the derivative appears to be 0.1. These values suggest an inverse relationship between x and $f'(x)$, namely $f'(x) = \frac{1}{x}$.

26. We know that $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$. For this problem, we'll take the average of the values obtained for $h = 1$ and $h = -1$; that's the average of $f(x+1) - f(x)$ and $f(x) - f(x-1)$ which equals $\dfrac{f(x+1) - f(x-1)}{2}$. Thus,
$f'(0) \approx f(1) - f(0) = 13 - 18 = -5$.
$f'(1) \approx (f(2) - f(0))/2 = (10 - 18)/2 = -4$.
$f'(2) \approx (f(3) - f(1))/2 = (9 - 13)/2 = -2$.
$f'(3) \approx (f(4) - f(2))/2 = (9 - 10)/2 = -0.5$.
$f'(4) \approx (f(5) - f(3))/2 = (11 - 9)/2 = 1$.
$f'(5) \approx (f(6) - f(4))/2 = (15 - 9)/2 = 3$.
$f'(6) \approx (f(7) - f(5))/2 = (21 - 11)/2 = 5$.

$f'(7) \approx (f(8) - f(6))/2 = (30 - 15)/2 = 7.5.$
$f'(8) \approx f(8) - f(7) = 30 - 21 = 9.$
The rate of change of $f(x)$ is positive for $4 \le x \le 8$, negative for $0 \le x \le 3$. The rate of change is greatest at about $x = 8$.

27. The value of $g(x)$ is increasing at a decreasing rate for $2.7 < x < 4.2$ and increasing at an increasing rate for $x > 4.2$.

$$\frac{\Delta y}{\Delta x} = \frac{7.4 - 6.0}{5.2 - 4.7} = 2.8 \qquad \text{between } x = 4.7 \text{ and } x = 5.2$$
$$\frac{\Delta y}{\Delta x} = \frac{9.0 - 7.4}{5.7 - 5.2} = 3.2 \qquad \text{between } x = 5.2 \text{ and } x = 5.7$$

Thus $g'(x)$ should be close to 3 near $x = 5.2$.

28. (a) x_3 (b) x_4 (c) x_5 (d) x_3

29. This is a line with slope 1, so the derivative is the constant function $f'(x) = 1$. The graph is the horizontal line $y = 1$. See Figure 2.43.

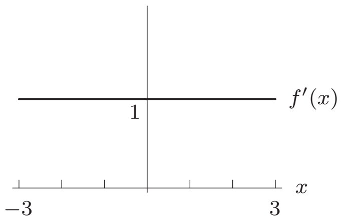


Figure 2.43

30. This is a line with slope -2, so the derivative is the constant function $f'(x) = -2$. The graph is a horizontal line at $y = -2$. See Figure 2.44.

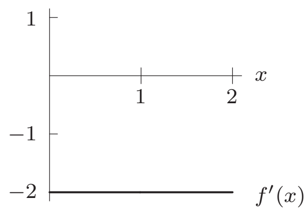


Figure 2.44

31. See Figure 2.45.

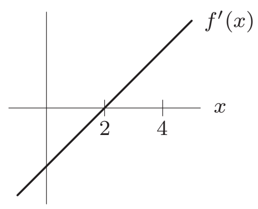


Figure 2.45

32. See Figure 2.46.

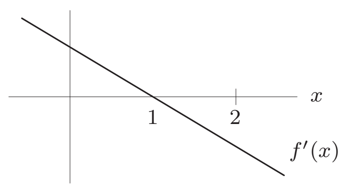


Figure 2.46

33. See Figure 2.47.

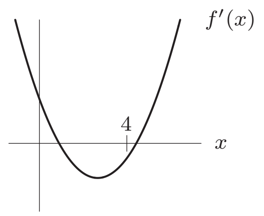


Figure 2.47

34. See Figure 2.48.

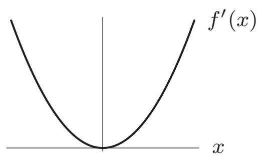


Figure 2.48

35. See Figure 2.49.

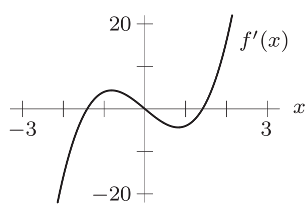


Figure 2.49

36. One possible graph is shown in Figure 2.50. Notice that as x gets large, the graph of $f(x)$ gets more and more horizontal. Thus, as x gets large, $f'(x)$ gets closer and closer to 0.

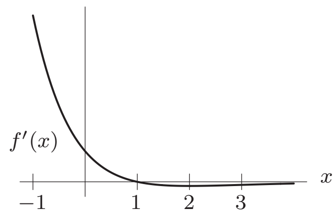


Figure 2.50

37. See Figure 2.51.

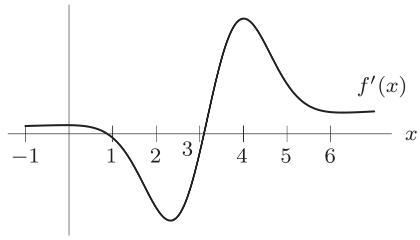


Figure 2.51

38. See Figure 2.52.

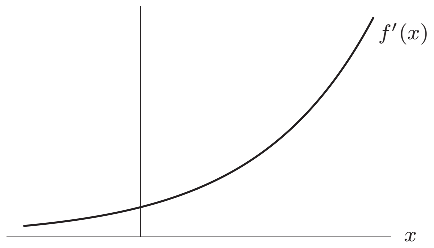


Figure 2.52

39. See Figure 2.53.

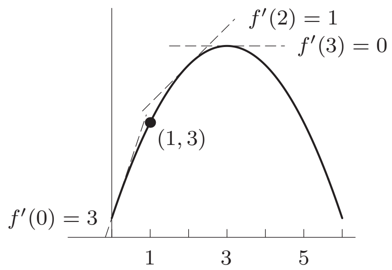
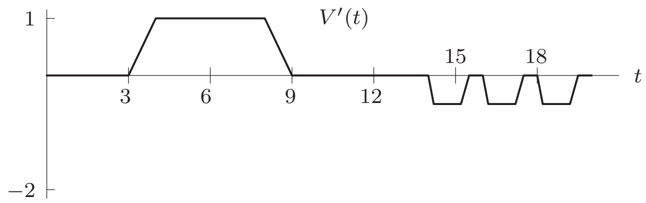


Figure 2.53

40. (a) Graph II
(b) Graph I
(c) Graph III

41. (a) $t = 3$
(b) $t = 9$
(c) $t = 14$
(d)



42. The derivative is zero whenever the graph of the original function is horizontal. Since the current is proportional to the derivative of the voltage, segments where the current is zero alternate with positive segments where the voltage is increasing and negative segments where the voltage is decreasing. See Figure 2.54. Note that the derivative does not exist where the graph has a corner.

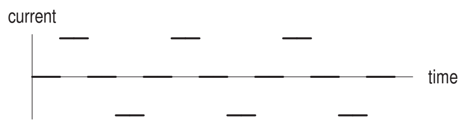


Figure 2.54

43. (a) The function f is increasing where f' is positive, so for $x_1 < x < x_3$.
(b) The function f is decreasing where f' is negative, so for $0 < x < x_1$ or $x_3 < x < x_5$.

44. On intervals where $f' = 0$, f is not changing at all, and is therefore constant. On the small interval where $f' > 0$, f is increasing; at the point where f' hits the top of its spike, f is increasing quite sharply. So f should be constant for a while, have a sudden increase, and then be constant again. A possible graph for f is shown in Figure 2.55.

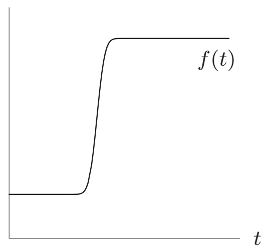
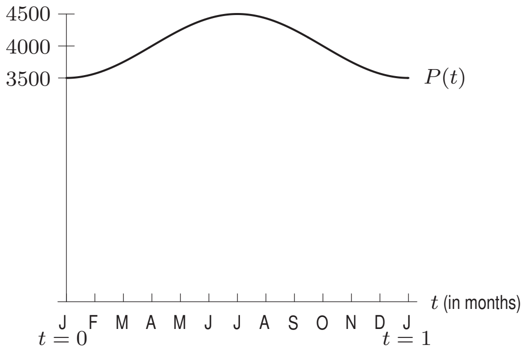


Figure 2.55: Step function

45. (a) The population varies periodically with a period of 1 year. See below.



(b) The population is at a maximum on July 1^{st}. At this time $\sin(2\pi t - \frac{\pi}{2}) = 1$, so the actual maximum population is $4000 + 500(1) = 4500$. Similarly, the population is at a minimum on January 1^{st}. At this time, $\sin(2\pi t - \frac{\pi}{2}) = -1$, so the minimum population is $4000 + 500(-1) = 3500$.
(c) The rate of change is most positive about April 1^{st} and most negative around October 1^{st}.
(d) Since the population is at its maximum around July 1^{st}, its rate of change is about 0 then.

46. The derivative of the accumulated federal debt with respect to time is shown in Figure 2.56. The derivative represents the rate of change of the federal debt with respect to time and is measured in trillions of dollars per year.

Figure 2.56

47. From the given information we know that f is increasing for values of x less than -2, is decreasing between $x = -2$ and $x = 2$, and is constant for $x > 2$. Figure 2.57 shows a possible graph—yours may be different.

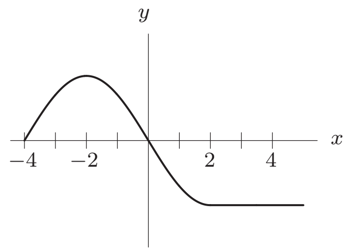


Figure 2.57

48. Since $f'(x) > 0$ for $1 < x < 3$, we see that $f(x)$ is increasing on this interval.
Since $f'(x) < 0$ for $x < 1$ and for $x > 3$, we see that $f(x)$ is decreasing on these intervals.
Since $f'(x) = 0$ for $x = 1$ and $x = 3$, the tangent to $f(x)$ will be horizontal at these x's.
One of many possible shapes of $y = f(x)$ is shown in Figure 2.58.

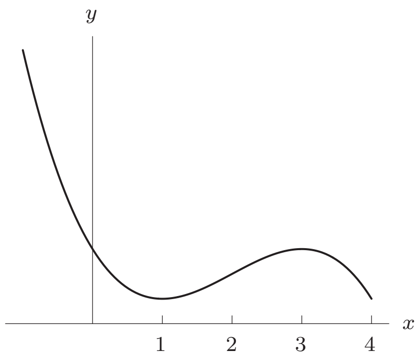


Figure 2.58

49. If $\lim_{x\to\infty} f(x) = 50$ and $f'(x)$ is positive for all x, then $f(x)$ increases to 50, but never rises above it. A possible graph of $f(x)$ is shown in Figure 2.59. If $\lim_{x\to\infty} f'(x)$ exists, it must be zero, since f looks more and more like a horizontal line. If $f'(x)$ approached another positive value c, then f would look more and more like a line with positive slope c, which would eventually go above $y = 50$.

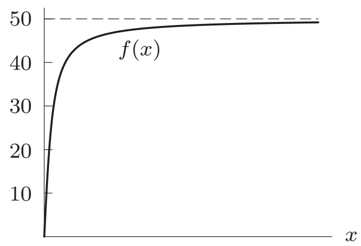
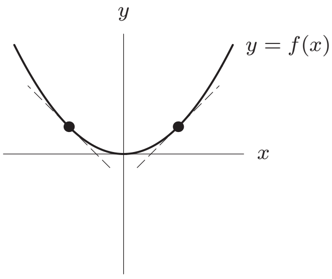
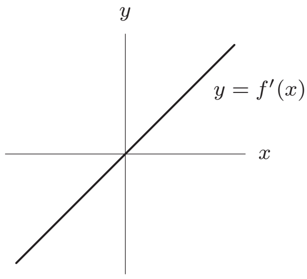


Figure 2.59

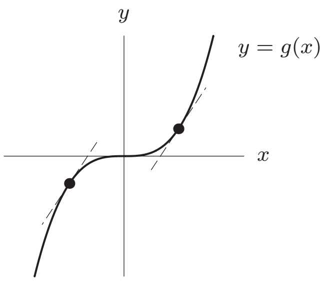
50. If $f(x)$ is even, its graph is symmetric about the y-axis. So the tangent line to f at $x = x_0$ is the same as that at $x = -x_0$ reflected about the y-axis.

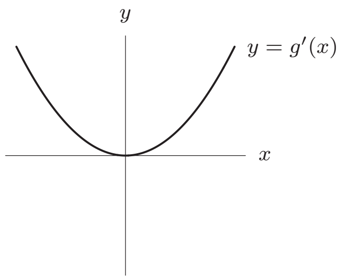




So the slopes of these two tangent lines are opposite in sign, so $f'(x_0) = -f'(-x_0)$, and f' is odd.

51. If $g(x)$ is odd, its graph remains the same if you rotate it 180° about the origin. So the tangent line to g at $x = x_0$ is the tangent line to g at $x = -x_0$, rotated 180°.





But the slope of a line stays constant if you rotate it 180°. So $g'(x_0) = g'(-x_0)$; g' is even.

Strengthen Your Understanding

52. Since $f(x) = \cos x$ is decreasing on some intervals, its derivative $f'(x)$ is negative on those intervals, and the graph of $f'(x)$ is below the x-axis where $\cos x$ is decreasing.

53. In order for $f'(x)$ to be greater than zero, the slope of $f(x)$ has to be greater than zero. For example, $f(x) = e^{-x}$ is positive for all x but since the graph is decreasing everywhere, $f(x)$ has negative derivative for all x.

54. Two different functions can have the same rate of change. For example, $f(x) = 1, g(x) = 2$ both are constant, so $f'(x) = g'(x) = 0$ but $f(x) \neq g(x)$.

55. $f(t) = t(1-t)$. We have $f(t) = t - t^2$, so $f'(t) = 1 - 2t$ so the velocity is positive for $0 < t < 0.5$ and negative for $0.5 < t < 1$.

Many other answers are possible.

56. Every linear function is of the form $f(x) = b + mx$ and has derivative $f'(x) = m$. One family of functions with the same derivative is $f(x) = b + 2x$.

57. True. The graph of a linear function $f(x) = mx + b$ is a straight line with the same slope m at every point. Thus $f'(x) = m$ for all x.

58. True. Shifting a graph vertically does not change the shape of the graph and so it does not change the slopes of the tangent lines to the graph.

59. False. If $f'(x)$ is increasing then $f(x)$ is concave up. However, $f(x)$ may be either increasing or decreasing. For example, the exponential decay function $f(x) = e^{-x}$ is decreasing but $f'(x)$ is increasing because the graph of f is concave up.

60. False. A counterexample is given by $f(x) = 5$ and $g(x) = 10$, two different functions with the same derivatives: $f'(x) = g'(x) = 0$.

Solutions for Section 2.4

Exercises

1. (a) The statement $f(200) = 1300$ means that it costs \$1300 to produce 200 gallons of the chemical.

(b) The statement $f'(200) = 6$ means that when the number of gallons produced is 200, costs are increasing at a rate of \$6 per gallon. In other words, it costs about \$6 to produce the next (the 201^{st}) gallon of the chemical.

2. (a) The statement $f(5) = 18$ means that when 5 milliliters of catalyst are present, the reaction will take 18 minutes. Thus, the units for 5 are ml while the units for 18 are minutes.

(b) As in part (a), 5 is measured in ml. Since f' tells how fast T changes per unit a, we have f' measured in minutes/ml. If the amount of catalyst increases by 1 ml (from 5 to 6 ml), the reaction time decreases by about 3 minutes.

3. (Note that we are considering the average temperature of the yam, since its temperature is different at different points inside it.)

(a) It is positive, because the temperature of the yam increases the longer it sits in the oven.

(b) The units of $f'(20)$ are °F/min. The statement $f'(20) = 2$ means that at time $t = 20$ minutes, the temperature T would increase by approximately 2°F if the yam is in the oven an additional minute.

4. (a) As the cup of coffee cools, the temperature decreases, so $f'(t)$ is negative.

(b) Since $f'(t) = dH/dt$, the units are degrees Celsius per minute. The quantity $f'(20)$ represents the rate at which the coffee is cooling, in degrees per minute, 20 minutes after the cup is put on the counter.

5. (a) The function f takes quarts of ice cream to cost in dollars, so 200 is the amount of ice cream, in quarts, and \$600 is the corresponding cost, in dollars. It costs \$600 to produce 200 quarts of ice cream.

(b) Here, 200 is in quarts, but the 2 is in dollars/quart. After producing 200 quarts of ice cream, the cost to produce one additional quart is about \$2.

6. (a) If the price is \$150, then 2000 items will be sold.

(b) If the price goes up from \$150 by \$1 per item, about 25 fewer items will be sold. Equivalently, if the price is decreased from \$150 by \$1 per item, about 25 more items will be sold.

7. Units of $C'(r)$ are dollars/percent. Approximately, $C'(r)$ means the additional amount needed to pay off the loan when the interest rate is increased by 1%. The sign of $C'(r)$ is positive, because increasing the interest rate will increase the amount it costs to pay off a loan.

8. The units of $f'(x)$ are feet/mile. The derivative, $f'(x)$, represents the rate of change of elevation with distance from the source, so if the river is flowing downhill everywhere, the elevation is always decreasing and $f'(x)$ is always negative. (In fact, there may be some stretches where the elevation is more or less constant, so $f'(x) = 0$.)

9. Units of $P'(t)$ are dollars/year. The practical meaning of $P'(t)$ is the rate at which the monthly payments change as the duration of the mortgage increases. Approximately, $P'(t)$ represents the change in the monthly payment if the duration is increased by one year. $P'(t)$ is negative because increasing the duration of a mortgage decreases the monthly payments.

10. Since B is measured in dollars and t is measured in years, dB/dt is measured in dollars per year. We can interpret dB as the extra money added to your balance in dt years. Therefore dB/dt represents how fast your balance is growing, in units of dollars/year.

11. (a) This means that investing the \$1000 at 5% would yield \$1649 after 10 years.

(b) Writing $g'(r)$ as dB/dr, we see that the units of dB/dr are dollars per percent (interest). We can interpret dB as the extra money earned if interest rate is increased by dr percent. Therefore $g'(5) = \frac{dB}{dr}|_{r=5} \approx 165$ means that the balance, at 5% interest, would increase by about \$165 if the interest rate were increased by 1%. In other words, $g(6) \approx g(5) + 165 = 1649 + 165 = 1814$.

12. (a) The units of lapse rate are the same as for the derivative dT/dz, namely (units of T)/(units of z) $=$ $^\circ C$/km.

(b) Since the lapse rate is 6.5, the derivative of T with respect to z is $dT/dz = -6.5^\circ C$/km. The air temperature drops about 6.5° for every kilometer you go up.

Problems

13. (a) Since $W = f(c)$ where W is weight in pounds and c is the number of Calories consumed per day:

$f(1800) = 155$	means that	consuming 1800 Calories per day results in a weight of 155 pounds.
$f'(2000) = 0$	means that	consuming 2000 Calories per day causes neither weight gain nor loss.
$f^{-1}(162) = 2200$	means that	a weight of 162 pounds is caused by a consumption of 2200 Calories per day.

(b) The units of dW/dc are pounds/(Calories/day).

14. (a) Let $f(t)$ be the volume, in cubic km, of the Greenland Ice Sheet t years since 2011 (Alternatively, in year t). We are given information about $f'(t)$, which has unit km^3 per year.
(b) If t is in years since 2011, we know $f'(0)$ is between -224 and -82 cubic km/year. (Alternatively, $f'(2011)$ is between -224 and -82.)

15. The graph is increasing for $0 < t < 15$ and is decreasing for $15 < t < 30$. One possible graph is shown in Figure 2.60. The units on the horizontal axis are years and the units on the vertical axis are people.

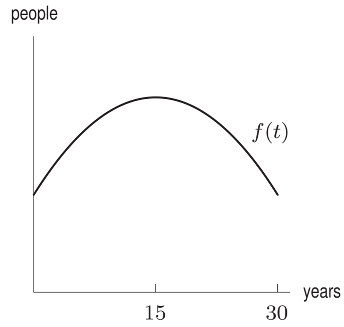


Figure 2.60

The derivative is positive for $0 < t < 15$ and negative for $15 < t < 30$. Two possible graphs are shown in Figure 2.61. The units on the horizontal axes are years and the units on the vertical axes are people per year.

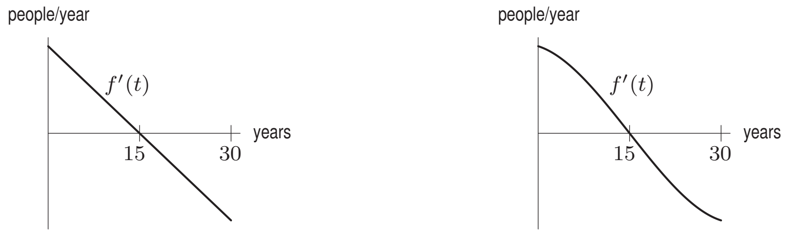


Figure 2.61

16. Since $f(t) = 1.34(1.004)^t$, we have
$$f(9) = 1.34(1.004)^9 = 1.389.$$
To estimate $f'(9)$, we use a small interval around 9:
$$f'(9) \approx \frac{f(9.001) - f(9)}{9.001 - 9} = \frac{1.34(1.004)^{9.001} - 1.34(1.004)^9}{0.001} = 0.0055.$$
We see that $f(9) = 1.389$ billion people and $f'(9) = 0.0055$ billion (that is, 5.5 million) people per year. Since $t = 9$ in 2020, this model predicts that the population of China will be about 1,389,000,000 people in 2009 and growing at a rate of about 5,500,000 people per year at that time.

17. $f(10) = 240{,}000$ means that if the commodity costs \$10, then 240,000 units of it will be sold. $f'(10) = -29{,}000$ means that if the commodity costs \$10 now, each \$1 increase in price will cause a decline in sales of 29,000 units.

18. Let p be the rating points earned by the CBS Evening News, let R be the revenue earned in millions of dollars, and let $R = f(p)$. When $p = 4.3$,
$$\text{Rate of change of revenue } \approx \frac{\$5.5 \text{ million}}{0.1 \text{ point}} = 55 \text{ million dollars/point.}$$
Thus
$$f'(4.3) \approx 55.$$

19. (a) The units of P are millions of people, the units of t are years, so the units of $f'(t)$ are millions of people per year. Therefore the statement $f'(6) = 2$ tells us that at $t = 6$ (that is, in 1986), the population of Mexico was increasing at 2 million people per year.
(b) The statement $f^{-1}(95.5) = 16$ tells us that the year when the population was 95.5 million was $t = 16$ (that is, in 1996).
(c) The units of $(f^{-1})'(P)$ are years per million of population. The statement $(f^{-1})'(95.5) = 0.46$ tells us that when the population was 95.5 million, it took about 0.46 years for the population to increase by 1 million.

20. (a) When $t = 10$, that is, at 10 am, 3.1 cm of rain has fallen.
(b) We are told that when 5 cm of rain has fallen, 16 hours have passed ($t = 16$); that is, 5 cm of rain has fallen by 4 pm.
(c) The rate at which rain is falling is 0.4 cm/hr at $t = 10$, that is, at 10 am.
(d) The units of $(f^{-1})'(5)$ are hours/cm. Thus, we are being told that when 5 cm of rain has fallen, rain is falling at a rate such that it will take 2 additional hours for another centimeter to fall.

21. (a) The depth of the water is 3 feet at time $t = 5$ hours.
(b) The depth of the water is increasing at 0.7 feet/hour at time $t = 5$ hours.
(c) When the depth of the water is 5 feet, the time is $t = 7$ hours.
(d) Since 5 is the depth in feet and $h^{-1}(5)$ is time in hours, the units of $(h^{-1})'$ are hours/feet. Thus, $(h^{-1})'(5) = 1.2$ tells us that when the water depth is 5 feet, the rate of change of time with depth is 1.2 hours per foot. In other words, when the depth is 5 feet, water is entering at a rate such that it takes 1.2 hours to add an extra foot of water.

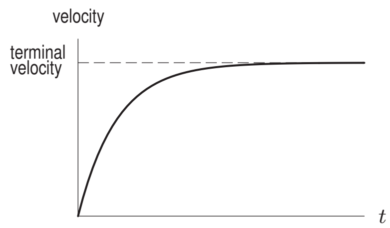
22. (a) The pressure in dynes/cm^2 at a depth of 100 meters.
(b) The depth of water in meters giving a pressure of $1.2 \cdot 10^6$ dynes/cm^2.
(c) The pressure at a depth of h meters plus a pressure of 20 dynes/cm^2.
(d) The pressure at a depth of 20 meters below the diver.
(e) The rate of increase of pressure with respect to depth, at 100 meters, in units of dynes/cm^2 per meter. Approximately, $p'(100)$ represents the increase in pressure in going from 100 meters to 101 meters.
(f) The depth, in meters, at which the rate of change of pressure with respect to depth is 100,000 dynes/cm^2 per meter.

23. The units of $g'(t)$ are inches/year. The quantity $g'(10)$ represents how fast Amelia Earhart was growing at age 10, so we expect $g'(10) > 0$. The quantity $g'(30)$ represents how fast she was growing at age 30, so we expect $g'(30) = 0$ because she was probably not growing taller at that age.

24. Units of $g'(55)$ are mpg/mph. The statement $g'(55) = -0.54$ means that at 55 miles per hour the fuel efficiency (in miles per gallon, or mpg) of the car decreases at a rate of approximately one half mpg as the velocity increases by one mph.

25. Units of dP/dt are barrels/year. dP/dt is the change in quantity of petroleum per change in time (a year). This is negative. We could estimate it by finding the amount of petroleum used worldwide over a short period of time.

26. (a)



(b) The graph should be concave down because air resistance decreases your acceleration as you speed up, and so the slope of the graph of velocity is decreasing.
(c) The slope represents the acceleration due to gravity.

27. (a) The derivative, dW/dt, measures the rate of change of water in the bathtub in gallons per minute.
(b) (i) The interval $t_0 < t < t_p$ represents the time before the plug is pulled. At that time, the rate of change of W is 0 since the amount of water in the tub is not changing.
(ii) Since dW/dt represents the rate at which the amount of water in the tub is changing, after the plug is pulled and water is leaving the tub, the sign of dW/dt is negative.
(iii) Once all the water has drained from the tub, the amount of water in the tub is not changing, so $dW/dt = 0$.

28. (a) The company hopes that increased advertising always brings in more customers instead of turning them away. Therefore, it hopes $f'(a)$ is always positive.
(b) If $f'(100) = 2$, it means that if the advertising budget is \$100,000, each extra dollar spent on advertising will bring in about \$2 worth of sales. If $f'(100) = 0.5$, each dollar above \$100 thousand spent on advertising will bring in about \$0.50 worth of sales.

(c) If $f'(100) = 2$, then as we saw in part (b), spending slightly more than \$100,000 will increase revenue by an amount greater than the additional expense, and thus more should be spent on advertising. If $f'(100) = 0.5$, then the increase in revenue is less than the additional expense, hence too much is being spent on advertising. The optimum amount to spend is an amount that makes $f'(a) = 1$. At this point, the increases in advertising expenditures just pay for themselves. If $f'(a) < 1$, too much is being spent; if $f'(a) > 1$, more should be spent.

29. (a) The derivative has units of people/second, so we find the rate of births, deaths, and migrations per second and combine them.

$$\begin{aligned}\text{Birth rate } &= \frac{1}{8} \text{ people per second}\\ \text{Death rate } &= \frac{1}{13} \text{ people per second}\\ \text{Migration rate } &= \frac{1}{27} \text{ people per second}\end{aligned}$$

Thus

$$f'(0) = \text{ Rate of change of population } = \frac{1}{8} - \frac{1}{13} + \frac{1}{27} = 0.0851 \text{ people/second.}$$

In other words, the population is increasing at 0.0851 people per second.

(b) From the answer to part (a), we see that it took $1/0.0851 = 11.75 \approx 12$ seconds to add one person.

30. Since $O'(2000) = -1,25$, we know the ODGI is decreasing at 1.25 units per year. To reduce the ODGI from 95 to 0 will take $95/1.25 = 76$ years. Thus the ozone hole is predicted to recover by 2076.

31. Since

$$\frac{P(67) - P(66)}{67 - 66} \approx P'(66),$$

we may think of $P'(66)$ as an estimate of $P(67) - P(66)$, and the latter is the number of people between 66 and 67 inches tall. Alternatively, since

$$\frac{P(66.5) - P(65.5)}{66.5 - 65.5} \text{ is a better estimate of } P'(66),$$

we may regard $P'(66)$ as an estimate of the number of people of height between 65.5 and 66.5 inches. The units for $P'(x)$ are people per inch. Since there are about 300 million people in the US, we guess that there are about 250 million full-grown persons in the US whose heights are distributed between 60 inches (5 ft) and 75 inches (6 ft 3 in). There are probably quite a few people of height 66 inches—between one and two times what we would expect from an even, or uniform, distribution— because 66 inches is nearly average. An even distribution would yield

$$P'(66) = \frac{250 \text{ million}}{15 \text{ ins}} \approx 17 \text{ million people per inch,}$$

so we expect $P'(66)$ to be between 17 and 34 million people per inch.

The value of $P'(x)$ is never negative because $P(x)$ is never decreasing. To see this, let's look at an example involving a particular value of x, say $x = 70$. The value $P(70)$ represents the number of people whose height is less than or equal to 70 inches, and $P(71)$ represents the number of people whose height is less than or equal to 71 inches. Since everyone shorter than 70 inches is also shorter than 71 inches, $P(70) \leq P(71)$. In general, $P(x)$ is 0 for small x, and increases as x increases, and is eventually constant (for large enough x).

32. (a) The units of compliance are units of volume per units of pressure, or liters per centimeter of water.

(b) The increase in volume for a 5 cm reduction in pressure is largest between 10 and 15 cm. Thus, the compliance appears maximum between 10 and 15 cm of pressure reduction. The derivative is given by the slope, so

$$\text{Compliance } \approx \frac{0.70 - 0.49}{15 - 10} = 0.042 \text{ liters per centimeter.}$$

(c) When the lung is nearly full, it cannot expand much more to accommodate more air.

33. Solving for $dp/d\delta$, we get

$$\frac{dp}{d\delta} = \left(\frac{p}{\delta + (p/c^2)}\right)\gamma.$$

(a) For $\delta \approx 10$ g/cm^3, we have $\log \delta \approx 1$, so, from Figure 2.38 in the text, we have $\gamma \approx 2.6$ and $\log p \approx 13$.

Thus $p \approx 10^{13}$, so $p/c^2 \approx 10^{13}/(9 \cdot 10^{20}) \approx 10^{-8}$, and

$$\frac{dp}{d\delta} \approx \frac{10^{13}}{10 + 10^{-8}} 2.6 \approx 2.6 \cdot 10^{12}.$$

The derivative can be interpreted as the ratio between a change in pressure and the corresponding change in density. The fact that it is so large says that a very large change in pressure brings about a very small change in density. This says that cold iron is not a very compressible material.

(b) For $\delta \approx 10^6$, we have $\log \delta \approx 6$, so, from Figure 2.38 in the text, $\gamma \approx 1.5$ and $\log p \approx 23$.
Thus $p \approx 10^{23}$, so $p/c^2 \approx 10^{23}/(9 \cdot 10^{20}) \approx 10^2$, and

$$\frac{dp}{d\delta} \approx \frac{10^{23}}{10^6 + 10^2} 1.5 \approx 1.5 \cdot 10^{17}.$$

This tells us that the matter in a white dwarf is even less compressible than cold iron.

Strengthen Your Understanding

34. Since we are not given the units of either t or s we cannot conclude that the units of the derivative are meters/second.

35. Since air is leaking from the balloon, the radius of the balloon must be decreasing, so $r'(t) < 0$.

36. Since T has units of minutes, its derivative with respect to P will have units of minutes/page.

37. Let $T(P)$ be the time, in years, to repay a loan of P dollars, then the derivative dT/dP is given in years/dollar.
There are many other possible answers.

38. Let $m = f(t)$ be the total distance, in miles, driven in a car, t days since it was purchased. Then the derivative dm/dt is given in miles/day.
There are many other possible answers.

39. True. The two sides of the equation are different frequently used notations for the very same quantity, the derivative of f at the point a.

40. True. The derivatives $f'(t)$ and $g'(t)$ measure the same thing, the rate of chemical production at the same time t, but they measure it in different units. The units of $f'(t)$ are grams per minute, and the units of $g'(t)$ are kilograms per minute. To convert from kg/min to g/min, multiply by 1000.

41. False. The derivatives $f'(t)$ and $g'(t)$ measure different things because they measure the rate of chemical production at different times. There is no conversion possible from one to the other.

42. (b) and (e) (b), (e)

43. (b) and (d) are equivalent, with (d) containing the most information. Notice that (a) and (c) are wrong.

Solutions for Section 2.5

Exercises

1. (a) Increasing, concave up
(b) Decreasing, concave down

2. (a) Since the graph is below the x-axis at $x = 2$, the value of $f(2)$ is negative.
(b) Since $f(x)$ is decreasing at $x = 2$, the value of $f'(2)$ is negative.
(c) Since $f(x)$ is concave up at $x = 2$, the value of $f''(2)$ is positive.

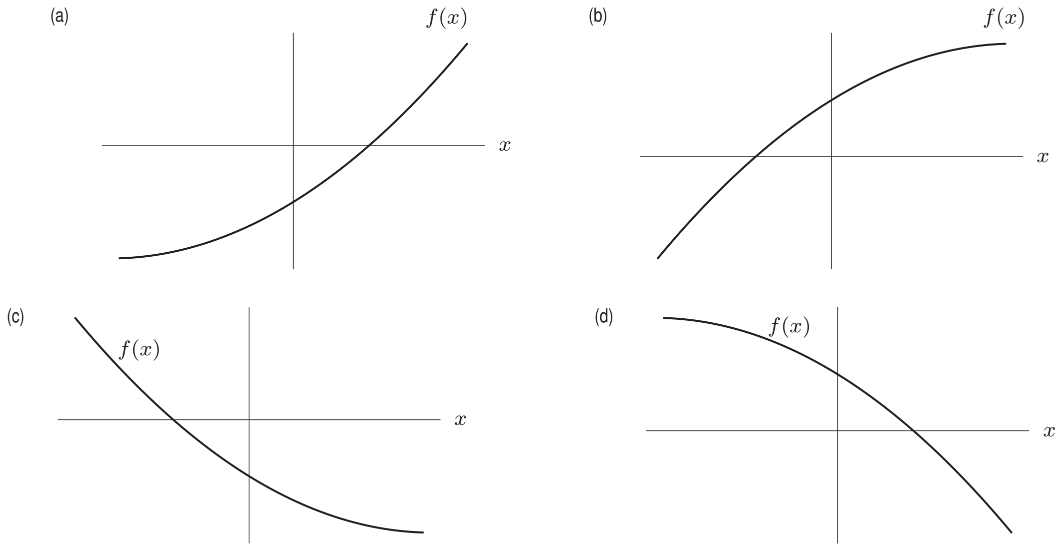
3. At B both dy/dx and d^2y/dx^2 could be positive because y is increasing and the graph is concave up there. At all the other points one or both of the derivatives could not be positive.

4. The two points at which $f' = 0$ are A and B. Since f' is nonzero at C and D and f'' is nonzero at all four points, we get the completed Table 2.6:

Table 2.6

Point	f	f'	f''
A	$-$	0	$+$
B	$+$	0	$-$
C	$+$	$-$	$-$
D	$-$	$+$	$+$

5.



6. The graph must be everywhere decreasing and concave up on some intervals and concave down on other intervals. One possibility is shown in Figure 2.62.

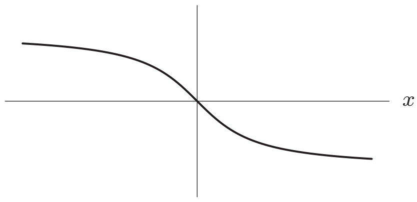


Figure 2.62

7. Since velocity is positive and acceleration is negative, we have $f' > 0$ and $f'' < 0$, and so the graph is increasing and concave down. See Figure 2.63.

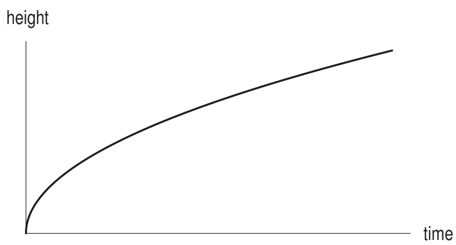


Figure 2.63

8. $f'(x) = 0$
$f''(x) = 0$

9. $f'(x) < 0$
$f''(x) = 0$

10. $f'(x) > 0$
$f''(x) > 0$

11. $f'(x) < 0$
$f''(x) > 0$

12. $f'(x) > 0$
$f''(x) < 0$

13. $f'(x) < 0$
$f''(x) < 0$

14. The velocity is the derivative of the distance, that is, $v(t) = s'(t)$. Therefore, we have

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h\to 0} \frac{(5(t+h)^2 + 3) - (5t^2 + 3)}{h} \\ &= \lim_{h\to 0} \frac{10th + 5h^2}{h} \\ &= \lim_{h\to 0} \frac{h(10t + 5h)}{h} = \lim_{h\to 0} (10t + 5h) = 10t \end{aligned}$$

The acceleration is the derivative of velocity, so $a(t) = v'(t)$:

$$\begin{aligned} a(t) &= \lim_{h\to 0} \frac{10(t+h) - 10t}{h} \\ &= \lim_{h\to 0} \frac{10h}{h} = 10. \end{aligned}$$

Problems

15. (a) The derivative, $f'(t)$, appears to be positive since the number of cars is increasing. The second derivative, $f''(t)$, appears to be negative during the period 1975–1990 because the rate of change is increasing. For example, between 1975 and 1980, the rate of change is $(121.6 - 106.7)/5 = 2.98$ million cars per year, while between 1985 and 1990, the rate of change is 1.16 million cars per year.

(b) The derivative, $f'(t)$, appears to be negative between 1990 and 1995 since the number of cars is decreasing, but increasing between 1995 and 2000. The second derivative, $f''(t)$, appears to be positive during the period 1990–2000 because the rate of change is increasing. For example, between 1990 and 1995, the rate of change is $(128.4 - 133.7)/5 = -1.06$ million cars per year, while between 1995 and 2000, the rate of change is 1.04 million cars per year.

(c) To estimate $f'(2000)$ we consider the interval 2000–2005

$$f'(2005) \approx \frac{f(2005) - f(2000)}{2005 - 2000} \approx \frac{136.6 - 133.6}{5} = \frac{3}{5} = 0.6.$$

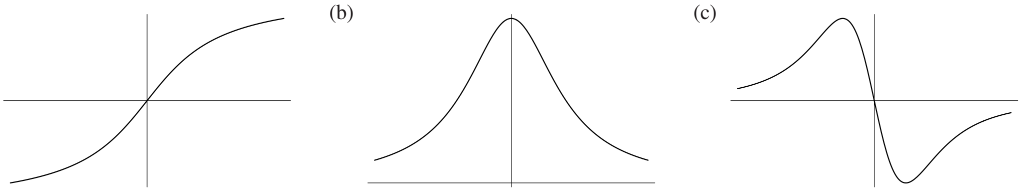
We estimate that $f'(2005) \approx 0.6$ million cars per year. The number of passenger cars in the US was increasing at a rate of about 600,000 cars per year in 2005.

16. To measure the average acceleration over an interval, we calculate the average rate of change of velocity over the interval. The units of acceleration are ft/sec per second, or (ft/sec)/sec, written ft/sec^2.

$$\begin{array}{c}\text{Average acceleration}\\ \text{for } 0 \le t \le 1\end{array} = \frac{\text{Change in velocity}}{\text{Time}} = \frac{v(1) - v(0)}{1} = \frac{30 - 0}{1} = 30 \text{ ft/sec}^2$$

$$\begin{array}{c}\text{Average acceleration}\\ \text{for } 1 \le t \le 2\end{array} = \frac{52 - 30}{2 - 1} = 22 \text{ ft/sec}^2$$

17. (a) (b) (c)



18. Since the graph of this function is a line, the second derivative (of any linear function) is 0. See Figure 2.64.

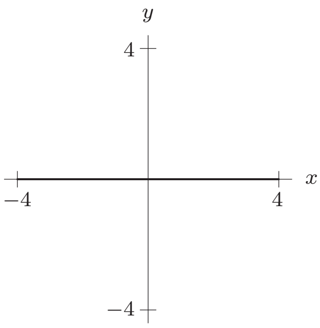


Figure 2.64

19. See Figure 2.65.

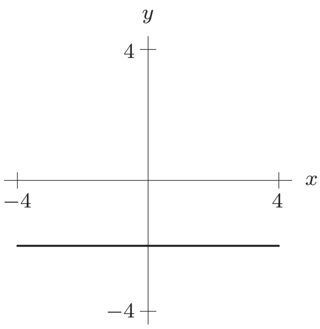


Figure 2.65

20. See Figure 2.66.

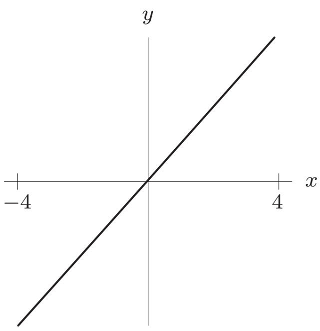


Figure 2.66

21. See Figure 2.67.

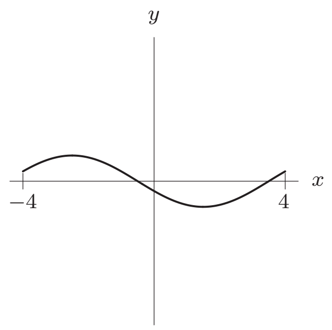


Figure 2.67

22. See Figure 2.68.

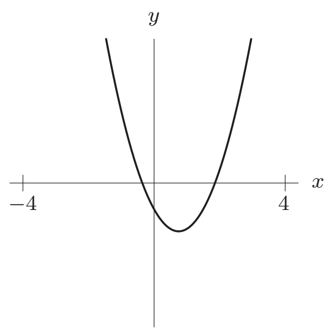


Figure 2.68

23. See Figure 2.69.

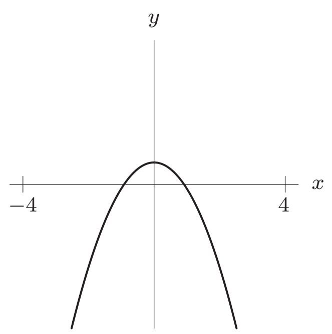
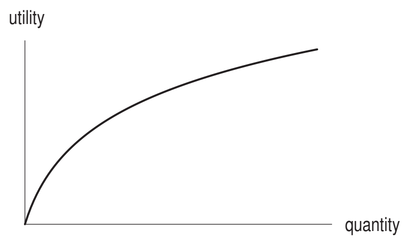


Figure 2.69

24. (a) $dP/dt > 0$ and $d^2P/dt^2 > 0$.

(b) $dP/dt < 0$ and $d^2P/dt^2 > 0$ (but dP/dt is close to zero).

25. (a)



(b) As a function of quantity, utility is increasing but at a decreasing rate; the graph is increasing but concave down. So the derivative of utility is positive, but the second derivative of utility is negative.

26. (a) Let $N(t)$ be the number of people below the poverty line. See Figure 2.70.

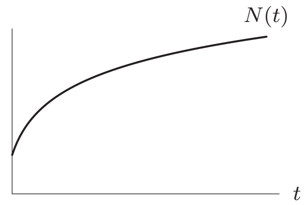


Figure 2.70

(b) dN/dt is positive, since people are still slipping below the poverty line. d^2N/dt^2 is negative, since the rate at which people are slipping below the poverty line, dN/dt, is decreasing.

27. (a) The EPA will say that the rate of discharge is still rising. The industry will say that the rate of discharge is increasing less quickly, and may soon level off or even start to fall.

(b) The EPA will say that the rate at which pollutants are being discharged is leveling off, but not to zero—so pollutants will continue to be dumped in the lake. The industry will say that the rate of discharge has decreased significantly.

28. (a) At x_4 and x_5, because the graph is below the x-axis there.
(b) At x_3 and x_4, because the graph is sloping down there.
(c) At x_3 and x_4, because the graph is sloping down there. This is the same condition as part (b).
(d) At x_2 and x_3, because the graph is bending downward there.
(e) At x_1, x_2, and x_5, because the graph is sloping upward there.
(f) At x_1, x_4, and x_5, because the graph is bending upward there.

29. (a) At t_3, t_4, and t_5, because the graph is above the t-axis there.
(b) At t_2 and t_3, because the graph is sloping up there.
(c) At t_1, t_2, and t_5, because the graph is concave up there
(d) At t_1, t_4, and t_5, because the graph is sloping down there.
(e) At t_3 and t_4, because the graph is concave down there.

30. Since f' is everywhere positive, f is everywhere increasing. Hence the greatest value of f is at x_6 and the least value of f is at x_1. Directly from the graph, we see that f' is greatest at x_3 and least at x_2. Since f'' gives the slope of the graph of f', f'' is greatest where f' is rising most rapidly, namely at x_6, and f'' is least where f' is falling most rapidly, namely at x_1.

31. To the right of $x = 5$, the function starts by increasing, since $f'(5) = 2 > 0$ (though f may subsequently decrease) and is concave down, so its graph looks like the graph shown in Figure 2.71. Also, the tangent line to the curve at $x = 5$ has slope 2 and lies above the curve for $x > 5$. If we follow the tangent line until $x = 7$, we reach a height of 24. Therefore, $f(7)$ must be smaller than 24, meaning 22 is the only possible value for $f(7)$ from among the choices given.

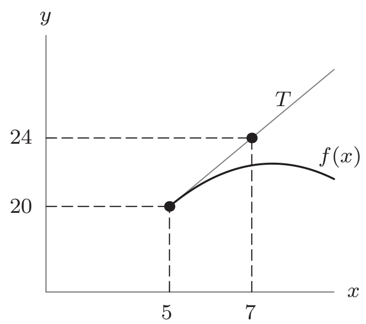


Figure 2.71

32. (a) From the information given, $C(1994) = 3200$ ppt and $C(2010) = 2750$ ppt.
(b) Since the change has been approximately linear, the rate of change is constant:

$$C'(1994) = C'(2010) = \frac{2750 - 3200}{2010 - 1994} = -28.125 \text{ ppt per year}.$$

(c) The slope is -28.125 and $C(1994) = 3200$.
If t is the year, we have

$$C(t) = 3200 - 28.125(t - 1994).$$

(d) We solve

$$\begin{aligned} 1850 &= 3200 - 28.125(t - 1994) \\ 28.125(t - 1994) &= 3200 - 1850 \\ t &= 1994 + \frac{3200 - 1850}{28.125} = 2042. \end{aligned}$$

The CFC level in the atmosphere above the US is predicted to return to the original level in 2042.
(e) Since $C''(t) > 0$, the graph bends upward, so the answer to part (d) is too early. The CFCs are expected to reach their original level later than 2042.

Strengthen Your Understanding

33. A linear function is neither concave up nor concave down.

34. When the acceleration of a car is zero, the car is not speeding up or slowing down. This happens whenever the velocity is constant. The car does not have to be stationary for this to happen.

35. One possibility is $f(x) = b + ax$, $a \neq 0$.

36. One possibility is $f(x) = x^2$. We have $f'(x) = 2x$, which is zero at $x = 0$ but $f''(x) = 2$.
There are many other possible answers.

37. True. The second derivative $f''(x)$ is the derivative of $f'(x)$. Thus the derivative of $f'(x)$ is positive, and so $f'(x)$ is increasing.

38. True. Instantaneous acceleration is a derivative, and all derivatives are limits of difference quotients. More precisely, instantaneous acceleration $a(t)$ is the derivative of the velocity $v(t)$, so

$$a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}.$$

39. True.

40. True; $f(x) = x^3$ is increasing over any interval.

41. False; $f(x) = x^2$ is monotonic on intervals which do not contain the origin (unless the origin is an endpoint).

Solutions for Section 2.6

Exercises

1. (a) Function f is not continuous at $x = 1$.
(b) Function f appears not differentiable at $x = 1, 2, 3$.

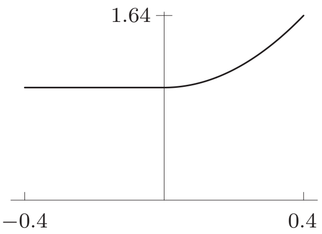
2. (a) Function g appears continuous at all x-values shown.
(b) Function g appears not differentiable at $x = 2, 4$. At $x = 2$, the curve is vertical, so the derivative does not exist. At $x = 4$, the graph has a corner, so the derivative does not exist.

3. No, there are sharp turning points.

4. Yes.

Problems

5. Yes, f is differentiable at $x = 0$, since its graph does not have a "corner" at $x = 0$. See below.



Another way to see this is by computing:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{(h + |h|)^2}{h} = \lim_{h\to 0} \frac{h^2 + 2h|h| + |h|^2}{h}.$$

Since $|h|^2 = h^2$, we have:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{2h^2 + 2h|h|}{h} = \lim_{h\to 0} 2(h + |h|) = 0.$$

So f is differentiable at 0 and $f'(0) = 0$.

6. As we can see in Figure 2.72, f oscillates infinitely often between the x-axis and the line $y = 2x$ near the origin. This means a line from $(0, 0)$ to a point $(h, f(h))$ on the graph of f alternates between slope 0 (when $f(h) = 0$) and slope 2 (when $f(h) = 2h$) infinitely often as h tends to zero. Therefore, there is no limit of the slope of this line as h tends to zero, and thus there is no derivative at the origin. Another way to see this is by noting that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h \sin(\frac{1}{h}) + h}{h} = \lim_{h\to 0} \left(\sin\left(\frac{1}{h}\right) + 1\right)$$

does not exist, since $\sin(\frac{1}{h})$ does not have a limit as h tends to zero. Thus, f is not differentiable at $x = 0$.

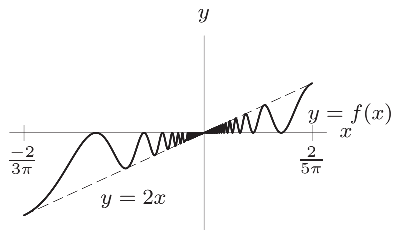


Figure 2.72

7. We can see from Figure 2.73 that the graph of f oscillates infinitely often between the curves $y = x^2$ and $y = -x^2$ near the origin. Thus the slope of the line from $(0, 0)$ to $(h, f(h))$ oscillates between h (when $f(h) = h^2$ and $\frac{f(h)-0}{h-0} = h$) and $-h$ (when $f(h) = -h^2$ and $\frac{f(h)-0}{h-0} = -h$) as h tends to zero. So, the limit of the slope as h tends to zero is 0, which is the derivative of f at the origin. Another way to see this is to observe that

$$\begin{aligned}\lim_{h\to 0} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0} \left(\frac{h^2 \sin(\frac{1}{h})}{h}\right) \\ &= \lim_{h\to 0} h \sin(\frac{1}{h}) \\ &= 0,\end{aligned}$$

since $\lim\limits_{h\to 0} h = 0$ and $-1 \le \sin(\frac{1}{h}) \le 1$ for any h. Thus f is differentiable at $x = 0$, and $f'(0) = 0$.

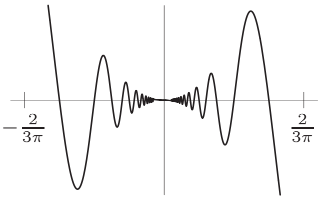


Figure 2.73

8. (a) The graph is concave up everywhere, except at $x = 2$ where the derivative is undefined. This is the case if the graph has a corner at $x = 2$. One possible graph is shown in Figure 2.74.
(b) The graph is concave up for $x < 2$ and concave down for $x > 2$, and the derivative is undefined at $x = 2$. This is the case if the graph is vertical at $x = 2$. One possible graph is shown in Figure 2.75.

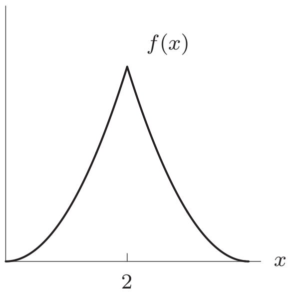


Figure 2.74

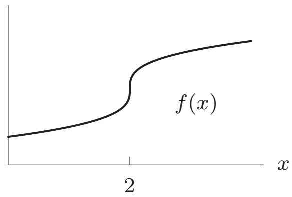


Figure 2.75

9. We want to look at

$$\lim_{h\to 0} \frac{(h^2 + 0.0001)^{1/2} - (0.0001)^{1/2}}{h}.$$

As $h \to 0$ from positive or negative numbers, the difference quotient approaches 0. (Try evaluating it for $h = 0.001$, 0.0001, etc.) So it appears there is a derivative at $x = 0$ and that this derivative is zero. How can this be if f has a corner at $x = 0$?

The answer lies in the fact that what appears to be a corner is in fact smooth—when you zoom in, the graph of f looks like a straight line with slope 0! See Figure 2.76.

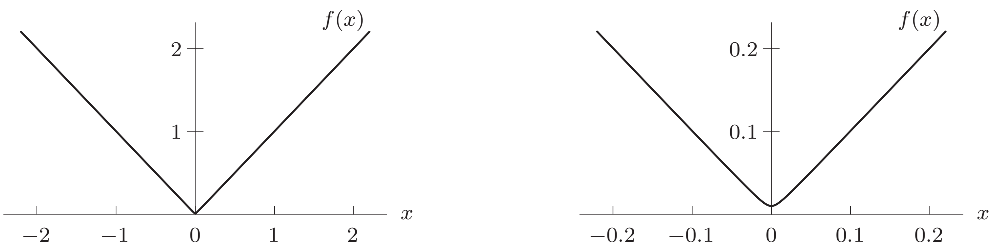


Figure 2.76: Close-ups of $f(x) = (x^2 + 0.0001)^{1/2}$ showing differentiability at $x = 0$

10. (a)

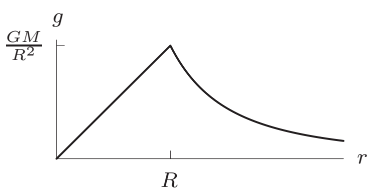


Figure 2.77

(b) The graph certainly looks continuous. The only point in question is $r = R$. Using the second formula with $r = R$ gives

$$g = \frac{GM}{R^2}.$$

Then, using the first formula with r approaching R from below, we see that as we get close to the surface of the earth

$$g \approx \frac{GMR}{R^3} = \frac{GM}{R^2}.$$

Since we get the same value for g from both formulas, g is continuous.

(c) For $r < R$, the graph of g is a line with a positive slope of $= \frac{GM}{R^3}$. For $r > R$, the graph of g looks like $1/x^2$, and so has a negative slope. Therefore the graph has a "corner" at $r = R$ and so is not differentiable there.

11. (a) The graph of Q against t does not have a break at $t = 0$, so Q appears to be continuous at $t = 0$. See Figure 2.78.

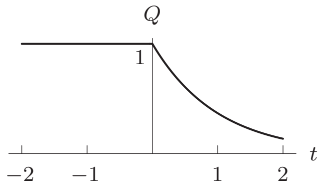


Figure 2.78

(b) The slope dQ/dt is zero for $t < 0$, and negative for all $t > 0$. At $t = 0$, there appears to be a corner, which does not disappear as you zoom in, suggesting that I is defined for all times t except $t = 0$.

12. (a) Notice that B is a linear function of r for $r \leq r_0$ and a reciprocal for $r > r_0$. The constant B_0 is the value of B at $r = r_0$ and the maximum value of B. See Figure 2.79.

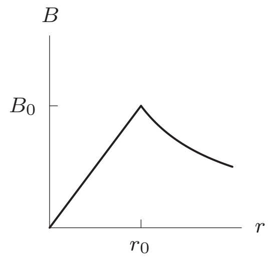


Figure 2.79

(b) B is continuous at $r = r_0$ because there is no break in the graph there. Using the formula for B, we have

$$\lim_{r \to r_0^-} B = \frac{r_0}{r_0} B_0 = B_0 \quad \text{and} \quad \lim_{r \to r_0^+} B = \frac{r_0}{r_0} B_0 = B_0.$$

(c) The function B is not differentiable at $r = r_0$ because the graph has a corner there. The slope is positive for $r < r_0$ and the slope is negative for $r > r_0$.

13. (a) Since

$$\lim_{r \to r_0^-} E = kr_0$$

and

$$\lim_{r \to r_0^+} E = \frac{kr_0^2}{r_0} = kr_0$$

and

$$E(r_0) = kr_0,$$

we see that E is continuous at r_0.

(b) The function E is not differentiable at $r = r_0$ because the graph has a corner there. The slope is positive for $r < r_0$ and the slope is negative for $r > r_0$.

(c) See Figure 2.80.

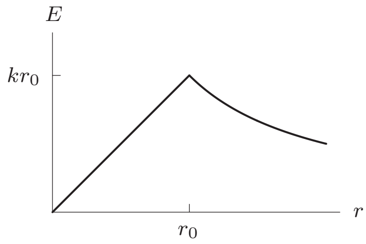


Figure 2.80

14. (a) The graph of $g(r)$ does not have a break or jump at $r = 2$, and so $g(r)$ is continuous there. See Figure 2.81. This is confirmed by the fact that

$$g(2) = 1 + \cos(\pi 2/2) = 1 + (-1) = 0$$

so the value of $g(r)$ as you approach $r = 2$ from the left is the same as the value when you approach $r = 2$ from the right.

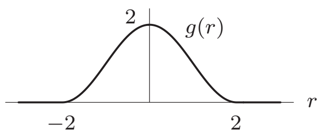


Figure 2.81

(b) The graph of $g(r)$ does not have a corner at $r = 2$, even after zooming in, so $g(r)$ appears to be differentiable at $r = 0$. This is confirmed by the fact that $\cos(\pi r/2)$ is at the bottom of a trough at $r = 2$, and so its slope is 0 there. Thus the slope to the left of $r = 2$ is the same as the slope to the right of $r = 2$.

15. (a) The graph of ϕ does not have a break at $y = 0$, and so ϕ appears to be continuous there. See figure Figure 2.82.

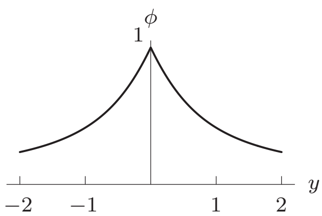


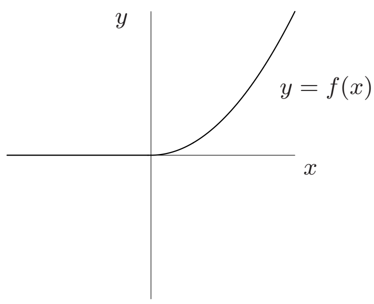
Figure 2.82

(b) The graph of ϕ has a corner at $y = 0$ which does not disappear as you zoom in. Therefore ϕ appears not be differentiable at $y = 0$.

16. (a) The graph of

$$f(x) = \begin{cases} 0 & \text{if } x < 0. \\ x^2 & \text{if } x \geq 0. \end{cases}$$

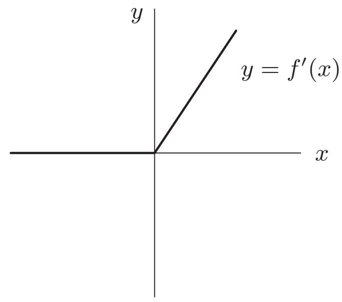
is shown to the right. The graph is continuous and has no vertical segments or corners, so $f(x)$ is differentiable everywhere.



By Example 4 on page 94,

$$f'(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

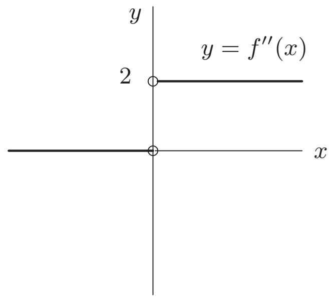
So its graph is shown to the right.



(b) The graph of the derivative has a corner at $x = 0$ so $f'(x)$ is not differentiable at $x = 0$. The graph of

$$f''(x) = \begin{cases} 0 \text{ if } x < 0 \\ 2 \text{ if } x > 0 \end{cases}$$

looks like:



The second derivative is not defined at $x = 0$. So it is certainly neither differentiable nor continuous at $x = 0$.

Strengthen Your Understanding

17. There are several ways in which a function can fail to be differentiable at a point, one of which is because the graph has a sharp corner at the point. Other cases are when the function is not continuous at a point or if the graph has a vertical tangent line.

18. The converse of this statement is true. However, a function can be continuous and not differentiable at a point; for example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

19. $f(x) = |x - 2|$. This is continuous but not differentiable at $x = 2$.

20. $f(x) = \sqrt{x}$, $x \geq 0$. This is invertible but $f'(x) = 1/(2\sqrt{x})$, which is not defined at $x = 0$.

21. Let

$$f(x) = \frac{x^2 - 1}{x^2 - 4}.$$

Since $x^2 - 1 = (x - 1)(x + 1)$, this function has zeros at $x = \pm 1$. However, at $x = \pm 2$, the denominator $x^2 - 4 = 0$, so $f(x)$ is undefined and not differentiable.

22. True. Let $f(x) = |x - 3|$. Then $f(x)$ is continuous for all x but not differentiable at $x = 3$ because its graph has a corner there. Other answers are possible.

23. True. If a function is differentiable at a point, then it is continuous at that point. For example, $f(x) = x^2$ is both differentiable and continuous on any interval. However, *one* example does not establish the truth of this statement; it merely illustrates the statement.

24. False. Being continuous does not imply differentiability. For example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

25. True. If a function were differentiable, then it would be continuous. For example,

$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ is neither differentiable nor continuous at $x = 0$. However, *one* example does not establish the truth of this statement; it merely illustrates the statement.

26. False. For example, $f(x) = |x|$ is not differentiable at $x = 0$, but it is continuous at $x = 0$.

27. (a) This is not a counterexample, since it does not satisfy the conditions of the statement, and therefore does not have the potential to contradict the statement.

(b) This contradicts the statement, because it satisfies its conditions but not its conclusion. Hence it is a counterexample. Notice that this counterexample could not actually exist, since the statement is true.

(c) This is an example illustrating the statement; it is not a counterexample.

(d) This is not a counterexample, for the same reason as in part (a).

Solutions for Chapter 2 Review

Exercises

1. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we find the values of $s(3) = 72$ and $s(10) = 144$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(10) - s(3)}{10 - 3} = \frac{144 - 72}{7} = \frac{72}{7} = 10.286 \text{ cm/sec}.$$

2. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = 12 \cdot 3 - 3^2 = 27$ and $s(1) = 12 \cdot 1 - 1^2 = 11$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{27 - 11}{2} = 8 \text{ mm/sec}.$$

3. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = \ln 3$ and $s(1) = \ln 1$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{\ln 3 - \ln 1}{2} = \frac{\ln 3}{2} = 0.549 \text{ mm/sec}.$$

4. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = 11$ and $s(1) = 3$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{11 - 3}{2} = 4 \text{ mm/sec}.$$

5. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values of $s(3) = 4$ and $s(1) = 4$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{4 - 4}{2} = 0 \text{ mm/sec}.$$

Though the particle moves, its average velocity over the interval is zero, since it is at the same position at $t = 1$ and $t = 3$.

6. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values on the graph of $s(3) = 2$ and $s(1) = 3$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{2 - 3}{2} = -\frac{1}{2} \text{ mm/sec}.$$

7. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the position of the particle, we find the values on the graph of $s(3) = 2$ and $s(1) = 2$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{2 - 2}{2} = 0 \text{ mm/sec}.$$

Though the particle moves, its average velocity over the interval is zero, since it is at the same position at $t = 1$ and $t = 3$.

8. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{7.84 - 7}{0.1} = 8.4 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{7.0804 - 7}{0.01} = 8.04 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{7.008004 - 7}{0.001} = 8.004 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 8, so we estimate the instantaneous velocity at $t = 1$ to be 8 m/sec.

9. See Figure 2.83.

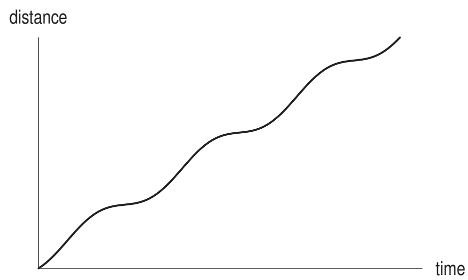


Figure 2.83

10. See Figure 2.84.

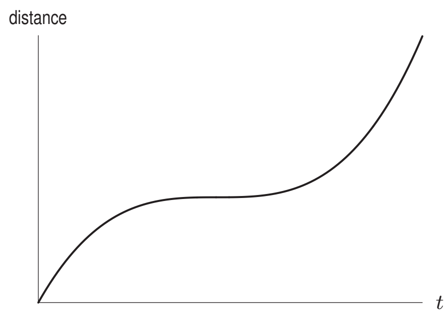


Figure 2.84

11. (a) Figure 2.85 shows a graph of $f(x) = x \sin x$.

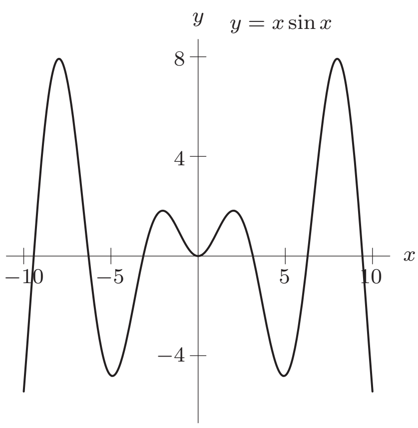


Figure 2.85

(b) Seven, since $x \sin x = 0$ at $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi$.
(c) From the graph, we see $x \sin x$ is increasing at $x = 1$, decreasing at $x = 4$.
(d) We calculate both average rates of change

$$\frac{f(2) - f(0)}{(2 - 0)} = \frac{2\sin 2 - 0}{2} = \sin 2 \approx 0.91$$
$$\frac{f(8) - f(6)}{(8 - 6)} = \frac{8\sin 8 - 6\sin 6}{2} \approx 4.80.$$

So the average rate of change over $6 \le x \le 8$ is greater.
(e) From the graph, we see the slope is greater at $x = -9$.

12. (a) Using the difference quotient

$$f'(0.6) \approx \frac{f(0.8) - f(0.4)}{0.8 - 0.4} = \frac{0.5}{0.4} = 1.25.$$

Substituting $x = 0.6$, we have $y = 3.9$, so the tangent line is $y - 3.9 = 1.25(x - 0.6)$, that is $y = 1.25x + 3.15$.
(b) The equation from part (a) gives

$$f(0.7) \approx 1.25(0.7) + 3.15 = 4.025$$
$$f(1.2) \approx 1.25(1.2) + 3.15 = 4.65$$
$$f(1.4) \approx 1.25(1.4) + 3.15 = 4.9$$

The estimate for $f(0.7)$ is likely to be reliable as 0.7 is close to 0.6 (and $f(0.8) = 4$, which is not too far off). The estimate for $f(1.2)$ is less reliable as 1.2 is outside the given data (from 0 to 1.0). The estimate for $f(1.4)$ less reliable still.

13. See Figure 2.86.

Figure 2.86

14. See Figure 2.87.

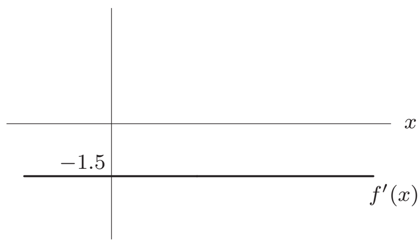


Figure 2.87

15. See Figure 2.88.

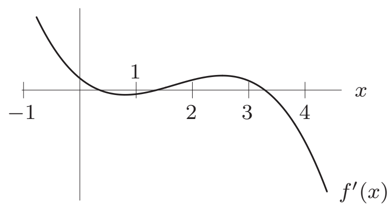


Figure 2.88

16. See Figure 2.89.

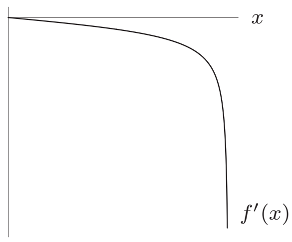


Figure 2.89

17. See Figure 2.90.

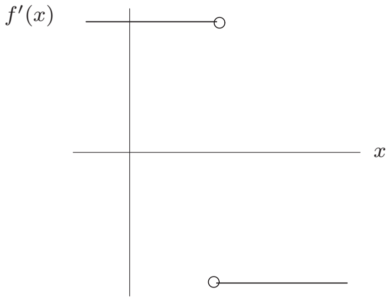


Figure 2.90

18. See Figure 2.91.

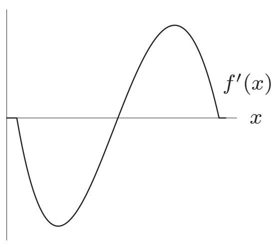


Figure 2.91

19. See Figure 2.92.

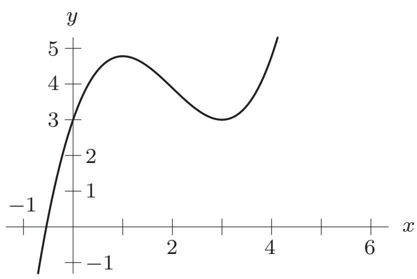


Figure 2.92

20. See Figure 2.93.

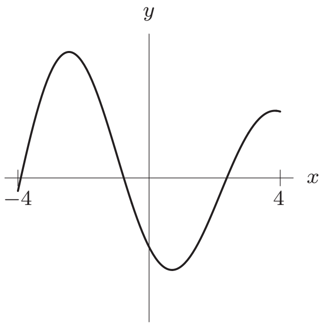


Figure 2.93

21. See Figure 2.94.

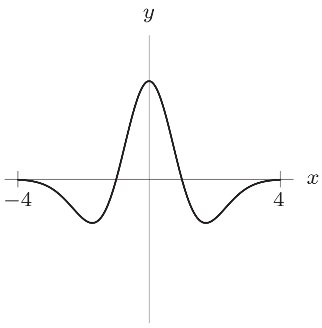


Figure 2.94

22. Using the definition of the derivative

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{5(x+h)^2+x+h-(5x^2+x)}{h} \\ &= \lim_{h\to 0} \frac{5(x^2+2xh+h^2)+x+h-5x^2-x}{h} \\ &= \lim_{h\to 0} \frac{10xh+5h^2+h}{h} \\ &= \lim_{h\to 0} (10x+5h+1) = 10x+1 \end{aligned}$$

23. Using the definition of the derivative, we have

$$\begin{aligned} n'(x) &= \lim_{h\to 0} \frac{n(x+h)-n(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\left(\frac{1}{x+h}+1\right)-\left(\frac{1}{x}+1\right)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) \\ &= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}. \end{aligned}$$

24. We need to look at the difference quotient and take the limit as h approaches zero. The difference quotient is

$$\frac{f(3+h)-f(3)}{h} = \frac{[(3+h)^2+1]-10}{h} = \frac{9+6h+h^2+1-10}{h} = \frac{6h+h^2}{h} = \frac{h(6+h)}{h}.$$

Since $h \neq 0$, we can divide by h in the last expression to get $6+h$. Now the limit as h goes to 0 of $6+h$ is 6, so

$$f'(3) = \lim_{h\to 0}\frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6.$$

So at $x=3$, the slope of the tangent line is 6. Since $f(3) = 3^2+1 = 10$, the tangent line passes through $(3,10)$, so its equation is

$$y-10 = 6(x-3), \quad \text{or} \quad y = 6x-8.$$

25. By joining consecutive points we get a line whose slope is the average rate of change. The steeper this line, the greater the average rate of change. See Figure 2.95.

(a) (i) C and D. Steepest slope.
 (ii) B and C. Slope closest to 0.

(b) A and B, and C and D. The two slopes are closest to each other.

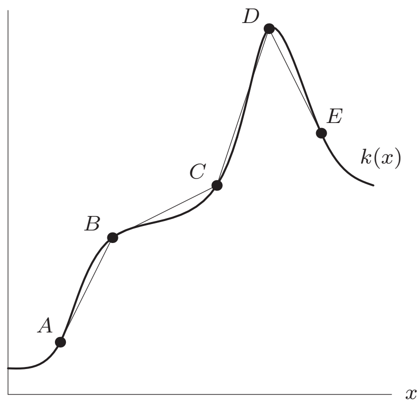


Figure 2.95

26. Using the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(3(x+h)-1)-(3x-1)}{h} \\ &= \lim_{h\to 0}\frac{3x+3h-1-3x+1}{h} \\ &= \lim_{h\to 0}\frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3. \end{aligned}$$

27. Using the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(5(x+h)^2)-(5x^2)}{h} \\ &= \lim_{h\to 0}\frac{5(x^2+2xh+h^2)-5x^2}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h} \\
&= \lim_{h\to 0} \frac{10xh + 5h^2}{h} \\
&= \lim_{h\to 0} (10x + 5h) \\
&= 10x.
\end{aligned}
$$

28. Using the definition of the derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^2 + 4) - (x^2 + 4)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 4 - x^2 - 4}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x.
\end{aligned}
$$

29. Using the definition of the derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(3(x+h)^2 - 7) - (3x^2 - 7)}{h} \\
&= \lim_{h\to 0} \frac{(3(x^2 + 2xh + h^2) - 7) - (3x^2 - 7)}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 7 - 3x^2 + 7}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h\to 0} (6x + 3h) \\
&= 6x.
\end{aligned}
$$

30. Using the definition of the derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2.
\end{aligned}
$$

31. $\displaystyle \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0} (2a + h) = 2a$

32. $\displaystyle \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{a+h} - \frac{1}{a}\right) = \lim_{h\to 0} \frac{a - (a+h)}{(a+h)ah} = \lim_{h\to 0} \frac{-1}{(a+h)a} = \frac{-1}{a^2}$

33. $\displaystyle \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(a+h)^2} - \frac{1}{a^2}\right) = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{(a+h)^2a^2h} = \lim_{h\to 0} \frac{(-2a - h)}{(a+h)^2a^2} = \frac{-2}{a^3}$

34. $\sqrt{a+h}-\sqrt{a} = \dfrac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{\sqrt{a+h}+\sqrt{a}} = \dfrac{a+h-a}{\sqrt{a+h}+\sqrt{a}} = \dfrac{h}{\sqrt{a+h}+\sqrt{a}}.$

Therefore $\lim_{h\to 0} \dfrac{\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h\to 0} \dfrac{1}{\sqrt{a+h}+\sqrt{a}} = \dfrac{1}{2\sqrt{a}}$

35. We combine terms in the numerator and multiply top and bottom by $\sqrt{a}+\sqrt{a+h}$.

$$\begin{aligned}\frac{1}{\sqrt{a+h}} - \frac{1}{\sqrt{a}} &= \frac{\sqrt{a}-\sqrt{a+h}}{\sqrt{a+h}\sqrt{a}} = \frac{(\sqrt{a}-\sqrt{a+h})(\sqrt{a}+\sqrt{a+h})}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})}\\ &= \frac{a-(a+h)}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})}\end{aligned}$$

Therefore $\lim_{h\to 0}\dfrac{1}{h}\left(\dfrac{1}{\sqrt{a+h}} - \dfrac{1}{\sqrt{a}}\right) = \lim_{h\to 0}\dfrac{-1}{\sqrt{a+h}\sqrt{a}(\sqrt{a}+\sqrt{a+h})} = \dfrac{-1}{2(\sqrt{a})^3}$

Problems

36. The function is everywhere increasing and concave up. One possible graph is shown in Figure 2.96.

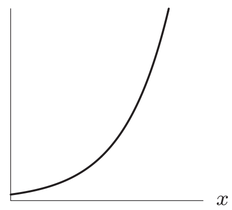


Figure 2.96

37. First note that the line $y = t$ has slope 1. From the graph, we see that

$$0 < \text{Slope at } C < \text{Slope at } B < \text{Slope between } A \text{ and } B < 1 < \text{Slope at } A.$$

Since instantaneous velocity is represented by the slope at a point and average velocity is represented by the slope between two points, we have

$$0 < \text{ Inst. vel. at } C \ < \text{ Inst. vel. at } B \ < \text{ Av. vel. between } A \text{ and } B \ < 1 < \text{ Inst. vel. at } A.$$

38. (a) The only graph in which the slope is 1 for all x is Graph (III).
(b) The only graph in which the slope is positive for all x is Graph (III).
(c) Graphs where the slope is 1 at $x = 2$ are Graphs (III) and (IV).
(d) Graphs where the slope is 2 at $x = 1$ are Graphs (II) and (IV).

39. (a) Velocity is zero at points A, C, F, and H.
(b) These are points where the acceleration is zero, at which the particle switches from speeding up to slowing down or vice versa.

40. (a) The derivative, $f'(t)$, appears to be positive between 2003–2005 and 2006–2007, since the number of cars increased in these intervals. The derivative, $f'(t)$, appears to be negative from 2005–2006, since the number of cars decreased then.
(b) We use the average rate of change formula on the interval 2005 to 2007 to estimate $f'(2006)$:

$$f'(2006) \approx \frac{135.9 - 136.6}{2007 - 2005} = \frac{-0.7}{2} = -0.35.$$

We see that $f'(2006) \approx -0.35$ million cars per year. The number of passenger cars in the US was decreasing at a rate of about 0.35 million, or 350,000, cars per year in 2006.

41. (a) If $f'(t) > 0$, the depth of the water is increasing. If $f'(t) < 0$, the depth of the water is decreasing.
(b) The depth of the water is increasing at 20 cm/min when $t = 30$ minutes.
(c) We use 1 meter $=$ 100 cm, 1 hour $=$ 60 min. At time $t = 30$ minutes

$$\text{Rate of change of depth} = 20\frac{\text{cm}}{\text{min}} = 20\frac{\text{cm}}{\text{min}}\cdot\frac{60 \text{ min}}{1 \text{ hr}}\cdot\frac{1 \text{ m}}{100 \text{ cm}} = 12 \text{ meters/hour}.$$

42. Since $f(t) = 45.7e^{-0.0061t}$, we have

$$f(6) = 45.7e^{-0.0061\cdot 6} = 44.058.$$

To estimate $f'(6)$, we use a small interval around 6:

$$f'(6) \approx \frac{f(6.001) - f(6)}{6.001 - 6} = \frac{45.7e^{-0.0061\cdot 6.001} - 45.7e^{-0.0061\cdot 6}}{0.001} = -0.269.$$

We see that $f(6) = 44.058$ million people and $f'(6) = -0.269$ million (that is, $-269{,}000$) people per year. Since $t = 6$ in 2015, this model predicts that the population of Ukraine will be about $44{,}058{,}000$ people in 2015 and declining at a rate of about 269,000 people per year at that time.

43. (a) The units of $R'(3)$ are thousands of dollars per (dollar per gallon).
The derivative $R'(3)$ tells us the rate of change of revenue with price. That is, $R'(3)$ gives approximately how much the revenue changes if the gas price increases by \$1 per gallon from \$3 per gallon.

(b) The units of $R^{-1}(5)$ are dollars per gallon. Thus, the units of $(R^{-1})'(5)$ are dollars/gallon per thousand dollars.
The derivative $(R^{-1})'(5)$ tells us the rate of change of price with revenue. That is, $(R^{-1})'(5)$ gives approximately how much the price of gas changes if the revenue increases by \$1000 from \$5000 to \$6000.

44. (a) A possible example is $f(x) = 1/|x-2|$ as $\lim_{x\to 2} 1/|x-2| = \infty$.

(b) A possible example is $f(x) = -1/(x-2)^2$ as $\lim_{x\to 2} -1/(x-2)^2 = -\infty$.

45. For $x < -2$, f is increasing and concave up. For $-2 < x < 1$, f is increasing and concave down. At $x = 1$, f has a maximum. For $x > 1$, f is decreasing and concave down. One such possible f is in Figure 2.97.

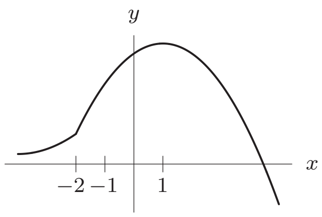


Figure 2.97

46. Since $f(2) = 3$ and $f'(2) = 1$, near $x = 2$ the graph looks like the segment shown in Figure 2.98.

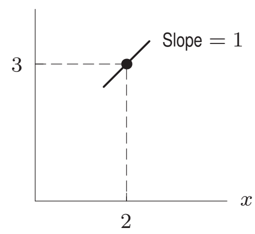


Figure 2.98

(a) If $f(x)$ is even, then the graph of $f(x)$ near $x = 2$ and $x = -2$ looks like Figure 2.99. Thus $f(-2) = 3$ and $f'(-2) = -1$.

(b) If $f(x)$ is odd, then the graph of $f(x)$ near $x = 2$ and $x = -2$ looks like Figure 2.100. Thus $f(-2) = -3$ and $f'(-2) = 1$.

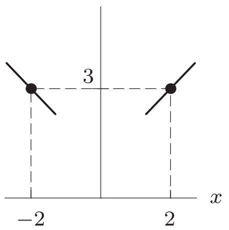


Figure 2.99: For f even

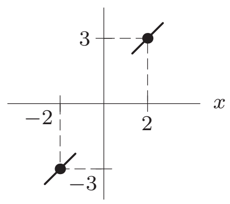


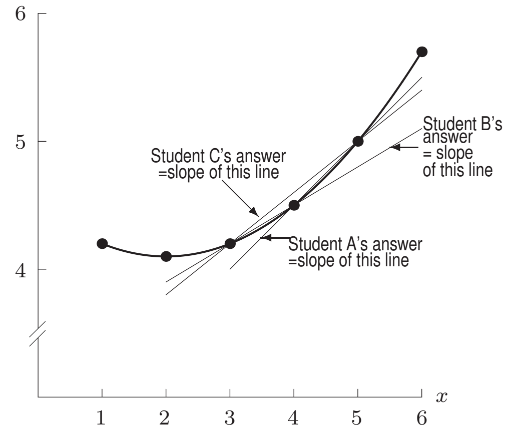
Figure 2.100: For f odd

47. The slopes of the lines drawn through successive pairs of points are negative but increasing, suggesting that $f''(x) > 0$ for $1 \le x \le 3.3$ and that the graph of $f(x)$ is concave up.

48. Using the approximation $\Delta y \approx f'(x)\Delta x$ with $\Delta x = 2$, we have $\Delta y \approx f'(20) \cdot 2 = 6 \cdot 2$, so

$$f(22) \approx f(20) + f'(20) \cdot 2 = 345 + 6 \cdot 2 = 357.$$

49. (a)

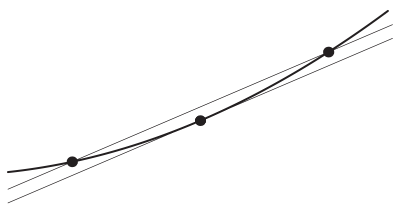


(b) The slope of f appears to be somewhere between student A's answer and student B's, so student C's answer, halfway in between, is probably the most accurate.

(c) Student A's estimate is $f'(x) \approx \frac{f(x+h)-f(x)}{h}$, while student B's estimate is $f'(x) \approx \frac{f(x)-f(x-h)}{h}$. Student C's estimate is the average of these two, or

$$f'(x) \approx \frac{1}{2}\left[\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}\right] = \frac{f(x+h)-f(x-h)}{2h}.$$

This estimate is the slope of the chord connecting $(x-h, f(x-h))$ to $(x+h, f(x+h))$. Thus, we estimate that the tangent to a curve is nearly parallel to a chord connecting points h units to the right and left, as shown below.



50. (a) Since the point $A = (7, 3)$ is on the graph of f, we have $f(7) = 3$.

(b) The slope of the tangent line touching the curve at $x = 7$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{3.8-3}{7.2-7} = \frac{0.8}{0.2} = 4.$$

Thus, $f'(7) = 4$.

51. At point A, we are told that $x = 1$ and $f(1) = 3$. Since $A = (x_2, y_2)$, we have $x_2 = 1$ and $y_2 = 3$. Since $h = 0.1$, we know $x_1 = 1 - 0.1 = 0.9$ and $x_3 = 1 + 0.1 = 1.1$.

Now consider Figure 2.101. Since $f'(1) = 2$, the slope of the tangent line AD is 2. Since $AB = 0.1$,

$$\frac{\text{Rise}}{\text{Run}} = \frac{BD}{0.1} = 2,$$

so $BD = 2(0.1) = 0.2$. Therefore $y_1 = 3 - 0.2 = 2.8$ and $y_3 = 3 + 0.2 = 3.2$.

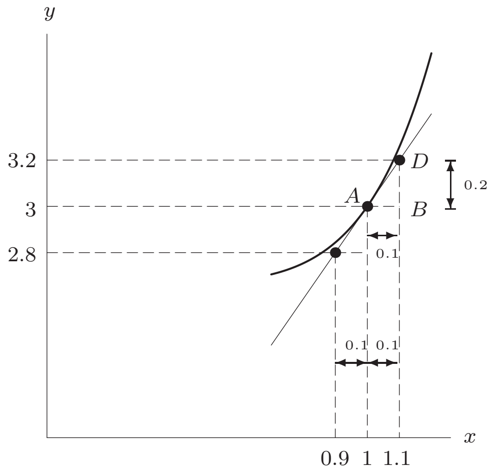


Figure 2.101

52. A possible graph of $y = f(x)$ is shown in Figure 2.102.

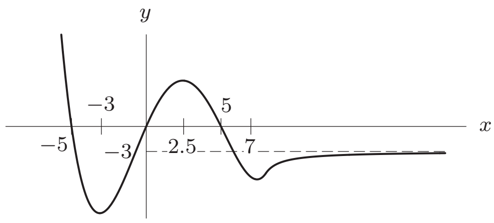


Figure 2.102

53. (a) Negative.
(b) $dw/dt = 0$ for t bigger than some t_0 (the time when the fire stops burning).
(c) $|dw/dt|$ increases, so dw/dt decreases since it is negative.

54. (a) The yam is cooling off so T is decreasing and $f'(t)$ is negative.
(b) Since $f(t)$ is measured in degrees Fahrenheit and t is measured in minutes, df/dt must be measured in units of $^\circ$F/min.

55. (a) The statement $f(140) = 120$ means that a patient weighing 140 pounds should receive a dose of 120 mg of the painkiller. The statement $f'(140) = 3$ tells us that if the weight of a patient increases by one pound (from 140 pounds), the dose should be increased by about 3 mg.
(b) Since the dose for a weight of 140 lbs is 120 mg and at this weight the dose goes up by about 3 mg for one pound, a 145 lb patient should get about an additional $3(5) = 15$ mg. Thus, for a 145 lb patient, the correct dose is approximately

$$f(145) \approx 120 + 3(5) = 135 \text{ mg}.$$

56. Suppose $p(t)$ is the average price level at time t. Then, if $t_0 =$ April 1991,
"Prices are still rising" means $p'(t_0) > 0$.
"Prices rising less fast than they were" means $p''(t_0) < 0$.
"Prices rising not as much less fast as everybody had hoped" means $H < p''(t_0)$, where H is the rate of change in rate of change of prices that people had hoped for.

57. The rate of change of the US population is $P'(t)$, so

$$P'(t) = 0.8\% \cdot \text{ Current population } = 0.008P(t).$$

58. (a) See Figure 2.103.

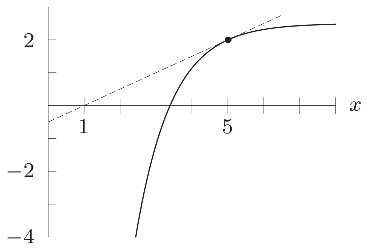


Figure 2.103

(b) Exactly one. There can't be more than one zero because f is increasing everywhere. There does have to be one zero because f stays below its tangent line (dotted line in above graph), and therefore f must cross the x-axis.

(c) The equation of the (dotted) tangent line is $y = \frac{1}{2}x - \frac{1}{2}$, and so it crosses the x-axis at $x = 1$. Therefore the zero of f must be between $x = 1$ and $x = 5$.

(d) $\lim_{x\to-\infty} f(x) = -\infty$, because f is increasing and concave down. Thus, as $x \to -\infty$, $f(x)$ decreases, at a faster and faster rate.

(e) Yes.

(f) No. The slope is decreasing since f is concave down, so $f'(1) > f'(5)$, i.e. $f'(1) > \frac{1}{2}$.

59. (a) $f'(0.6) \approx \dfrac{f(0.8) - f(0.6)}{0.8 - 0.6} = \dfrac{4.0 - 3.9}{0.2} = 0.5.$ $\quad f'(0.5) \approx \dfrac{f(0.6) - f(0.4)}{0.6 - 0.4} = \dfrac{0.4}{0.2} = 2.$

(b) Using the values of f' from part (a), we get $f''(0.6) \approx \dfrac{f'(0.6) - f'(0.5)}{0.6 - 0.5} = \dfrac{0.5 - 2}{0.1} = \dfrac{-1.5}{0.1} = -15.$

(c) The maximum value of f is probably near $x = 0.8$. The minimum value of f is probably near $x = 0.3$.

60. (a) Slope of tangent line $= \lim_{h\to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h}$. Using $h = 0.001$, $\frac{\sqrt{4.001}-\sqrt{4}}{0.001} = 0.249984$. Hence the slope of the tangent line is about 0.25.

(b)

$$
\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - 2 &= 0.25(x - 4)\\
y - 2 &= 0.25x - 1\\
y &= 0.25x + 1
\end{aligned}
$$

(c) $f(x) = kx^2$
If $(4, 2)$ is on the graph of f, then $f(4) = 2$, so $k \cdot 4^2 = 2$. Thus $k = \frac{1}{8}$, and $f(x) = \frac{1}{8}x^2$.

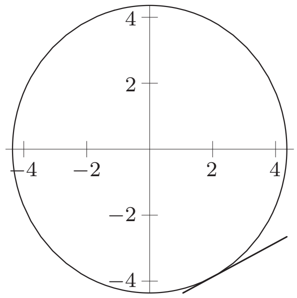
(d) To find where the graph of f crosses then line $y = 0.25x + 1$, we solve:

$$
\begin{aligned}
\frac{1}{8}x^2 &= 0.25x + 1\\
x^2 &= 2x + 8\\
x^2 - 2x - 8 &= 0\\
(x - 4)(x + 2) &= 0\\
x = 4 &\text{ or } x = -2\\
f(-2) &= \frac{1}{8}(4) = 0.5
\end{aligned}
$$

Therefore, $(-2, 0.5)$ is the other point of intersection. (Of course, $(4, 2)$ is a point of intersection; we know that from the start.)

61. (a) The slope of the tangent line at $(0, \sqrt{19})$ is zero: it is horizontal.
The slope of the tangent line at $(\sqrt{19}, 0)$ is undefined: it is vertical.

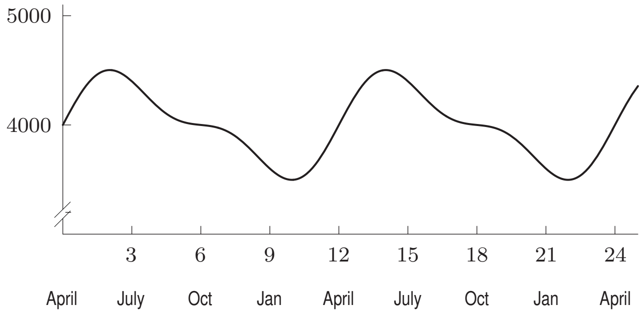
(b) The slope appears to be about $\frac{1}{2}$. (Note that when x is 2, y is about -4, but when x is 4, y is approximately -3.)



(c) Using symmetry we can determine: Slope at $(-2, \sqrt{15})$: about $\frac{1}{2}$. Slope at $(-2, -\sqrt{15})$: about $-\frac{1}{2}$. Slope at $(2, \sqrt{15})$: about $-\frac{1}{2}$.

62. (a) IV, (b) III, (c) II, (d) I, (e) IV, (f) II

63. (a) The population varies periodically with a period of 12 months (i.e. one year).



(b) The herd is largest about June 1^{st} when there are about 4500 deer.
(c) The herd is smallest about February 1^{st} when there are about 3500 deer.
(d) The herd grows the fastest about April 1^{st}. The herd shrinks the fastest about July 15 and again about December 15.
(e) It grows the fastest about April 1^{st} when the rate of growth is about 400 deer/month, i.e about 13 new fawns per day.

64. (a) The graph looks straight because the graph shows only a small part of the curve magnified greatly.
(b) The month is March: We see that about the 21^{st} of the month there are twelve hours of daylight and hence twelve hours of night. This phenomenon (the length of the day equaling the length of the night) occurs at the equinox, midway between winter and summer. Since the length of the days is increasing, and Madrid is in the northern hemisphere, we are looking at March, not September.
(c) The slope of the curve is found from the graph to be about 0.04 (the rise is about 0.8 hours in 20 days or 0.04 hours/day). This means that the amount of daylight is increasing by about 0.04 hours (about $2\frac{1}{2}$ minutes) per calendar day, or that each day is $2\frac{1}{2}$ minutes longer than its predecessor.

65. (a) A possible graph is shown in Figure 2.104. At first, the yam heats up very quickly, since the difference in temperature between it and its surroundings is so large. As time goes by, the yam gets hotter and hotter, its rate of temperature increase slows down, and its temperature approaches the temperature of the oven as an asymptote. The graph is thus concave down. (We are considering the average temperature of the yam, since the temperature in its center and on its surface will vary in different ways.)

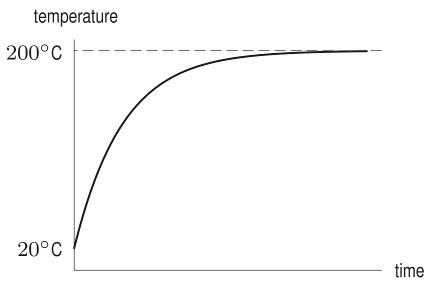


Figure 2.104

(b) If the rate of temperature increase were to remain 2°/min, in ten minutes the yam's temperature would increase 20°, from 120° to 140°. Since we know the graph is not linear, but concave down, the actual temperature is between 120° and 140°.
(c) In 30 minutes, we know the yam increases in temperature by 45° at an average rate of $45/30 = 1.5^\circ$/min. Since the graph is concave down, the temperature at $t = 40$ is therefore between $120 + 1.5(10) = 135^\circ$ and 140°.
(d) If the temperature increases at 2°/minute, it reaches 150° after 15 minutes, at $t = 45$. If the temperature increases at 1.5°/minute, it reaches 150° after 20 minutes, at $t = 50$. So t is between 45 and 50 mins.

66. (a) We construct the difference quotient using $\operatorname{erf}(0)$ and each of the other given values:

$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(1) - \operatorname{erf}(0)}{1-0} = 0.84270079$$
$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(0.1) - \operatorname{erf}(0)}{0.1-0} = 1.1246292$$
$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(0.01) - \operatorname{erf}(0)}{0.01-0} = 1.128342.$$

Based on these estimates, the best estimate is $\operatorname{erf}'(0) \approx 1.12$; the subsequent digits have not yet stabilized.
(b) Using $\operatorname{erf}(0.001)$, we have

$$\operatorname{erf}'(0) \approx \frac{\operatorname{erf}(0.001) - \operatorname{erf}(0)}{0.001-0} = 1.12838$$

and so the best estimate is now 1.1283.

67. (a)

Table 2.7

x	$\frac{\sinh(x+0.001)-\sinh(x)}{0.001}$	$\frac{\sinh(x+0.0001)-\sinh(x)}{0.0001}$	so $f'(0) \approx$	$\cosh(x)$
0	1.00000	1.00000	1.00000	1.00000
0.3	1.04549	1.04535	1.04535	1.04534
0.7	1.25555	1.25521	1.25521	1.25517
1	1.54367	1.54314	1.54314	1.54308

(b) It seems that they are approximately the same, i.e. the derivative of $\sinh(x) = \cosh(x)$ for x = 0, 0.3, 0.7, and 1.

68. (a) Since the sea level is rising, we know that $a'(t) > 0$ and $m'(t) > 0$. Since the rate is accelerating, we know that $a''(t) > 0$ and $m''(t) > 0$.
(b) The rate of change of sea level for the mid-Atlantic states is between 2 and 4, we know $2 < a'(t) < 4$. (Possibly also $a'(t) = 2$ or $a'(t) = 4$.)
Similarly, $2 < m'(t) < 10$. (Possibly also $m'(t) = 2$ or $m'(t) = 10$.)
(c) (i) If $a'(t) = 2$, then sea level rise $= 2 \cdot 100 = 200$ mm.
If $a'(t) = 4$, then sea level rise $= 4 \cdot 100 = 400$ mm.
So sea level rise is between 200 mm and 400 mm.
(ii) The shortest amount of time for the sea level in the Gulf of Mexico to rise 1 meter occurs when the rate is largest, 10 mm per year. Since 1 meter $=$ 1000 mm,
shortest time to rise 1 meter $= 1000/10 = 100$ years.

CAS Challenge Problems

69. The CAS says the derivative is zero. This can be explained by the fact that $f(x) = \sin^2 x + \cos^2 x = 1$, so $f'(x)$ is the derivative of the constant function 1. The derivative of a constant function is zero.

70. (a) The CAS gives $f'(x) = 2\cos^2 x - 2\sin^2 x$. Form of answers may vary.
(b) Using the double angle formulas for sine and cosine, we have

$$f(x) = 2\sin x \cos x = \sin(2x)$$
$$f'(x) = 2\cos^2 x - 2\sin^2 x = 2(\cos^2 x - \sin^2 x) = 2\cos(2x).$$

Thus we get

$$\frac{d}{dx}\sin(2x) = 2\cos(2x).$$

71. (a) The first derivative is $g'(x) = -2axe^{-ax^2}$, so the second derivative is

$$g''(x) = \frac{d^2}{dx^2}e^{-ax^2} = \frac{-2\,a}{e^{ax^2}} + \frac{4a^2x^2}{e^{ax^2}}.$$

Form of answers may vary.

(b) Both graphs get narrow as a gets larger; the graph of g'' is below the x-axis along the interval where g is concave down, and is above the x-axis where g is concave up. See Figure 2.105.

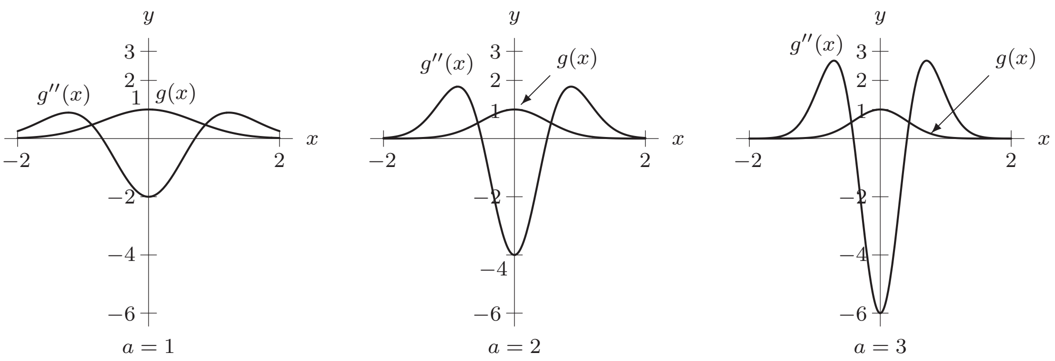


Figure 2.105

(c) The second derivative of a function is positive when the graph of the function is concave up and negative when it is concave down.

72. (a) The CAS gives the same derivative, $1/x$, in all three cases.

(b) From the properties of logarithms, $g(x) = \ln(2x) = \ln 2 + \ln x = f(x) + \ln 2$. So the graph of g is the same shape as the graph of f, only shifted up by $\ln 2$. So the graphs have the same slope everywhere, and therefore the two functions have the same derivative. By the same reasoning, $h(x) = f(x) + \ln 3$, so h and f have the same derivative as well.

73. (a) The computer algebra system gives

$$\frac{d}{dx}(x^2+1)^2 = 4x(x^2+1)$$
$$\frac{d}{dx}(x^2+1)^3 = 6x(x^2+1)^2$$
$$\frac{d}{dx}(x^2+1)^4 = 8x(x^2+1)^3$$

(b) The pattern suggests that

$$\frac{d}{dx}(x^2+1)^n = 2nx(x^2+1)^{n-1}.$$

Taking the derivative of $(x^2+1)^n$ with a CAS confirms this.

74. (a) Using a CAS, we find

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}(\sin x\cos x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1.$$

(b) The product of the derivatives of $\sin x$ and $\cos x$ is $\cos x(-\sin x) = -\cos x \sin x$. On the other hand, the derivative of the product is $\cos^2 x - \sin^2 x$, which is not the same. So no, the derivative of a product is not always equal to the product of the derivatives.

PROJECTS FOR CHAPTER TWO

1. (a) $S(0) = 12$ since the days are always 12 hours long at the equator.

(b) Since $S(0) = 12$ from part (a) and the formula gives $S(0) = a$, we have $a = 12$. Since $S(x)$ must be continuous at $x = x_0$, and the formula gives $S(x_0) = a + b\arcsin(1) = 12 + b\left(\frac{\pi}{2}\right)$ and also $S(x_0) = 24$, we must have $12 + b\left(\frac{\pi}{2}\right) = 24$ so $b\left(\frac{\pi}{2}\right) = 12$ and $b = \frac{24}{\pi} \approx 7.64$.

(c) $S(32°13') \approx 14.12$ and $S(46°4') \approx 15.58$.

(d)

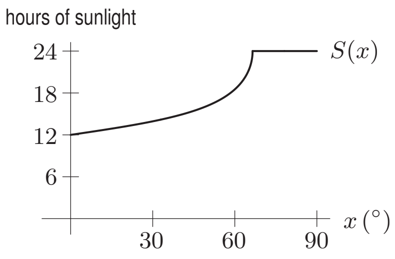


Figure 2.106

(e) The graph in Figure 2.106 appears to have a corner at $x_0 = 66°30'$. We compare the slope to the right of x_0 and to the left of x_0. To the right of S_0, the function is constant, so $S'(x) = 0$ for $x > 66°30'$.

We estimate the slope immediately to the left of x_0. We want to calculate the following:

$$\lim_{h\to 0^-} \frac{S(x_0+h) - S(x_0)}{h}.$$

We approximate it by taking $x_0 = 66.5$ and $h = -0.1,\ -0.01,\ -0.001$:

$$\frac{S(66.49) - S(66.5)}{-0.1} \approx \frac{22.3633 - 24}{-0.1} = 16.38,$$

$$\frac{S(66.499) - S(66.5)}{-0.01} \approx \frac{23.4826 - 24}{-0.01} = 51.83,$$

$$\frac{S(66.4999) - S(66.5)}{-0.001} \approx \frac{23.8370 - 24}{-0.001} = 163.9.$$

These approximations suggest that, for $x_0 = 66.5$,

$$\lim_{h\to 0^-} \frac{S(x_0+h) - S(x_0)}{h} \quad \text{does not exist.}$$

This evidence suggests that $S(x)$ is not differentiable at x_0. A proof requires the techniques found in Chapter 3.

2. (a) (i) Estimating derivatives using difference quotients (but other answers are possible):

$$P'(1900) \approx \frac{P(1910) - P(1900)}{10} = \frac{92.0 - 76.0}{10} = 1.6 \text{ million people per year}$$

$$P'(1945) \approx \frac{P(1950) - P(1940)}{10} = \frac{150.7 - 131.7}{10} = 1.9 \text{ million people per year}$$

$$P'(2000) \approx \frac{P(2000) - P(1990)}{10} = \frac{281.4 - 248.7}{10} = 3.27 \text{ million people per year}$$

(ii) The population growth rate was at its greatest at some time between 1950 and 1960.

(iii) $P'(1950) \approx \dfrac{P(1960) - P(1950)}{10} = \dfrac{179.0 - 150.7}{10} = 2.83$ million people per year,
so $P(1956) \approx P(1950) + P'(1950)(1956 - 1950) = 150.7 + 2.83(6) \approx 167.7$ million people.

(iv) If the growth rate between 2000 and 2010 was the same as the growth rate from 1990 to 2000, then the total population should be about 314 million people in 2010.

(b) (i) $f^{-1}(100)$ is the point in time when the population of the US was 100 million people (somewhere between 1910 and 1920).

(ii) The derivative of $f^{-1}(P)$ at $P = 100$ represents the ratio of change in time to change in population, and its units are years per million people. In other words, this derivative represents about how long it took for the population to increase by 1 million, when the population was 100 million.

(iii) Since the population increased by $105.7 - 92.0 = 13.7$ million people in 10 years, the average rate of increase is 1.37 million people per year. If the rate is fairly constant in that period, the amount of time it would take for an increase of 8 million people (100 million $-$ 92.0 million) would be

$$\frac{\text{8 million people}}{\text{1.37 million people/year}} \approx \text{5.8 years} \approx \text{6 years}$$

Adding this to our starting point of 1910, we estimate that the population of the US reached 100 million around 1916, i.e. $f^{-1}(100) \approx 1916$.

(iv) Since it took 10 years between 1910 and 1920 for the population to increase by $105.7 - 92.0 = 13.7$ million people, the derivative of $f^{-1}(P)$ at $P = 100$ is approximately

$$\frac{\text{10 years}}{\text{13.7 million people}} = 0.73 \text{ years/million people}$$

(c) (i) Clearly the population of the US at any instant is an integer that varies up and down every few seconds as a child is born, a person dies, or a new immigrant arrives. So $f(t)$ has "jumps;" it is not a smooth function. But these jumps are small relative to the values of f, so f appears smooth unless we zoom in very closely on its graph (to within a few seconds).

Major land acquisitions such as the Louisiana Purchase caused larger jumps in the population, but since the census is taken only every ten years and the territories acquired were rather sparsely populated, we cannot see these jumps in the census data.

(ii) We can regard rate of change of the population for a particular time t as representing an estimate of how much the population will increase during the year after time t.

(iii) Many economic indicators are treated as smooth, such as the Gross National Product, the Dow Jones Industrial Average, volumes of trading, and the price of commodities like gold. But these figures only change in increments, not continuously.