# 2 ☐ LIMITS

## 2.1 Limits of Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

   (b) The terms $a_n$ approach $8$ as $n$ becomes large. In fact, we can make $a_n$ as close to $8$ as we like by taking $n$ sufficiently large.

   (c) The terms $a_n$ become large as $n$ becomes large. In fact, we can make $a_n$ as large as we like by taking $n$ sufficiently large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

   (b) A divergent sequence is a sequence for which $\lim\limits_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

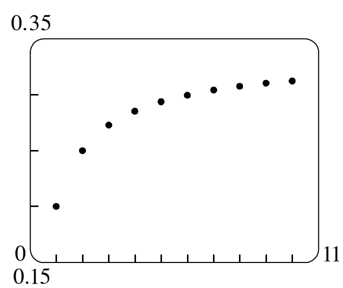
3. The graph shows a decline in the world record for the men's 100-meter sprint as $t$ increases. It is tempting to say that this sequence will approach zero, however, it is important to remember that the sequence represents data from a physical competition. Thus, the sequence likely has a nonzero limit as $t \to \infty$ since human physiology will ultimately limit how fast a human can sprint 100-meters. This means that there is a certain world record time which athletes can never surpass.

4. (a) If the sequence does not have a limit as $t \to \infty$, then the world record distances for the women's hammer throw may increase indefinitely as $t \to \infty$. That is, the sequence is divergent.

   (b) It seems unlikely that the world record hammer throw distance will increase indefinitely. Human physiology will ultimately limit the maximum distance a woman can throw. Therefore, barring evolutionary changes to human physiology, it seems likely that the sequence will converge.

5.

| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 1 | 0.2000 | 6 | 0.3000 |
| 2 | 0.2500 | 7 | 0.3043 |
| 3 | 0.2727 | 8 | 0.3077 |
| 4 | 0.2857 | 9 | 0.3103 |
| 5 | 0.2941 | 10 | 0.3125 |

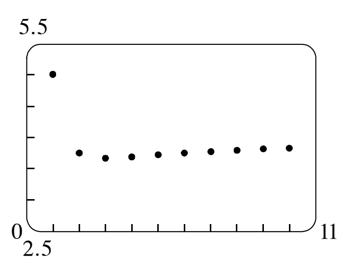



The sequence appears to converge to a number between $0.30$ and $0.35$. Calculating the limit gives

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n+3n^2} = \lim_{n\to\infty} \frac{\dfrac{n^2}{n^2}}{\dfrac{2n+3n^2}{n^2}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} \dfrac{2}{n} + \lim\limits_{n\to\infty} 3} = \frac{1}{0+3} = \frac{1}{3}.$$ This agrees with the value predicted from the data.

6.

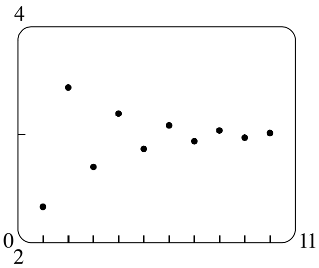
| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 1 | 5.0000 | 6 | 3.7500 |
| 2 | 3.7500 | 7 | 3.7755 |
| 3 | 3.6667 | 8 | 3.7969 |
| 4 | 3.6875 | 9 | 3.8148 |
| 5 | 3.7200 | 10 | 3.8300 |




The sequence appears to converge to a number between $3.9$ and $4.0$. Calculating the limit gives $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(4 - \dfrac{2}{n} + \dfrac{3}{n^2}\right) = 4 - 0 + 0 = 4$. So we expect the sequence to converge to 4 as we plot more terms.

7.

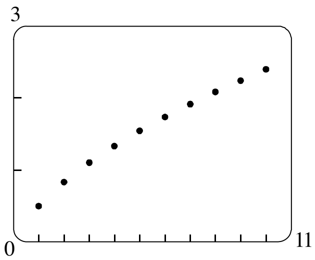
| $n$ | $a_n$ |
|---|---|
| 1 | 2.3333 |
| 2 | 3.4444 |
| 3 | 2.7037 |
| 4 | 3.1975 |
| 5 | 2.8683 |
| 6 | 3.0878 |
| 7 | 2.9415 |
| 8 | 3.0390 |
| 9 | 2.9740 |
| 10 | 3.0173 |




The sequence appears to converge to approximately 3. Calculating the limit gives $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(3 + \left(-\frac{2}{3}\right)^n\right) = 3 + 0 = 3$. This agrees with the value predicted from the data.

8.

| $n$ | $a_n$ |
|---|---|
| 1 | 0.5000 |
| 2 | 0.8284 |
| 3 | 1.0981 |
| 4 | 1.3333 |
| 5 | 1.5451 |
| 6 | 1.7394 |
| 7 | 1.9200 |
| 8 | 2.0896 |
| 9 | 2.2500 |
| 10 | 2.4025 |




The sequence does not appear to converge since the values of $a_n$ do not approach a fixed number. We can verify this by trying to calculate the limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n}+1}{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}} + \frac{1}{n}}.$$

The denominator approaches 0 while the numerator remains constant so the limit does not exist, as expected.

9. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{3n^4} = \dfrac{1}{3} \lim\limits_{n\to\infty} \dfrac{1}{n^4} = 0.$ Converges

10. $a_n = \dfrac{5}{3^n}$ is a geometric sequence with $r = \dfrac{1}{3}$. So $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{5}{3^n} = 5 \lim\limits_{n\to\infty} \left(\dfrac{1}{3}\right)^n = 5 \cdot 0 = 0$ Converges

11. $a_n = \dfrac{2n^2 + n - 1}{n^2} = 2 + \dfrac{1}{n} - \dfrac{1}{n^2}$ so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty} \dfrac{1}{n} - \lim\limits_{n\to\infty} \dfrac{1}{n^2} = 2 + 0 - 0 = 2$ Converges

12. $a_n = \dfrac{n^3 - 1}{n} = n^2 - \dfrac{1}{n}$ so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^2 - \lim\limits_{n\to\infty} \dfrac{1}{n} = \lim\limits_{n\to\infty} n^2$ When $n$ is large, $n^2$ is large so $\lim\limits_{n\to\infty} a_n = \infty$ and the sequence diverges.

13. $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3+5n}{2+7n} = \lim_{n\to\infty} \frac{\frac{3+5n}{n}}{\frac{2+7n}{n}} = \lim_{n\to\infty} \frac{\frac{3}{n}+5}{\frac{2}{n}+7} = \frac{\lim\limits_{n\to\infty} \frac{3}{n} + \lim\limits_{n\to\infty} 5}{\lim\limits_{n\to\infty} \frac{2}{n} + \lim\limits_{n\to\infty} 7} = \frac{0+5}{0+7} = \frac{5}{7}$$ Converges

14. $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3-1}{n^3+1} = \lim_{n\to\infty} \frac{\frac{n^3-1}{n^3}}{\frac{n^3+1}{n^3}} = \lim_{n\to\infty} \frac{1-\frac{1}{n^3}}{1+\frac{1}{n^3}} = \frac{\lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \frac{1}{n^3}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n^3}} = \frac{1-0}{1+0} = 1$$ Converges

15. $a_n = 1 - (0.2)^n$, so $\lim\limits_{n\to\infty} a_n = 1 - 0 = 1$ [by (3) with $r = 0.2$]. Converges

16. $a_n = 2^{-n} + 6^{-n} = \left(\frac{1}{2}\right)^n + \left(\frac{1}{6}\right)^n$ so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n + \lim\limits_{n\to\infty}\left(\frac{1}{6}\right)^n = 0 + 0 = 0$
[by (3) with $r = \frac{1}{2}$ and $r = \frac{1}{6}$] Converges

17. $a_n = \dfrac{n^2}{\sqrt{n^3+4n}} = \dfrac{n^2/\sqrt{n^3}}{\sqrt{n^3+4n}/\sqrt{n^3}} = \dfrac{\sqrt{n}}{\sqrt{1+4/n^2}}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim\limits_{n\to\infty}\sqrt{n} = \infty$ and $\lim\limits_{n\to\infty}\sqrt{1+4/n^2} = 1$. Diverges

18. $a_n = \sin(n\pi/2) \Rightarrow a_1 = \sin(\pi/2) = 1,\ a_2 = \sin(\pi) = 0,\ a_3 = \sin(3\pi/2) = -1,\ a_4 = \sin(2\pi) = 0,$ $a_5 = \sin(5\pi/2) = 1$. Observe that $a_n$ cycles between the values 1,0, and -1 as $n$ increases. Hence the sequence does not converge.

19. $a_n = \cos(n\pi/2) \Rightarrow a_1 = \cos(\pi/2) = 0,\ a_2 = \cos(\pi) = -1,\ a_3 = \cos(3\pi/2) = 0,\ a_4 = \cos(2\pi) = 1,$ $a_5 = \cos(5\pi/2) = 0$. Observe that $a_n$ cycles between the values 1, 0, and $-1$ as $n$ increases. Hence the sequence does not converge.

20. $a_n = \dfrac{\pi^n}{3^n} = \left(\dfrac{\pi}{3}\right)^n$ so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\dfrac{\pi}{3}\right)^n = \infty$ since $\dfrac{\pi}{3} \approx 1.05 > 1$ Diverges

21. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\dfrac{10^n}{1+9^n} = \lim\limits_{n\to\infty}\dfrac{\frac{10^n}{10^n}}{\frac{1+9^n}{10^n}} = \lim\limits_{n\to\infty}\dfrac{1}{\frac{1}{10^n}+\left(\frac{9}{10}\right)^n} = \dfrac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty}\left(\frac{1}{10}\right)^n + \lim\limits_{n\to\infty}\left(\frac{9}{10}\right)^n} = \infty$ because the denominator approaches 0 while the numerator remains constant. Diverges

22. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n}}{\sqrt{n}+\sqrt[4]{n}} = \lim\limits_{n\to\infty}\dfrac{n^{1/3}}{n^{1/2}+n^{1/4}} = \lim\limits_{n\to\infty}\dfrac{\frac{n^{1/3}}{n^{1/2}}}{\frac{n^{1/2}+n^{1/4}}{n^{1/2}}} = \dfrac{\lim\limits_{n\to\infty}\frac{1}{n^{1/6}}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\frac{1}{n^{1/4}}} = \dfrac{0}{1+0} = 0$

Converges

23. $a_n = \ln(2n^2+1) - \ln(n^2+1) = \ln\left(\dfrac{2n^2+1}{n^2+1}\right) = \ln\left(\dfrac{2+1/n^2}{1+1/n^2}\right) \to \ln 2$ as $n \to \infty$. Converges

24. $a_n = \dfrac{3^{n+2}}{5^n} = \dfrac{3^2 3^n}{5^n} = 9\left(\frac{3}{5}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 9\lim\limits_{n\to\infty}\left(\frac{3}{5}\right)^n = 9 \cdot 0 = 0$ by (3) with $r = \frac{3}{5}$. Converges

25. $a_n = \dfrac{e^n + e^{-n}}{e^{2n}-1} \cdot \dfrac{e^{-n}}{e^{-n}} = \dfrac{1+e^{-2n}}{e^n - e^{-n}} \to 0$ as $n \to \infty$ because $1 + e^{-2n} \to 1$ and $e^n - e^{-n} \to \infty$. Converges

26. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1+\dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Converges

27.

| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 1 | 1.0000 | 5 | 1.9375 |
| 2 | 1.5000 | 6 | 1.9688 |
| 3 | 1.7500 | 7 | 1.9844 |
| 4 | 1.8750 | 8 | 1.9922 |

The sequence appears to converge to 2. Assume the limit exists so that $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = a$, then $a_{n+1} = \frac{1}{2}a_n + 1 \Rightarrow$ $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty}\left(\frac{1}{2}a_n + 1\right) \Rightarrow a = \frac{1}{2}a + 1 \Rightarrow a = 2$

Therefore, $\lim\limits_{n\to\infty} a_n = 2$.

28.

| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 1 | 2.0000 | 5 | 0.7654 |
| 2 | 0.3333 | 6 | 0.7449 |
| 3 | 0.8889 | 7 | 0.7517 |
| 4 | 0.7037 | 8 | 0.7494 |

The sequence appears to converge to $0.75$. Assume the limit exists so that

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = a$, then $a_{n+1} = 1 - \frac{1}{3}a_n \quad \Rightarrow$

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \left(1 - \frac{1}{3}a_n\right) \quad \Rightarrow \quad a = 1 - \frac{1}{3}a \quad \Rightarrow \quad a = 3/4$

Therefore, $\lim\limits_{n\to\infty} a_n = \frac{3}{4}$.

29.

| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 1 | 2.0000 | 5 | 17.0000 |
| 2 | 3.0000 | 6 | 33.0000 |
| 3 | 5.0000 | 7 | 65.0000 |
| 4 | 9.0000 | 8 | 129.0000 |

The sequence is divergent.

30.

| $n$ | $a_n$ |
|---|---|
| 1 | 1.0000 |
| 2 | 2.2361 |
| 3 | 3.3437 |
| 4 | 4.0888 |
| 5 | 4.5215 |
| 6 | 4.7547 |
| 7 | 4.8758 |
| 8 | 4.9375 |

The sequence appears to converge to $5$. Assume the limit exists so that

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = a$, then $a_{n+1} = \sqrt{5a_n} \quad \Rightarrow \quad \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{5a_n} \quad \Rightarrow$

$a = \sqrt{5a} \quad \Rightarrow \quad a^2 = 5a \quad \Rightarrow \quad a(a-5) = 0 \quad \Rightarrow \quad a = 0$ or $a = 5$

Therefore, if the limit exists it will be either $0$ or $5$. Since the first $8$ terms of the sequence appear to approach 5, we surmise that $\lim\limits_{n\to\infty} a_n = 5$.

31.

| $n$ | $a_n$ |
|---|---|
| 1 | 1.0000 |
| 2 | 3.0000 |
| 3 | 1.5000 |
| 4 | 2.4000 |
| 5 | 1.7647 |
| 6 | 2.1702 |
| 7 | 1.8926 |
| 8 | 2.0742 |

The sequence appears to converge to $2$. Assume the limit exists so that

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = a$, then $a_{n+1} = \dfrac{6}{1+a_n} \quad \Rightarrow \quad \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{6}{1+a_n} \quad \Rightarrow$

$a = \dfrac{6}{1+a} \quad \Rightarrow \quad a^2 + a - 6 = 0 \quad \Rightarrow \quad (a-2)(a+3) = 0 \quad \Rightarrow \quad a = -3$ or $a = 2$

Therefore, if the limit exists it will be either $-3$ or $2$, but since all terms of the sequence are positive, we see that $\lim\limits_{n\to\infty} a_n = 2$.

32.

| $n$ | $a_n$ |
|---|---|
| 1 | 3.0000 |
| 2 | 5.0000 |
| 3 | 3.0000 |
| 4 | 5.0000 |
| 5 | 3.0000 |
| 6 | 5.0000 |
| 7 | 3.0000 |
| 8 | 5.0000 |

The sequence cycles between $3$ and $5$, hence it is divergent.

**33.**

| $n$ | $a_n$ |
|---|---|
| 1 | 1.0000 |
| 2 | 1.7321 |
| 3 | 1.9319 |
| 4 | 1.9829 |
| 5 | 1.9957 |
| 6 | 1.9989 |
| 7 | 1.9997 |
| 8 | 1.9999 |

The sequence appears to converge to $2$. Assume the limit exists so that

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = a$, then $a_{n+1} = \sqrt{2+a_n} \quad\Rightarrow\quad \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{2+a_n} \quad\Rightarrow$

$a = \sqrt{2+a} \quad\Rightarrow\quad a^2 - a - 2 = 0 \quad\Rightarrow\quad (a-2)(a+1) = 0 \quad\Rightarrow\quad a = -1$ or $a = 2$

Therefore, if the limit exists it will be either $-1$ or $2$, but since all terms of the sequence are positive, we see that $\lim\limits_{n\to\infty} a_n = 2$.

**34.**

| $n$ | $a_n$ |
|---|---|
| 1 | 100.0000 |
| 2 | 50.1250 |
| 3 | 25.3119 |
| 4 | 13.1498 |
| 5 | 7.5255 |
| 6 | 5.4238 |
| 7 | 5.0166 |
| 8 | 5.0000 |

The sequence appears to converge to $5$. Assume the limit exists so that

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = a$, then

$a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{25}{a_n}\right) \quad\Rightarrow\quad \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{2}\left(a_n + \dfrac{25}{a_n}\right) \quad\Rightarrow\quad a =$

$\dfrac{1}{2}\left(a + \dfrac{25}{a}\right) \quad\Rightarrow\quad 2a = a + \dfrac{25}{a} \quad\Rightarrow\quad a^2 = 25 \quad\Rightarrow\quad a = -5$ or $a = 5$

Therefore, if the limit exists it will be either $-5$ or $5$, but since all terms of the sequence are positive, we see that $\lim\limits_{n\to\infty} a_n = 5$.

**35.** (a) The quantity of the drug in the body after the first tablet is $100\ \mathrm{mg}$. After the second tablet, there is $100\ \mathrm{mg}$ plus $20\%$ of the first $100$-$\mathrm{mg}$ tablet, that is, $[100 + 100(0.20)] = 120\ \mathrm{mg}$. After the third tablet, the quantity is $[100 + 120(0.20)] = 124\ \mathrm{mg}$.

(b) After the $n^{\mathrm{th}} + 1$ tablet, there is $100\ \mathrm{mg}$ plus 20% of the $n^{\mathrm{th}}$ tablet, so that $Q_{n+1} = 100 + (0.20)\, Q_n$

(c) From Formula (6), the solution to $Q_{n+1} = 100 + (0.20)\, Q_n$, $Q_0 = 0\ \mathrm{mg}$ is

$$Q_n = (0.20)^n\,(0) + 100\left(\frac{1-0.20^n}{1-0.20}\right) = \frac{100}{0.80}\,(1-0.20^n) = 125\,(1-0.20^n)$$

(d) In the long run, we have $\lim\limits_{n\to\infty} Q_n = \lim\limits_{n\to\infty} 125\,(1-0.20^n) = 125\left(\lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} 0.20^n\right) = 125\,(1-0) = 125\ \mathrm{mg}$

**36.** (a) The concentration of the drug in the body after the first injection is 1.5 mg/mL. After the second injection, there is 1.5 mg/mL plus $10\%$ ($90\%$ reduction) of the concentration from the first injection, that is, $[1.5 + 1.5(0.10)] = 1.65$ mg/mL. After the third injection, the concentration is $[1.5 + 1.65(0.10)] = 1.665$ mg/mL.

(b) The drug concentration is $0.1C_n$ ($90\%$ reduction) just before the $n^{\mathrm{th}} + 1$ injection, after which the concentration increases by 1.5 mg/mL. Hence $C_{n+1} = 0.1C_n + 1.5$.

(c) From Formula (6), the solution to $C_{n+1} = 0.1C_n + 1.5$, $C_0 = 0$ mg/mL is

$$C_n = (0.1)^n\,(0) + 1.5\left(\frac{1-0.1^n}{1-0.1}\right) = \frac{1.5}{0.9}\,(1-0.1^n) = \frac{5}{3}\,(1-0.1^n)$$

(d) The limiting value of the concentration is

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{5}{3}\,(1-0.1^n) = \frac{5}{3}\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} 0.1^n\right) = \frac{5}{3}\,(1-0) = \frac{5}{3} \approx 1.667 \text{ mg/mL}.$$

37. (a) The quantity of the drug in the body after the first tablet is $150\ \text{mg}$. After the second tablet, there is $150\ \text{mg}$ plus $5\%$ of the first $150$-$\text{mg}$ tablet, that is, $[150 + 150(0.05)]\ \text{mg}$. After the third tablet, the quantity is $[150 + 150(0.05) + 150(0.05)^2] = 157.875\ \text{mg}$. After $n$ tablets, the quantity (in $\text{mg}$) is $150 + 150(0.05) + \cdots + 150(0.05)^{n-1}$. We can use Formula 5 to write this as $\dfrac{150(1-0.05^n)}{1-0.05} = \dfrac{3000}{19}(1-0.05^n)$.

(b) The number of milligrams remaining in the body in the long run is $\lim\limits_{n\to\infty}\left[\frac{3000}{19}(1-0.05^n)\right] = \frac{3000}{19}(1-0) \approx 157.895$, only $0.02\ \text{mg}$ more than the amount after 3 tablets.

38. (a) The residual concentration just before the second injection is $De^{-aT}$; before the third, $De^{-aT} + De^{-a2T}$; before the $(n+1)$st, $De^{-aT} + De^{-a2T} + \cdots + De^{-anT}$. This sum is equal to $\dfrac{De^{-aT}\left(1-e^{-anT}\right)}{1-e^{-aT}}$ [Formula 3].

(b) The limiting pre-injection concentration is $\lim\limits_{n\to\infty} \dfrac{De^{-aT}\left(1-e^{-anT}\right)}{1-e^{-aT}} = \dfrac{De^{-aT}(1-0)}{1-e^{-aT}} \cdot \dfrac{e^{aT}}{e^{aT}} = \dfrac{D}{e^{aT}-1}$.

(c) $\dfrac{D}{e^{aT}-1} \ge C \quad\Rightarrow\quad D \ge C\left(e^{aT}-1\right)$, so the minimal dosage is $D = C\left(e^{aT}-1\right)$.

39. (a) Many people would guess that $x < 1$, but note that $x$ consists of an infinite number of 9s.

(b) $x = 0.99999\ldots = \dfrac{9}{10} + \dfrac{9}{100} + \dfrac{9}{1000} + \dfrac{9}{10{,}000} + \cdots = \sum\limits_{n=1}^{\infty} \dfrac{9}{10^n}$, which is a geometric series with $a_1 = 0.9$ and $r = 0.1$. Its sum is $\dfrac{0.9}{1-0.1} = \dfrac{0.9}{0.9} = 1$, that is, $x = 1$.

(c) The number 1 has two decimal representations, $1.00000\ldots$ and $0.99999\ldots$.

(d) Except for 0, all rational numbers that have a terminating decimal representation can be written in more than one way. For example, $0.5$ can be written as $0.49999\ldots$ as well as $0.50000\ldots$.

40. $a_n = (5-n)a_{n-1}$, $a_1 = 1 \quad\Rightarrow\quad a_2 = (5-2)(1) = 3$, $a_3 = (5-3)(3) = 6$, $a_4 = (5-4)(6) = 6$, $a_5 = (5-5)(6) = 0$, $a_6 = (5-6)(0) = 0$, .and so on. Observe that the fifth term and higher will all be zero. So the sum of all the terms in the sequence is found by adding the first four terms: $a_1 + a_2 + a_3 + a_4 = 1 + 3 + 6 + 6 = 16$.

41. $0.\overline{8} = \dfrac{8}{10} + \dfrac{8}{10^2} + \cdots$ is a geometric series with $a = \dfrac{8}{10}$ and $r = \dfrac{1}{10}$. It converges to $\dfrac{a}{1-r} = \dfrac{8/10}{1-1/10} = \dfrac{8}{9}$.

42. $0.\overline{46} = \dfrac{46}{100} + \dfrac{46}{100^2} + \cdots$ is a geometric series with $a = \dfrac{46}{100}$ and $r = \dfrac{1}{100}$. It converges to $\dfrac{a}{1-r} = \dfrac{46/100}{1-1/100} = \dfrac{46}{99}$.

43. $2.\overline{516} = 2 + \dfrac{516}{10^3} + \dfrac{516}{10^6} + \cdots$. Now $\dfrac{516}{10^3} + \dfrac{516}{10^6} + \cdots$ is a geometric series with $a = \dfrac{516}{10^3}$ and $r = \dfrac{1}{10^3}$. It converges to $\dfrac{a}{1-r} = \dfrac{516/10^3}{1-1/10^3} = \dfrac{516/10^3}{999/10^3} = \dfrac{516}{999}$. Thus, $2.\overline{516} = 2 + \dfrac{516}{999} = \dfrac{2514}{999} = \dfrac{838}{333}$.

44. $10.1\overline{35} = 10.1 + \dfrac{35}{10^3} + \dfrac{35}{10^5} + \cdots$. Now $\dfrac{35}{10^3} + \dfrac{35}{10^5} + \cdots$ is a geometric series with $a = \dfrac{35}{10^3}$ and $r = \dfrac{1}{10^2}$. It converges to $\dfrac{a}{1-r} = \dfrac{35/10^3}{1-1/10^2} = \dfrac{35/10^3}{99/10^2} = \dfrac{35}{990}$. Thus, $10.1\overline{35} = 10.1 + \dfrac{35}{990} = \dfrac{9999+35}{990} = \dfrac{10{,}034}{990} = \dfrac{5017}{495}$.

45. $1.53\overline{42} = 1.53 + \dfrac{42}{10^4} + \dfrac{42}{10^6} + \cdots$. Now $\dfrac{42}{10^4} + \dfrac{42}{10^6} + \cdots$ is a geometric series with $a = \dfrac{42}{10^4}$ and $r = \dfrac{1}{10^2}$.

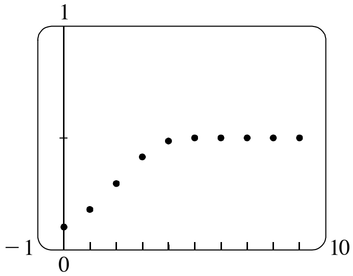
It converges to $\dfrac{a}{1-r} = \dfrac{42/10^4}{1-1/10^2} = \dfrac{42/10^4}{99/10^2} = \dfrac{42}{9900}$.

Thus, $1.53\overline{42} = 1.53 + \dfrac{42}{9900} = \dfrac{153}{100} + \dfrac{42}{9900} = \dfrac{15{,}147}{9900} + \dfrac{42}{9900} = \dfrac{15{,}189}{9900}$ or $\dfrac{5063}{3300}$.

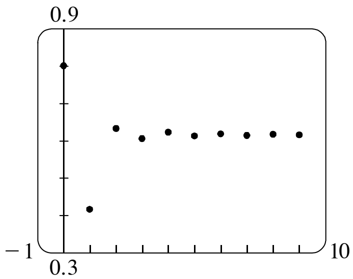
46. $7.\overline{12345} = 7 + \dfrac{12{,}345}{10^5} + \dfrac{12{,}345}{10^{10}} + \cdots$. Now $\dfrac{12{,}345}{10^5} + \dfrac{12{,}345}{10^{10}} + \cdots$ is a geometric series with $a = \dfrac{12{,}345}{10^5}$ and $r = \dfrac{1}{10^5}$.

It converges to $\dfrac{a}{1-r} = \dfrac{12{,}345/10^5}{1-1/10^5} = \dfrac{12{,}345/10^5}{99{,}999/10^5} = \dfrac{12{,}345}{99{,}999}$.

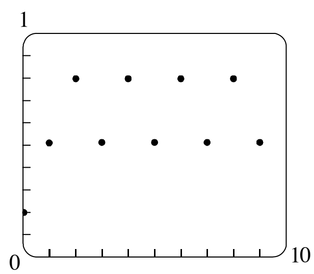
Thus, $7.\overline{12345} = 7 + \dfrac{12{,}345}{99{,}999} = \dfrac{699{,}993}{99{,}999} + \dfrac{12{,}345}{99{,}999} = \dfrac{712{,}338}{99{,}999}$ or $\dfrac{237{,}446}{33{,}333}$.

47. 

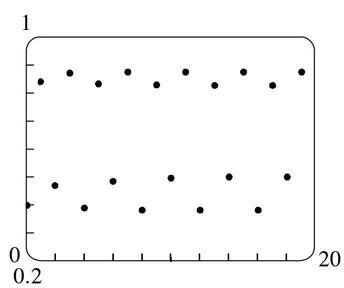

Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 2x_t(1-x_t)$, $x_0 = 0.1$. The sequence appears to converge to a value of 0.5. Assume the limit exists so that $\lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} x_t = x$, then $x_{t+1} = 2x_t(1-x_t) \;\Rightarrow\; \lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} 2x_t(1-x_t) \;\Rightarrow\; x = 2x(1-x) \;\Rightarrow\; x(1-2x) = 0 \;\Rightarrow\; x = 0$ or $x = 1/2$. Therefore, if the limit exists it will be either 0 or $\frac{1}{2}$. Since the graph of the sequence appears to approach $\frac{1}{2}$, we see that $\lim\limits_{t\to\infty} x_t = \frac{1}{2}$.

48. 


Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 2.6x_t(1-x_t)$, $x_0 = 0.8$. The sequence appears to converge to a value of 0.6. Assume the limit exists so that $\lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} x_t = x$, then $x_{t+1} = 2.6x_t(1-x_t) \;\Rightarrow\; \lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} 2.6x_t(1-x_t) \;\Rightarrow\; x = 2.6x(1-x) \;\Rightarrow\; x(1.6-2.6x) = 0 \;\Rightarrow\; x = 0$ or $x = \frac{8}{13} \approx 0.615$. Therefore, if the limit exists it will be either 0 or $\frac{8}{13}$. Since the graph of the sequence appears to approach $\frac{8}{13}$, we see that $\lim\limits_{t\to\infty} x_t = \frac{8}{13}$.

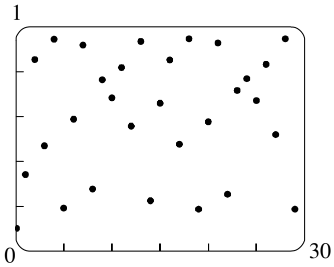
49. 


Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 3.2x_t(1-x_t)$, $x_0 = 0.2$. The sequence does not appear to converge to a fixed value. Instead, the terms oscillate between values near 0.5 and 0.8.

50. 

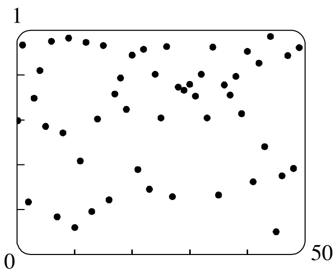

Computer software was used to plot the first 20 points of the recursion equation $x_{t+1} = 3.5x_t(1-x_t)$, $x_0 = 0.4$. The sequence does not appear to converge to a fixed value. Instead, the terms oscillate between values near 0.45 and 0.85.

51.



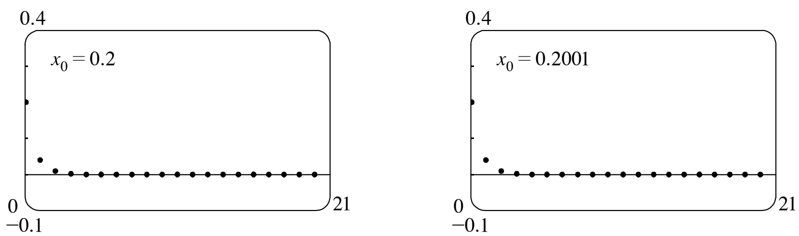

Computer software was used to plot the first 30 points of the recursion equation $x_{t+1} = 3.8x_t(1 - x_t)$, $x_0 = 0.1$. The sequence does not appear to converge to a fixed value. The terms fluctuate substantially in value exhibiting chaotic behavior.

52.

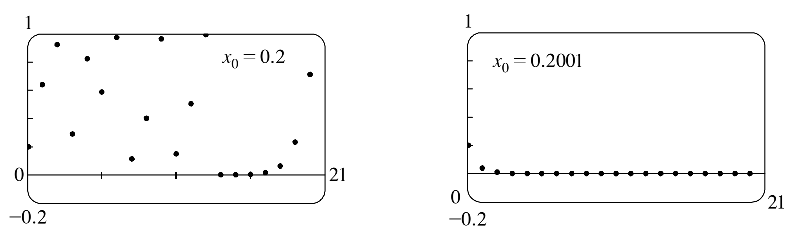



Computer software was used to plot the first 50 points of the recursion equation $x_{t+1} = 3.9x_t(1 - x_t)$, $x_0 = 0.6$. The sequence does not appear to converge to a fixed value. The terms fluctuate substantially in value exhibiting chaotic behavior.

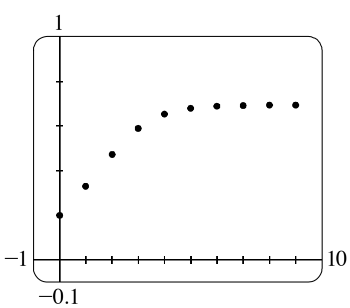
53. Computer software was used to plot the first 20 points of the recursion equation $x_{t+1} = \frac{1}{4}x_t(1 - x_t)$, with $x_0 = 0.2$ and $x_0 = 0.2001$. The plots indicate that the solutions are nearly identical, converging to zero as $t$ increases.




54. Computer software was used to plot the first 20 points of the recursion equation $x_{t+1} = 4x_t(1 - x_t)$, with $x_0 = 0.2$ and $x_0 = 0.2001$. The recursion with $x_0 = 0.2$ behaves chaotically whereas the recursion with $x_0 = 0.2001$ converges to zero. The plots indicate that a small change in initial conditions can significantly impact the behaviour of a recursive sequence.



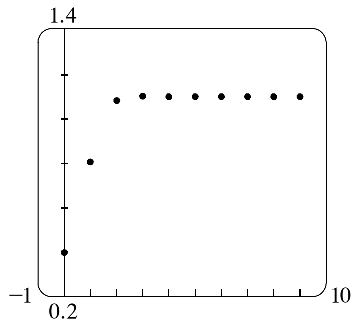

55.




Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 2x_te^{-x_t}$, $x_0 = 0.2$. The sequence appears to converge to a value near 0.7. Assume the limit exists so that $\lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} x_t = x$, then

$x_{t+1} = 2x_te^{-x_t} \quad\Rightarrow\quad \lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} 2x_te^{-x_t} \quad\Rightarrow\quad x = 2xe^{-x} \quad\Rightarrow$ $x\left(1 - 2e^{-x}\right) = 0 \quad\Rightarrow\quad x = 0$ or $x = \ln 2 \approx 0.693$. Therefore, if the limit exists it will be either 0 or $\ln 2$. Since the graph of the sequence appears to approach $\ln 2$, we see that $\lim\limits_{t\to\infty} x_t = \ln 2$.

56.

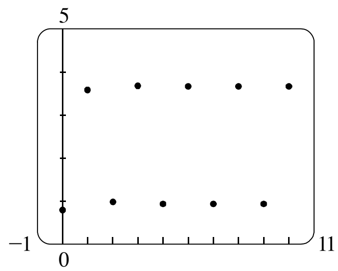



Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 3x_t e^{-x_t}$, $x_0 = 0.4$. The sequence appears to converge to a value of 1.1. Assume the limit exists so that $\lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} x_t = x$, then

$x_{t+1} = 3x_t e^{-x_t} \quad\Rightarrow\quad \lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} 3x_t e^{-x_t} \quad\Rightarrow\quad x = 3xe^{-x} \quad\Rightarrow$

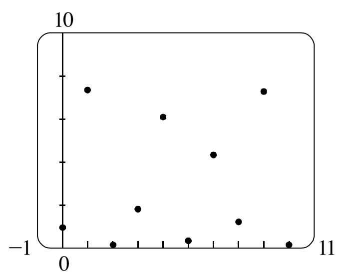
$x\left(1 - 3e^{-x}\right) = 0 \quad\Rightarrow\quad x = 0$ or $x = \ln 3 \approx 1.099$. Therefore, if the limit exists it will be either 0 or $\ln 3$. Since the graph of the sequence appears to approach $\ln 3$, we surmise that $\lim\limits_{t\to\infty} x_t = \ln 3$.

57.




Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 10x_t e^{-x_t}$, $x_0 = 0.8$. The sequence does not appear to converge to a fixed value of $x_t$. Instead, the terms oscillate between values near 0.9 and 3.7.

58.




Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 20x_t e^{-x_t}$, $x_0 = 0.9$. The sequence does not appear to converge to a fixed value of $x_t$. The terms fluctuate substantially in value exhibiting chaotic behaviour.

59. Let $A_n$ represent the removed area of the Sierpinski carpet after the $n$th step of construction. In the first step, one square of area $\frac{1}{9}$ is removed so $A_1 = \frac{1}{9}$. In the second step, 8 squares each of area $\frac{1}{9}\left(\frac{1}{9}\right) = \frac{1}{9^2}$ are removed, so

$A_2 = A_1 + \frac{8}{9^2} = \frac{1}{9} + \frac{8}{9^2} = \frac{1}{9}\left(1 + \frac{8}{9}\right)$. In the third step, 8 squares are removed for each of the 8 squares removed in the previous step. So there are a total of $8 \cdot 8 = 8^2$ squares removed each having an area of $\frac{1}{9}\left(\frac{1}{9^2}\right) = \frac{1}{9^3}$. This gives

$A_3 = A_2 + \frac{8^2}{9^3} = \frac{1}{9}\left(1 + \frac{8}{9}\right) + \frac{8}{9^2} = \frac{1}{9}\left[1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2\right]$. Observing the pattern in the first few terms of the sequence, we deduce the general formula for the $n$th term to be $A_n = \frac{1}{9}\left[1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2 + \ldots + \left(\frac{8}{9}\right)^{n-1}\right]$. The terms in parentheses represent the sum of a geometric sequence with $a = 1$ and $r = 8/9$. Using Equation (5), we can write

$A_n = \frac{1}{9}\left[\frac{1(1-(8/9)^n)}{1-8/9}\right] = 1 - \left(\frac{8}{9}\right)^n$. As $n$ increases, $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty}\left[1 - \left(\frac{8}{9}\right)^n\right] = 1$. Hence the area of the removed squares is 1 implying that the Sierpinski carpet has zero area.

60. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta$, .... Therefore,

$|CD| + |DE| + |EF| + |FG| + \cdots = b\sum_{n=1}^{\infty}\sin^n\theta = b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with $r = \sin\theta$

and $|\sin\theta| < 1$ $\quad\left[\text{because } 0 < \theta < \frac{\pi}{2}\right]$.

## PROJECT Modeling the Dynamics of Viral Infections

1. Viral replication is an example of exponential growth. The exponential growth recursion formula is $N(t+1) = RN(t)$ where $R$ is the growth rate and $N(t)$ is the number of viral particles at time $t$. In Section 1.6, we saw the general solution of this recursion is $N_t = N_0 \cdot R^t$. With $R = 3$ and $N_0 = 1$, the recursion equation is $N_{t+1} = 3N_t$ and the general solution is $N_t = 3^t$.

2. Let $t_1$ be the amount of time spent in phase 1 of the infection. Solving for $t_1$ in the equation $N_{t_1} = N_0 \cdot R^{t_1}$ using logarithms: $\ln\left(R^{t_1}\right) = \ln\left(N_{t_1}/N_0\right) \;\Rightarrow\; t_1 = \dfrac{\ln(N_{t_1}/N_0)}{\ln(R)}$. The immune response initiates when $N_{t_1} = 2 \cdot 10^6$. Therefore the time it takes for the immune response to kick in is $t_1 = \dfrac{\ln(2\cdot 10^6) - \ln(N_0)}{\ln(3)} \approx 13.2 - 0.91\ln(N_0)$. Hence, the larger the initial viral size the sooner the immune system responds.

3. Let $t_2$ be the amount of time since the immune response initiated, $R_{\text{immune}}$ be the replication rate during the immune response, and $d_{\text{immune}}$ be the number of viruses killed by the immune system at each timestep. The second phase of the infection is modeled by a two-step recursion. First, the virus replicates producing $N^* = R_{\text{immune}}N_{t_2}$ viruses. Then, the immune system kills viruses leaving $N_{t_2+1} = N^* - d_{\text{immune}}$ leftover. Combining the two steps gives the recursion formula $N_{t_2+1} = R_{\text{immune}}N_{t_2} - d_{\text{immune}}$.

4. The viral population will decrease over time if $\Delta N < 0$ at each timestep. Solving this inequality for $N_{t_2}$:

$$N_{t_2+1} - N_{t_2} < 0 \quad\Rightarrow\quad (R_{\text{immune}} - 1)N_{t_2} - d_{\text{immune}} < 0 \quad\Rightarrow\quad N_{t_2} < \frac{d_{\text{immune}}}{(R_{\text{immune}} - 1)}$$ where we assumed $R_{\text{immune}} > 1$.

Substituting the constants $R_{\text{immune}} = \frac{1}{2}\cdot 3 = 1.5$ and $d_{\text{immune}} = 500,000$ gives $N_{t_2} < 1,000,000$. Therefore, the immune response will cause the infection to subside over time if the viral count is less than one million. This is not possible since the immune response initiates only once the virus reaches two million copies.

5. The recursion for the third phase can be obtained from the second phase recursion formula by replacing the replication and death rates with the new values. This gives $N_{t_3+1} = R_{\text{drug}}N_{t_3} - d_{\text{drug}}$ where $t_3$ is the amount of time since the start of drug treatment.

6. Similar to Problem 4, we solve for $N_{t_3}$ in the inequality $\Delta N = N_{t_3+1} - N_{t_3} < 0$ and find that $N_{t_3} < \dfrac{d_{\text{drug}}}{(R_{\text{drug}} - 1)}$.

Substituting the constants $R_{\text{drug}} = 1.25$ and $d_{\text{drug}} = 25,000,000$ gives $N_{t_3} < 100,000,000$. Therefore, the drug and immune system will cause the infection to subside over time if the viral count is less than 100 million. This is possible provided drug treatment begins before the viral count reaches 100 million.

7. From Formula (6), the general solution to the recursion equation $N_{t_2+1} = R_{\text{immune}} N_{t_2} - d_{\text{immune}}$ is given by $N_{t_2} = R_{\text{immune}}^{t_2} N_0 - d_{\text{immune}} \left( \dfrac{1 - R_{\text{immune}}^{t_2}}{1 - R_{\text{immune}}} \right)$. Solving for $t_2$ in this expression gives

$N_{t_2} = R_{\text{immune}}^{t_2} \left( N_0 + \dfrac{d_{\text{immune}}}{1 - R_{\text{immune}}} \right) - \dfrac{d_{\text{immune}}}{1 - R_{\text{immune}}} \quad \Rightarrow \quad R_{\text{immune}}^{t_2} = \dfrac{N_{t_2} + d_{\text{immune}} (1 - R_{\text{immune}})^{-1}}{N_0 + d_{\text{immune}} (1 - R_{\text{immune}})^{-1}} \quad \Rightarrow$

$t_2 = \ln \left[ \dfrac{N_{t_2} + d_{\text{immune}} (1 - R_{\text{immune}})^{-1}}{N_0 + d_{\text{immune}} (1 - R_{\text{immune}})^{-1}} \right] \Big/ \ln R_{\text{immune}}$. Note that the number of viral particles at the start of phase two is $N_0 = 2 \cdot 10^6$. Substituting $R_{\text{immune}} = 1.5$, $d_{\text{immune}} = 500,000$ and the critical viral load $N_{t_2} = 100 \cdot 10^6$ into the equation gives $t_2 = \dfrac{\ln(99)}{\ln(1.5)} \approx 11.33\,\text{h}$. This is the amount of time spent in phase two after which the infection cannot be controlled. From Problem 2, phase two begins after $t_1 = \dfrac{\ln(2 \cdot 10^6) - \ln(1)}{\ln(3)} \approx 13.21\,\text{h}$. Thus, the total time is $t = t_1 + t_2 \approx 24.54\,\text{h}$. Hence, drug treatment must be started within approximately one day (24 hours) of the initial infection in order to control the viral count.

8. A general expression for the time it takes to reach the critical viral load is obtained by combining the expressions for $t_1$ and $t_2$ from Problems 2 and 7. This gives $t = t_1 + t_2 = \dfrac{\ln(2 \cdot 10^6)}{\ln(R)} - \dfrac{\ln(N_0)}{\ln(R)} + \dfrac{\ln \left[ \dfrac{N_{t_2} + d_{\text{immune}} (1 - R_{\text{immune}})^{-1}}{2 \cdot 10^6 + d_{\text{immune}} (1 - R_{\text{immune}})^{-1}} \right]}{\ln R_{\text{immune}}}$.

Substituting $R_{\text{immune}} = 0.5R$, $d_{\text{immune}} = 5 \cdot 10^5$, $N_0 = n_0$ and $N_{t_2} = 100 \cdot 10^6$ gives

$t = \dfrac{\ln(2 \cdot 10^6)}{\ln(R)} - \dfrac{\ln(n_0)}{\ln(R)} + \dfrac{\ln \left[ \dfrac{100 \cdot 10^6 + (5 \cdot 10^5)(1 - 0.5R)^{-1}}{2 \cdot 10^6 + (5 \cdot 10^5)(1 - 0.5R)^{-1}} \right]}{\ln(0.5R)}$. Note: We have inherently assumed that $n_0 < 2 \cdot 10^6$, so that some time is spent in phase 1.

9. After 24 hours, the infection has been in the immune response phase for $t_2 = 24 - 13.21 = 10.79\,\text{h}$.
Using the general expression for $N_{t_2}$ from Problem 7 the number of viruses after 24 hours is

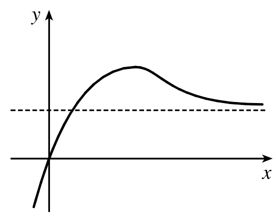
$N_{10.79} = (1.5^{10.79})(2 \cdot 10^6) - (5 \cdot 10^5) \left( \dfrac{1 - 1.5^{10.79}}{1 - 1.5} \right) \approx 80,555,008$. Since this is less than the critical viral load (100 million), drug intervention will be effective in controlling the virus. Rewriting the equation for $t_2$ for the drug phase gives

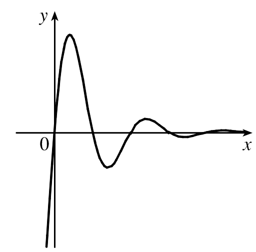
$t_3 = \ln \left[ \dfrac{N_{t_3} + d_{\text{drug}} (1 - R_{\text{drug}})^{-1}}{N_0 + d_{\text{drug}} (1 - R_{\text{drug}})^{-1}} \right] \Big/ \ln R_{\text{drug}}$ where $t_3$ is the amount of time since the drug treatment started. Substituting values $N_{t_3} = 0$, $N_0 = 80,555,008$, $R_{\text{drug}} = 1.25$ and $d_{\text{drug}} = 25,000,000$ yields $t_3 = 7.34\,\text{h}$. Therefore, it takes approximately 7 hours after starting the drug treatment to completely eliminate the virus.

## 2.2 Limits of Functions at Infinity

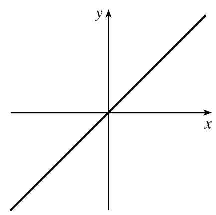
1. (a) As $x$ becomes large, the values of $f(x)$ approach 5.
   (b) As $x$ becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

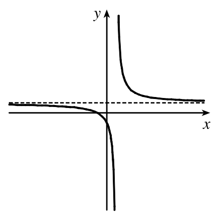


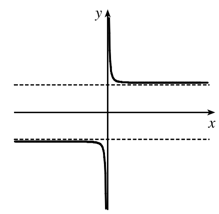

(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.




No horizontal asymptote




One horizontal asymptote




Two horizontal asymptotes

3. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.

It appears that $\lim\limits_{x\to\infty} \left(x^2/2^x\right) = 0$.

4. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b)

| $x$ | $f(x)$ |
|---|---|
| 10,000 | 0.135308 |
| 100,000 | 0.135333 |
| 1,000,000 | 0.135335 |

From the table, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

5. $$\lim_{x\to\infty} \frac{1}{2x+3} = \lim_{x\to\infty} \frac{1/x}{(2x+3)/x} = \frac{\lim\limits_{x\to\infty} (1/x)}{\lim\limits_{x\to\infty} (2+3/x)} = \frac{\lim\limits_{x\to\infty} (1/x)}{\lim\limits_{x\to\infty} 2 + 3 \lim\limits_{x\to\infty} (1/x)} = \frac{0}{2+3(0)} = \frac{0}{2} = 0$$

6. $$\lim_{x\to\infty} \frac{3x+5}{x-4} = \lim_{x\to\infty} \frac{(3x+5)/x}{(x-4)/x} = \lim_{x\to\infty} \frac{3+5/x}{1-4/x} = \frac{\lim\limits_{x\to\infty} 3 + 5 \lim\limits_{x\to\infty} \dfrac{1}{x}}{\lim\limits_{x\to\infty} 1 - 4 \lim\limits_{x\to\infty} \dfrac{1}{x}} = \frac{3+5(0)}{1-4(0)} = 3$$

7. $$\lim_{x\to\infty} \frac{3x-2}{2x+1} = \lim_{x\to\infty} \frac{(3x-2)/x}{(2x+1)/x} = \lim_{x\to\infty} \frac{3-2/x}{2+1/x} = \frac{\lim\limits_{x\to\infty} 3 - 2 \lim\limits_{x\to\infty} 1/x}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} 1/x} = \frac{3-2(0)}{2+0} = \frac{3}{2}$$

8. $$\lim_{x\to\infty} \frac{1-x^2}{x^3-x+1} = \lim_{x\to\infty} \frac{(1-x^2)/x^3}{(x^3-x+1)/x^3} = \lim_{x\to\infty} \frac{1/x^3 - 1/x}{1 - 1/x^2 + 1/x^3}$$
$$= \frac{\lim\limits_{x\to\infty} 1/x^3 - \lim\limits_{x\to\infty} 1/x}{\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} 1/x^2 + \lim\limits_{x\to\infty} 1/x^3} = \frac{0-0}{1-0+0} = 0$$

9. $\displaystyle\lim_{x\to-\infty}\frac{1-x-x^2}{2x^2-7}=\lim_{x\to-\infty}\frac{(1-x-x^2)/x^2}{(2x^2-7)/x^2}=\frac{\lim_{x\to-\infty}(1/x^2-1/x-1)}{\lim_{x\to-\infty}(2-7/x^2)}$

$$=\frac{\lim_{x\to-\infty}(1/x^2)-\lim_{x\to-\infty}(1/x)-\lim_{x\to-\infty}1}{\lim_{x\to-\infty}2-7\lim_{x\to-\infty}(1/x^2)}=\frac{0-0-1}{2-7(0)}=-\frac{1}{2}$$

10. $\displaystyle\lim_{x\to-\infty}\frac{4x^3+6x^2-2}{2x^3-4x+5}=\lim_{x\to-\infty}\frac{(4x^3+6x^2-2)/x^3}{(2x^3-4x+5)/x^3}=\lim_{x\to-\infty}\frac{4+6/x-2/x^3}{2-4/x^2+5/x^3}=\frac{4+0-0}{2-0+0}=2$

11. $\displaystyle\lim_{t\to-\infty}0.6^t=\lim_{t\to-\infty}\left(\frac{3}{5}\right)^t=\lim_{t\to-\infty}\left(\frac{5}{3}\right)^{-t}=\infty$ since $5/3>1$ and $-t\to\infty$ as $t\to-\infty$

12. $\displaystyle\lim_{r\to\infty}\frac{5}{10^r}=0$ since $10^r\to\infty$ as $r\to\infty$

13. $\displaystyle\lim_{t\to\infty}\frac{\sqrt{t}+t^2}{2t-t^2}=\lim_{t\to\infty}\frac{(\sqrt{t}+t^2)/t^2}{(2t-t^2)/t^2}=\lim_{t\to\infty}\frac{1/t^{3/2}+1}{2/t-1}=\frac{0+1}{0-1}=-1$

14. $\displaystyle\lim_{t\to\infty}\frac{t-t\sqrt{t}}{2t^{3/2}+3t-5}=\lim_{t\to\infty}\frac{\left(t-t\sqrt{t}\right)/t^{3/2}}{\left(2t^{3/2}+3t-5\right)/t^{3/2}}=\lim_{t\to\infty}\frac{1/t^{1/2}-1}{2+3/t^{1/2}-5/t^{3/2}}=\frac{0-1}{2+0-0}=-\frac{1}{2}$

15. $\displaystyle\lim_{x\to\infty}\frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}=\lim_{x\to\infty}\frac{(2x^2+1)^2/x^4}{[(x-1)^2(x^2+x)]/x^4}=\lim_{x\to\infty}\frac{[(2x^2+1)/x^2]^2}{[(x^2-2x+1)/x^2][(x^2+x)/x^2]}$

$$=\lim_{x\to\infty}\frac{(2+1/x^2)^2}{(1-2/x+1/x^2)(1+1/x)}=\frac{(2+0)^2}{(1-0+0)(1+0)}=4$$

16. $\displaystyle\lim_{x\to\infty}\frac{x^2}{\sqrt{x^4+1}}=\lim_{x\to\infty}\frac{x^2/x^2}{\sqrt{x^4+1}/x^2}=\lim_{x\to\infty}\frac{1}{\sqrt{(x^4+1)/x^4}}$ [since $x^2=\sqrt{x^4}$ for $x>0$]

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^4}}=\frac{1}{\sqrt{1+0}}=1$$

17. $\displaystyle\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$

$$=\lim_{x\to\infty}\frac{\left(9x^2+x\right)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

18. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$$=\lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

19. $\displaystyle\lim_{x\to\infty}\frac{6}{3+e^{-2x}}=\frac{6}{3+\lim_{x\to\infty}e^{-2x}}=\frac{6}{3+0}=\frac{6}{3}=2$

20. For $x>0$, $\sqrt{x^2+1}>\sqrt{x^2}=x$. So as $x\to\infty$, we have $\sqrt{x^2+1}\to\infty$, that is, $\lim_{x\to\infty}\sqrt{x^2+1}=\infty$.

21. $\lim\limits_{x\to\infty}\dfrac{x^4-3x^2+x}{x^3-x+2}=\lim\limits_{x\to\infty}\dfrac{(x^4-3x^2+x)/x^3}{(x^3-x+2)/x^3}$ $\left[\begin{array}{c}\text{divide by the highest power}\\ \text{of } x \text{ in the denominator}\end{array}\right]$ $=\lim\limits_{x\to\infty}\dfrac{x-3/x+1/x^2}{1-1/x^2+2/x^3}=\infty$

since the numerator increases without bound and the denominator approaches 1 as $x\to\infty$.

22. $\lim\limits_{x\to\infty}(e^{-x}+2\cos 3x)$ does not exist. $\lim\limits_{x\to\infty}e^{-x}=0$, but $\lim\limits_{x\to\infty}(2\cos 3x)$ does not exist because the values of $2\cos 3x$ oscillate between the values of $-2$ and 2 infinitely often, so the given limit does not exist.

23. $\lim\limits_{x\to-\infty}(x^4+x^5)=\lim\limits_{x\to-\infty}x^5(\frac{1}{x}+1)$ [factor out the largest power of $x$] $=-\infty$ because $x^5\to-\infty$ and $1/x+1\to1$ as $x\to-\infty$.

*Or:* $\lim\limits_{x\to-\infty}\left(x^4+x^5\right)=\lim\limits_{x\to-\infty}x^4\left(1+x\right)=-\infty$.

24. $\lim\limits_{x\to-\infty}\dfrac{1+x^6}{x^4+1}=\lim\limits_{x\to-\infty}\dfrac{(1+x^6)/x^4}{(x^4+1)/x^4}$ $\left[\begin{array}{c}\text{divide by the highest power}\\ \text{of } x \text{ in the denominator}\end{array}\right]$ $=\lim\limits_{x\to-\infty}\dfrac{1/x^4+x^2}{1+1/x^4}=\infty$

since the numerator increases without bound and the denominator approaches 1 as $x\to-\infty$.

25. As $t$ increases, $1/t^2$ approaches zero, so $\lim\limits_{t\to\infty}e^{-1/t^2}=e^{-(0)}=1$

26. Divide numerator and denominator by $e^{3x}$: $\lim\limits_{x\to\infty}\dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim\limits_{x\to\infty}\dfrac{1-e^{-6x}}{1+e^{-6x}}=\dfrac{1-0}{1+0}=1$

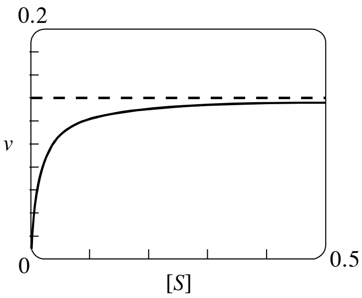
27. $\lim\limits_{x\to\infty}\dfrac{1-e^x}{1+2e^x}=\lim\limits_{x\to\infty}\dfrac{(1-e^x)/e^x}{(1+2e^x)/e^x}=\lim\limits_{x\to\infty}\dfrac{1/e^x-1}{1/e^x+2}=\dfrac{0-1}{0+2}=-\dfrac{1}{2}$

28. $\lim\limits_{x\to-\infty}\left[\ln(x^2)-\ln(x^2+1)\right]=\lim\limits_{x\to-\infty}\left[\ln\left(\dfrac{x^2}{x^2+1}\right)\right]=\lim\limits_{x\to-\infty}\left[\ln\left(\dfrac{1}{1+1/x^2}\right)\right]=\ln\left(\dfrac{1}{1+0}\right)=\ln(1)=0$

29. $R(N)=SN/(c+N)\quad\Rightarrow\quad R(c)=Sc/(c+c)=S/2$. Hence, $c$ is the nutrient concentration at which the growth rate is half of the maximum possible value. This is often referred to as the half-saturation constant.

30. (a) $\lim\limits_{[\mathrm{S}]\to\infty}v=\lim\limits_{[\mathrm{S}]\to\infty}\dfrac{0.14[\mathrm{S}]}{0.015+[\mathrm{S}]}=\lim\limits_{[\mathrm{S}]\to\infty}\dfrac{0.14}{0.015/[\mathrm{S}]+1}$ $\left[\begin{array}{c}\text{divide numerator and}\\ \text{denominator by [S]}\end{array}\right]$ $=\dfrac{0.14}{0+1}=0.14$. So the line $v=0.14$ is a horizontal asymptote. Therefore, as the concentration increases, the enzymatic reaction rate will approach 0.14. Note, we did not need to consider the limit as [S]$\to-\infty$ because concentrations must be positive in value.

(b)

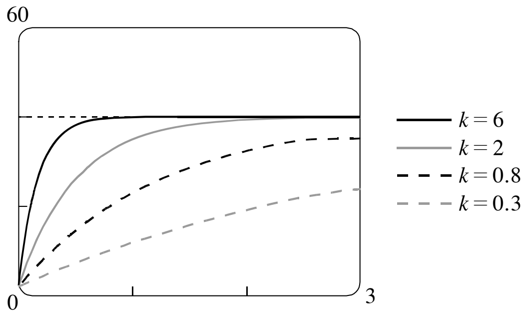



31. $\lim\limits_{v\to\infty}N(v)=\lim\limits_{v\to\infty}\dfrac{8v}{1+2v+v^2}=\lim\limits_{v\to\infty}\dfrac{(8v)/v^2}{(1+2v+v^2)/v^2}$ $\left[\begin{array}{c}\text{divide by the highest power}\\ \text{of } v \text{ in the denominator}\end{array}\right]$

$$=\lim_{v\to\infty}\frac{8/v}{1/v^2+2/v+1}=\frac{0}{0+0+1}=0$$

Therefore, as the mortality rate increases, the number of new infections approaches zero.

32. (a) $\lim_{t\to\infty} L(t) = \lim_{t\to\infty}\left[L_\infty - (L_\infty - L_0)e^{-kt}\right] = L_\infty - (L_\infty - L_0)(0) = L_\infty$. Therefore, as the fish ages the mean length approaches $L_\infty$.

(b) 


The constant $k$ affects the rate at which the function approaches the horizontal asymptote $L_\infty$. Increasing $k$ causes the Bertalanffy growth function to approach the horizontal asymptote at a faster rate.

33. $B(t) = \dfrac{8\times10^7}{1+3e^{-0.71t}} \quad\Rightarrow\quad \lim_{t\to\infty} B(t) = \lim_{t\to\infty}\dfrac{8\times10^7}{1+3e^{-0.71t}} = \dfrac{8\times10^7}{1+0} = 8\times10^7$. This means that in the long run the biomass of the Pacific halibut will tend to $8\times10^7$ kg.

34. (a) After $t$ minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000+25t)$ liters of water and $25t\cdot30 = 750t$ grams of salt. Therefore, the salt concentration at time $t$ will be $C(t) = \dfrac{750t}{5000+25t} = \dfrac{30t}{200+t}\ \dfrac{\text{g}}{\text{L}}$.

(b) $\lim_{t\to\infty} C(t) = \lim_{t\to\infty}\dfrac{30t}{200+t} = \lim_{t\to\infty}\dfrac{30t/t}{200/t+t/t} = \dfrac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

35. $e^{-x} < 0.0001 \quad\Rightarrow\quad \ln\left(e^{-x}\right) < \ln(0.0001) \quad\Rightarrow\quad -x < \ln(0.0001) \quad\Rightarrow\quad x > -\ln(0.0001) \approx 9.21$, so $x$ must be bigger than 9.21.

36. $f(x) = \dfrac{x}{x+1} \quad\Rightarrow\quad \lim_{x\to\infty} f(x) = \lim_{n\to\infty}\dfrac{x}{x+1} = \lim_{n\to\infty}\dfrac{\frac{x}{x}}{\frac{x+1}{x}} = \lim_{n\to\infty}\dfrac{1}{1+\frac{1}{x}} = \dfrac{1}{1+0} = 1$

$f(x) > 0.99 \quad\Rightarrow\quad \dfrac{x}{x+1} > 0.99 \quad\Rightarrow\quad x > 0.99(x+1) \quad\Rightarrow\quad 0.01x > 0.99 \quad\Rightarrow\quad x > 0.99/0.01 \quad\Rightarrow\quad x > 99$

37. (a) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} v^*\left(1-e^{-gt/v^*}\right) = v^*(1-0) = v^*$

(b) Substituting the values $v^* = 7.5$ and $v(t) = 0.99 * 7.5$ into the velocity function gives $0.99*7.5 = (7.5)(1-e^{-gt/v^*})$ $\Rightarrow\quad 0.99(7.5) = (7.5)\left(1-e^{-t(9.8)/(7.5)}\right) \quad\Rightarrow\quad e^{-t(9.8)/(7.5)} = 0.01 \quad\Rightarrow\quad t = -\dfrac{(7.5)\ln(0.01)}{9.8} = 3.52\,\text{s}$

## 2.3 Limits of Functions at Finite Numbers

1. As $x$ approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking $x$ sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As $x$ approaches 1 from the left, $f(x)$ approaches 3; and as $x$ approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim_{x\to-3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking $x$ sufficiently close to $-3$ (but not equal to $-3$).

(b) $\lim_{x\to4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking $x$ sufficiently close to 4 through values larger than 4.

4. (a) As $x$ approaches 2 from the left, the values of $f(x)$ approach 3, so $\lim\limits_{x\to 2^-} f(x) = 3$.

(b) As $x$ approaches 2 from the right, the values of $f(x)$ approach 1, so $\lim\limits_{x\to 2^+} f(x) = 1$.

(c) $\lim\limits_{x\to 2} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(d) When $x = 2$, $y = 3$, so $f(2) = 3$.

(e) As $x$ approaches 4, the values of $f(x)$ approach 4, so $\lim\limits_{x\to 4} f(x) = 4$.

(f) There is no value of $f(x)$ when $x = 4$, so $f(4)$ does not exist.

5. (a) As $x$ approaches 1, the values of $f(x)$ approach 2, so $\lim\limits_{x\to 1} f(x) = 2$.

(b) As $x$ approaches 3 from the left, the values of $f(x)$ approach 1, so $\lim\limits_{x\to 3^-} f(x) = 1$.

(c) As $x$ approaches 3 from the right, the values of $f(x)$ approach 4, so $\lim\limits_{x\to 3^+} f(x) = 4$.

(d) $\lim\limits_{x\to 3} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(e) When $x = 3$, $y = 3$, so $f(3) = 3$.

6. (a) $h(x)$ approaches 4 as $x$ approaches $-3$ from the left, so $\lim\limits_{x\to -3^-} h(x) = 4$.

(b) $h(x)$ approaches 4 as $x$ approaches $-3$ from the right, so $\lim\limits_{x\to -3^+} h(x) = 4$.

(c) $\lim\limits_{x\to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

(d) $h(-3)$ is not defined, so it doesn't exist.

(e) $h(x)$ approaches 1 as $x$ approaches 0 from the left, so $\lim\limits_{x\to 0^-} h(x) = 1$.

(f) $h(x)$ approaches $-1$ as $x$ approaches 0 from the right, so $\lim\limits_{x\to 0^+} h(x) = -1$.

(g) $\lim\limits_{x\to 0} h(x)$ does not exist because the limits in part (e) and part (f) are not equal.

(h) $h(0) = 1$ since the point $(0, 1)$ is on the graph of $h$.

(i) Since $\lim\limits_{x\to 2^-} h(x) = 2$ and $\lim\limits_{x\to 2^+} h(x) = 2$, we have $\lim\limits_{x\to 2} h(x) = 2$.

(j) $h(2)$ is not defined, so it doesn't exist.

7. (a) $P(t)$ approaches 260 as $x$ approaches 2 from the left, so $\lim\limits_{t\to 2^-} P(t) = 260$.

(b) $P(t)$ approaches 254 as $x$ approaches 2 from the right, so $\lim\limits_{t\to 2^+} P(t) = 254$.

(c) $\lim\limits_{t\to 2} P(t)$ does not exist because $\lim\limits_{t\to 2^-} P(t) \neq \lim\limits_{t\to 2^+} P(t)$.

(d) $P(t)$ approaches 254 as $x$ approaches 4 from the left, so $\lim\limits_{t\to 4^-} P(t) = 254$.

(e) $P(t)$ approaches 258 as $x$ approaches 4 from the right, so $\lim\limits_{t\to 4^+} P(t) = 258$.

(f) $\lim\limits_{t\to 4} P(t)$ does not exist because $\lim\limits_{t\to 4^-} P(t) \neq \lim\limits_{t\to 4^+} P(t)$.

(g) $\lim\limits_{x\to 5} P(t) = 258$ because $\lim\limits_{x\to 5^-} P(t) = 258 = \lim\limits_{x\to 5^+} P(t)$

(h) On June 3 ($t = 2$), the population decreased by 6. This could have been a result of deaths, emigration, or a combination of the two. On June 5 ($t = 4$), the population increased by 4. This could have been a result of births, immigration, or a combination of the two.

8. (a) $\lim\limits_{x\to 2} R(x) = -\infty$ (b) $\lim\limits_{x\to 5} R(x) = \infty$ (c) $\lim\limits_{x\to -3^-} R(x) = -\infty$ (d) $\lim\limits_{x\to -3^+} R(x) = \infty$

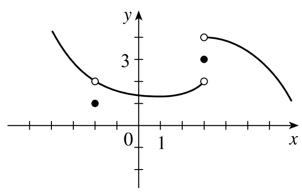
(e) The equations of the vertical asymptotes are $x = -3$, $x = 2$, and $x = 5$.

9. (a) $\lim\limits_{x\to 0} g(x) = -\infty$ (b) $\lim\limits_{x\to 2^-} g(x) = -\infty$ (c) $\lim\limits_{x\to 2^+} g(x) = \infty$

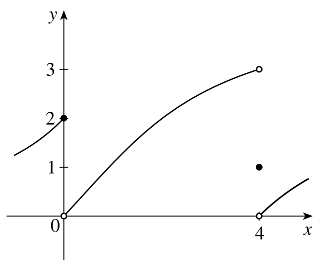
(d) $\lim\limits_{x\to \infty} g(x) = 2$ (e) $\lim\limits_{x\to -\infty} g(x) = -1$ (f) Vertical: $x = 0$, $x = 2$; horizontal: $y = -1$, $y = 2$

10. $\lim\limits_{t\to 12^-} f(t) = 150$ mg and $\lim\limits_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

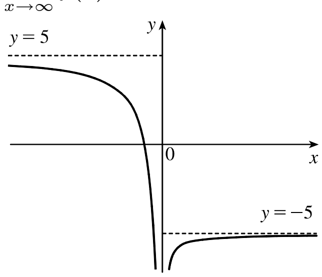
11. $\lim\limits_{x\to 3^+} f(x) = 4$, $\lim\limits_{x\to 3^-} f(x) = 2$, $\lim\limits_{x\to -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$



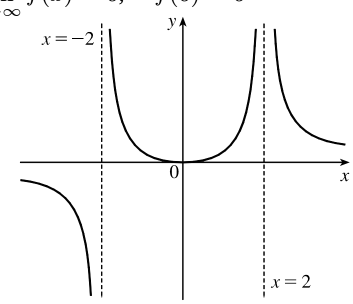
12. $\lim\limits_{x\to 0^-} f(x) = 2$, $\lim\limits_{x\to 0^+} f(x) = 0$, $\lim\limits_{x\to 4^-} f(x) = 3$, $\lim\limits_{x\to 4^+} f(x) = 0$, $f(0) = 2$, $f(4) = 1$



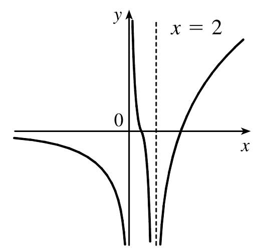
13. $\lim\limits_{x\to 0} f(x) = -\infty$, $\lim\limits_{x\to -\infty} f(x) = 5$, $\lim\limits_{x\to \infty} f(x) = -5$



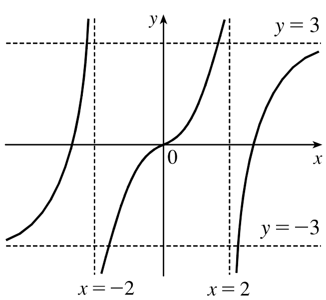

14. $\lim\limits_{x\to 2} f(x) = \infty$, $\lim\limits_{x\to -2^+} f(x) = \infty$, $\lim\limits_{x\to -2^-} f(x) = -\infty$, $\lim\limits_{x\to -\infty} f(x) = 0$, $\lim\limits_{x\to \infty} f(x) = 0$, $f(0) = 0$



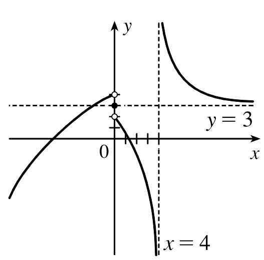

15. $\lim\limits_{x\to 2} f(x) = -\infty$, $\lim\limits_{x\to \infty} f(x) = \infty$, $\lim\limits_{x\to -\infty} f(x) = 0$, $\lim\limits_{x\to 0^+} f(x) = \infty$, $\lim\limits_{x\to 0^-} f(x) = -\infty$



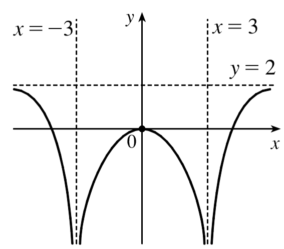

16. $\lim\limits_{x\to \infty} f(x) = 3$, $\lim\limits_{x\to 2^-} f(x) = \infty$, $\lim\limits_{x\to 2^+} f(x) = -\infty$, $f$ is odd




17. $f(0) = 3$, $\lim\limits_{x\to 0^-} f(x) = 4$, $\lim\limits_{x\to 0^+} f(x) = 2$, $\lim\limits_{x\to -\infty} f(x) = -\infty$, $\lim\limits_{x\to 4^-} f(x) = -\infty$, $\lim\limits_{x\to 4^+} f(x) = \infty$, $\lim\limits_{x\to \infty} f(x) = 3$




18. $\lim\limits_{x\to 3} f(x) = -\infty$, $\lim\limits_{x\to \infty} f(x) = 2$, $f(0) = 0$, $f$ is even



19. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

| $x$ | $f(x)$ |
|---|---|
| 2.5 | 0.714286 |
| 2.1 | 0.677419 |
| 2.05 | 0.672131 |
| 2.01 | 0.667774 |
| 2.005 | 0.667221 |
| 2.001 | 0.666778 |

| $x$ | $f(x)$ |
|---|---|
| 1.9 | 0.655172 |
| 1.95 | 0.661017 |
| 1.99 | 0.665552 |
| 1.995 | 0.666110 |
| 1.999 | 0.666556 |

It appears that $\lim\limits_{x \to 2} \dfrac{x^2 - 2x}{x^2 - x - 2} = 0.\bar{6} = \frac{2}{3}$.

20. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

| $x$ | $f(x)$ |
|---|---|
| 0 | 0 |
| $-0.5$ | $-1$ |
| $-0.9$ | $-9$ |
| $-0.95$ | $-19$ |
| $-0.99$ | $-99$ |
| $-0.999$ | $-999$ |

| $x$ | $f(x)$ |
|---|---|
| $-2$ | 2 |
| $-1.5$ | 3 |
| $-1.1$ | 11 |
| $-1.01$ | 101 |
| $-1.001$ | 1001 |

It appears that $\lim\limits_{x \to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$ *does not exist* since $f(x) \to \infty$ as $x \to -1^-$ and $f(x) \to -\infty$ as $x \to -1^+$.

21. For $f(t) = \dfrac{e^{5t} - 1}{t}$:

| $t$ | $f(t)$ |
|---|---|
| 0.5 | 22.364988 |
| 0.1 | 6.487213 |
| 0.01 | 5.127110 |
| 0.001 | 5.012521 |
| 0.0001 | 5.001250 |

| $t$ | $f(t)$ |
|---|---|
| $-0.5$ | 1.835830 |
| $-0.1$ | 3.934693 |
| $-0.01$ | 4.877058 |
| $-0.001$ | 4.987521 |
| $-0.0001$ | 4.998750 |

It appears that $\lim\limits_{t \to 0} \dfrac{e^{5t} - 1}{t} = 5$.

22. For $f(h) = \dfrac{(2+h)^5 - 32}{h}$:

| $h$ | $f(h)$ |
|---|---|
| 0.5 | 131.312500 |
| 0.1 | 88.410100 |
| 0.01 | 80.804010 |
| 0.001 | 80.080040 |
| 0.0001 | 80.008000 |

| $h$ | $f(h)$ |
|---|---|
| $-0.5$ | 48.812500 |
| $-0.1$ | 72.390100 |
| $-0.01$ | 79.203990 |
| $-0.001$ | 79.920040 |
| $-0.0001$ | 79.992000 |

It appears that $\lim\limits_{h \to 0} \dfrac{(2+h)^5 - 32}{h} = 80$.

23. For $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$:

| $x$ | $f(x)$ |
|---|---|
| 1 | 0.236068 |
| 0.5 | 0.242641 |
| 0.1 | 0.248457 |
| 0.05 | 0.249224 |
| 0.01 | 0.249844 |

| $x$ | $f(x)$ |
|---|---|
| $-1$ | 0.267949 |
| $-0.5$ | 0.258343 |
| $-0.1$ | 0.251582 |
| $-0.05$ | 0.250786 |
| $-0.01$ | 0.250156 |

It appears that $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4}$.

24. For $f(x) = \dfrac{\tan 3x}{\tan 5x}$:

| $x$ | $f(x)$ |
|---|---|
| $\pm 0.2$ | 0.439279 |
| $\pm 0.1$ | 0.566236 |
| $\pm 0.05$ | 0.591893 |
| $\pm 0.01$ | 0.599680 |
| $\pm 0.001$ | 0.599997 |

It appears that $\lim\limits_{x \to 0} \dfrac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5}$.

25. For $f(x) = \dfrac{x^6 - 1}{x^{10} - 1}$:

| $x$ | $f(x)$ |
|---|---|
| 0.5 | 0.985337 |
| 0.9 | 0.719397 |
| 0.95 | 0.660186 |
| 0.99 | 0.612018 |
| 0.999 | 0.601200 |

| $x$ | $f(x)$ |
|---|---|
| 1.5 | 0.183369 |
| 1.1 | 0.484119 |
| 1.05 | 0.540783 |
| 1.01 | 0.588022 |
| 1.001 | 0.598800 |

It appears that $\displaystyle\lim_{x\to 1} \frac{x^6 - 1}{x^{10} - 1} = 0.6 = \tfrac{3}{5}$.

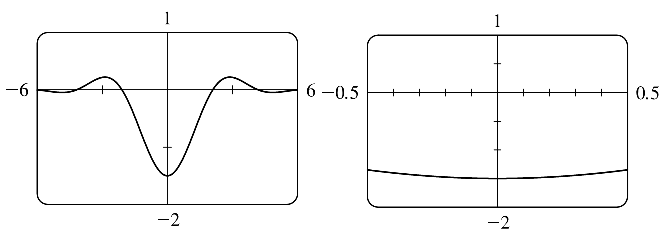
26. For $f(x) = \dfrac{9^x - 5^x}{x}$:

| $x$ | $f(x)$ |
|---|---|
| 0.5 | 1.527864 |
| 0.1 | 0.711120 |
| 0.05 | 0.646496 |
| 0.01 | 0.599082 |
| 0.001 | 0.588906 |

| $x$ | $f(x)$ |
|---|---|
| $-0.5$ | 0.227761 |
| $-0.1$ | 0.485984 |
| $-0.05$ | 0.534447 |
| $-0.01$ | 0.576706 |
| $-0.001$ | 0.586669 |

It appears that $\displaystyle\lim_{x\to 0} \frac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is $\ln(9/5)$.

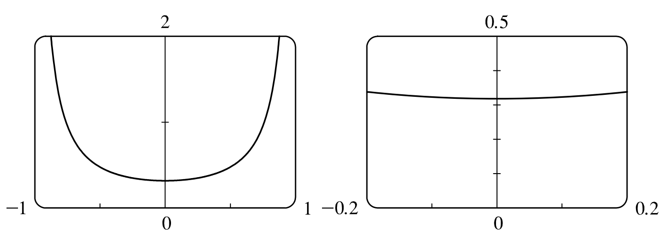
27. (a) From the graphs, it seems that $\displaystyle\lim_{x\to 0} \frac{\cos 2x - \cos x}{x^2} = -1.5$.




(b)

| $x$ | $f(x)$ |
|---|---|
| $\pm 0.1$ | $-1.493759$ |
| $\pm 0.01$ | $-1.499938$ |
| $\pm 0.001$ | $-1.499999$ |
| $\pm 0.0001$ | $-1.500000$ |

28. (a) From the graphs, it seems that $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sin \pi x} \approx 0.32$.




(b)

| $x$ | $f(x)$ |
|---|---|
| $\pm 0.1$ | 0.323068 |
| $\pm 0.01$ | 0.318357 |
| $\pm 0.001$ | 0.318310 |
| $\pm 0.0001$ | 0.318310 |

Later we will be able to show that the exact value is $\dfrac{1}{\pi}$.

29. $\displaystyle\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$ since the numerator is negative and the denominator approaches 0 from the positive side as $x \to -3^+$.

30. $\displaystyle\lim_{x\to -3^-} \frac{x+2}{x+3} = \infty$ since the numerator is negative and the denominator approaches 0 from the negative side as $x \to -3^-$.

31. $\displaystyle\lim_{x\to 1} \frac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

32. $\displaystyle\lim_{x\to 5^-} \frac{e^x}{(x-5)^3} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 5^-$.

33. Let $t = x^2 - 9$. Then as $x \to 3^+$, $t \to 0^+$, and $\displaystyle\lim_{x\to 3^+} \ln(x^2 - 9) = \lim_{t\to 0^+} \ln t = -\infty$ by (8).

34. $\displaystyle\lim_{x\to \pi^-} \cot x = \lim_{x\to \pi^-} \frac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \to \pi^-$.

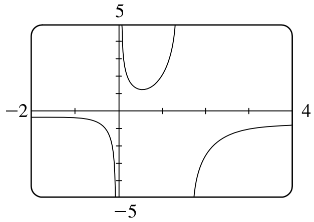
**35.** $\lim\limits_{x\to 2\pi^-} x\csc x = \lim\limits_{x\to 2\pi^-} \dfrac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2\pi^-$.

**36.** $\lim\limits_{x\to 2^-} \dfrac{x^2-2x}{x^2-4x+4} = \lim\limits_{x\to 2^-} \dfrac{x(x-2)}{(x-2)^2} = \lim\limits_{x\to 2^-} \dfrac{x}{x-2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2^-$.

**37.** $\lim\limits_{x\to 2^+} \dfrac{x^2-2x-8}{x^2-5x+6} = \lim\limits_{x\to 2^+} \dfrac{(x-4)(x+2)}{(x-3)(x-2)} = \infty$ since the numerator is negative and the denominator approaches 0 through negative values as $x \to 2^+$.

**38.** (a) The denominator of $y = \dfrac{x^2+1}{3x-2x^2} = \dfrac{x^2+1}{x(3-2x)}$ is equal to zero when $x = 0$ and $x = \frac{3}{2}$ (and the numerator is not), so $x = 0$ and $x = 1.5$ are vertical asymptotes of the function.

(b)




**39.** (a) $f(x) = \dfrac{1}{x^3-1}$.

| $x$ | $f(x)$ |
|---|---|
| 0.5 | −1.14 |
| 0.9 | −3.69 |
| 0.99 | −33.7 |
| 0.999 | −333.7 |
| 0.9999 | −3333.7 |
| 0.99999 | −33,333.7 |

| $x$ | $f(x)$ |
|---|---|
| 1.5 | 0.42 |
| 1.1 | 3.02 |
| 1.01 | 33.0 |
| 1.001 | 333.0 |
| 1.0001 | 3333.0 |
| 1.00001 | 33,333.3 |

From these calculations, it seems that

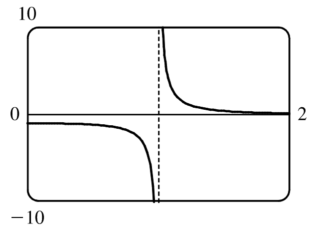
$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

(b) If $x$ is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim\limits_{x\to 1^-} f(x) = -\infty$.

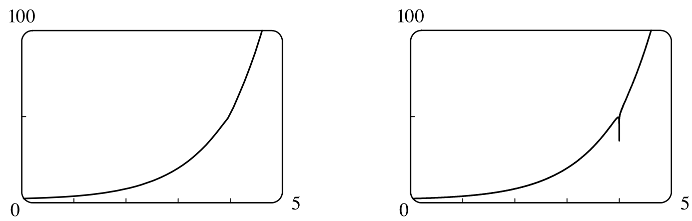
If $x$ is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim\limits_{x\to 1^+} f(x) = \infty$.

(c) It appears from the graph of $f$ that

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

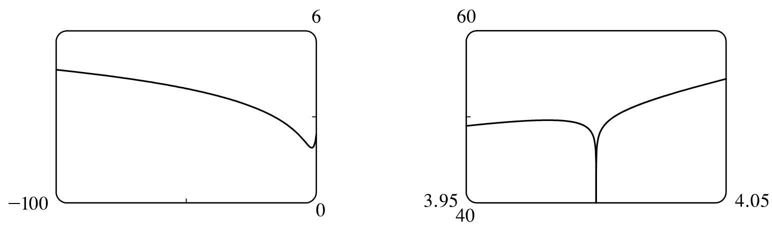



**40.** (a)

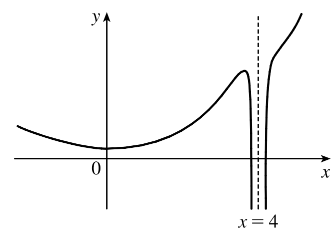



No, because the calculator-produced graph of $f(x) = e^x + \ln|x-4|$ looks like an exponential function, but the graph of $f$ has an infinite discontinuity at $x = 4$. A second graph, obtained by increasing the `numpoints` option in Maple, begins to reveal the discontinuity at $x = 4$.

(b) There isn't a single graph that shows all the features of $f$. Several graphs are needed since $f$ looks like $\ln|x-4|$ for large negative values of $x$ and like $e^x$ for $x > 5$, but yet has the infinite discontiuity at $x = 4$.



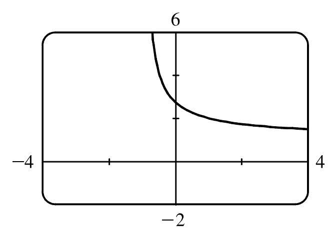

A hand-drawn graph, though distorted, might be better at revealing the main features of this function.




41. (a) Let $h(x) = (1+x)^{1/x}$.

| $x$ | $h(x)$ |
|---|---|
| $-0.001$ | $2.71964$ |
| $-0.0001$ | $2.71842$ |
| $-0.00001$ | $2.71830$ |
| $-0.000001$ | $2.71828$ |
| $0.000001$ | $2.71828$ |
| $0.00001$ | $2.71827$ |
| $0.0001$ | $2.71815$ |
| $0.001$ | $2.71692$ |

(b)




It appears that $\lim\limits_{x\to 0}(1+x)^{1/x} \approx 2.71828$ which is approximately $e$.

In Section 3.7 we will see that the value of the limit is exactly $e$.

42. For $f(x) = x^2 - (2^x/1000)$:

(a)

| $x$ | $f(x)$ |
|---|---|
| $1$ | $0.998000$ |
| $0.8$ | $0.638259$ |
| $0.6$ | $0.358484$ |
| $0.4$ | $0.158680$ |
| $0.2$ | $0.038851$ |
| $0.1$ | $0.008928$ |
| $0.05$ | $0.001465$ |

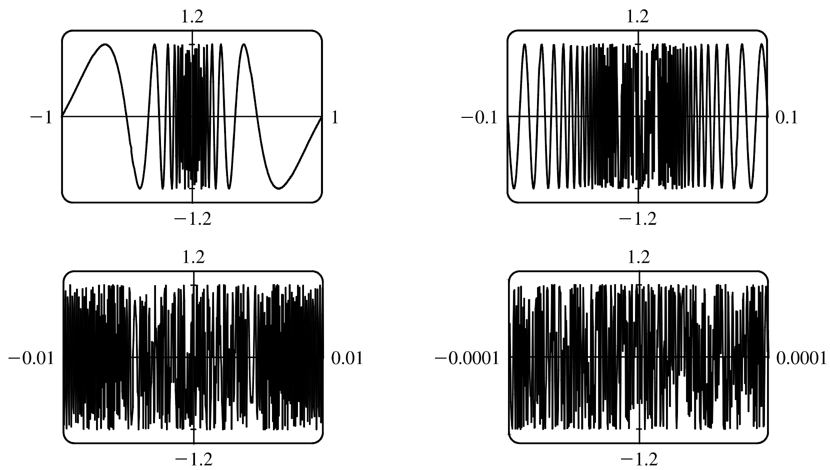
It appears that $\lim\limits_{x\to 0} f(x) = 0$.

(b)

| $x$ | $f(x)$ |
|---|---|
| $0.04$ | $0.000572$ |
| $0.02$ | $-0.000614$ |
| $0.01$ | $-0.000907$ |
| $0.005$ | $-0.000978$ |
| $0.003$ | $-0.000993$ |
| $0.001$ | $-0.001000$ |

It appears that $\lim\limits_{x\to 0} f(x) = -0.001$.

43. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.




44. $\lim\limits_{v \to c^-} m = \lim\limits_{v \to c^-} \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. As $v \to c^-$, $\sqrt{1 - v^2/c^2} \to 0^+$, and $m \to \infty$.

## 2.4 Limits: Algebraic Methods

1. (a) $\lim\limits_{x \to 2} [f(x) + 5g(x)] = \lim\limits_{x \to 2} f(x) + \lim\limits_{x \to 2} [5g(x)]$ [Limit Law 1]

$= \lim\limits_{x \to 2} f(x) + 5 \lim\limits_{x \to 2} g(x)$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim\limits_{x \to 2} [g(x)]^3 = \left[\lim\limits_{x \to 2} g(x)\right]^3$ [Limit Law 6]

$= (-2)^3 = -8$

(c) $\lim\limits_{x \to 2} \sqrt{f(x)} = \sqrt{\lim\limits_{x \to 2} f(x)}$ [Limit Law 11]

$= \sqrt{4} = 2$

(d) $\lim\limits_{x \to 2} \dfrac{3f(x)}{g(x)} = \dfrac{\lim\limits_{x \to 2} [3f(x)]}{\lim\limits_{x \to 2} g(x)}$ [Limit Law 5]

$= \dfrac{3 \lim\limits_{x \to 2} f(x)}{\lim\limits_{x \to 2} g(x)}$ [Limit Law 3]

$= \dfrac{3(4)}{-2} = -6$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim\limits_{x \to 2} \dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim\limits_{x \to 2} \dfrac{g(x)\,h(x)}{f(x)} = \dfrac{\lim\limits_{x \to 2} [g(x)\,h(x)]}{\lim\limits_{x \to 2} f(x)}$ [Limit Law 5]

$= \dfrac{\lim\limits_{x \to 2} g(x) \cdot \lim\limits_{x \to 2} h(x)}{\lim\limits_{x \to 2} f(x)}$ [Limit Law 4]

$= \dfrac{-2 \cdot 0}{4} = 0$

2. (a) $\lim\limits_{x \to 2} [f(x) + g(x)] = \lim\limits_{x \to 2} f(x) + \lim\limits_{x \to 2} g(x) = 2 + 0 = 2$

(b) $\lim\limits_{x \to 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

(c) $\lim\limits_{x \to 0} [f(x)g(x)] = \lim\limits_{x \to 0} f(x) \cdot \lim\limits_{x \to 0} g(x) = 0 \cdot 1.3 = 0$

(d) Since $\lim_{x\to -1} g(x) = 0$ and $g$ is in the denominator, but $\lim_{x\to -1} f(x) = -1 \neq 0$, the given limit does not exist.

(e) $\lim_{x\to 2} x^3 f(x) = \left[\lim_{x\to 2} x^3\right]\left[\lim_{x\to 2} f(x)\right] = 2^3 \cdot 2 = 16$

(f) $\lim_{x\to 1} \sqrt{3 + f(x)} = \sqrt{3 + \lim_{x\to 1} f(x)} = \sqrt{3+1} = 2$

3. $$\begin{aligned} \lim_{x\to -2}(3x^4 + 2x^2 - x + 1) &= \lim_{x\to -2} 3x^4 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && \text{[Limit Laws 1 and 2]} \\ &= 3\lim_{x\to -2} x^4 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && [3] \\ &= 3(-2)^4 + 2(-2)^2 - (-2) + (1) && \text{[9, 8, and 7]} \\ &= 48 + 8 + 2 + 1 = 59 \end{aligned}$$

4. $$\begin{aligned} \lim_{t\to -1}(t^2+1)^3(t+3)^5 &= \lim_{t\to -1}(t^2+1)^3 \cdot \lim_{t\to -1}(t+3)^5 && \text{[Limit Law 4]} \\ &= \left[\lim_{t\to -1}(t^2+1)\right]^3 \cdot \left[\lim_{t\to -1}(t+3)\right]^5 && [6] \\ &= \left[\lim_{t\to -1} t^2 + \lim_{t\to -1} 1\right]^3 \cdot \left[\lim_{t\to -1} t + \lim_{t\to -1} 3\right]^5 && [1] \\ &= \left[(-1)^2+1\right]^3 \cdot [-1+3]^5 = 8 \cdot 32 = 256 && \text{[9, 7, and 8]} \end{aligned}$$

5. $$\begin{aligned} \lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} &= \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}} && \text{[Limit Law 11]} \\ &= \sqrt{\frac{\lim_{x\to 2}(2x^2+1)}{\lim_{x\to 2}(3x-2)}} && [5] \\ &= \sqrt{\frac{2\lim_{x\to 2} x^2 + \lim_{x\to 2} 1}{3\lim_{x\to 2} x - \lim_{x\to 2} 2}} && \text{[1, 2, and 3]} \\ &= \sqrt{\frac{2(2)^2+1}{3(2)-2}} = \sqrt{\frac{9}{4}} = \frac{3}{2} && \text{[9, 8, and 7]} \end{aligned}$$

6. $$\begin{aligned} \lim_{x\to 0}\frac{\cos^4 x}{5+2x^3} &= \frac{\lim_{x\to 0}\cos^4 x}{\lim_{x\to 0}(5+2x^3)} && [5] \\ &= \frac{\left(\lim_{x\to 0}\cos x\right)^4}{\lim_{x\to 0} 5 + 2\lim_{x\to 0} x^3} && \text{[6, 1, and 3]} \\ &= \frac{1^4}{5+2(0)^3} = \frac{1}{5} && \text{[7, 9, and Equation 5]} \end{aligned}$$

7. $$\begin{aligned} \lim_{\theta\to\pi/2}\theta\sin\theta &= \left(\lim_{\theta\to\pi/2}\theta\right)\left(\lim_{\theta\to\pi/2}\sin\theta\right) && [4] \\ &= \frac{\pi}{2}\cdot\sin\frac{\pi}{2} && \text{[8 and Direct Substitution Property]} \\ &= \frac{\pi}{2} \end{aligned}$$

8. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

   (b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x \to a} f(x)$, we never consider $x = a$.

9. $\displaystyle \lim_{x \to 5} \frac{x^2 - 6x + 5}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x-1)}{x-5} = \lim_{x \to 5}(x-1) = 5 - 1 = 4$

10. $\displaystyle \lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{x(x-4)}{(x-4)(x+1)} = \lim_{x \to 4} \frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}$

11. $\displaystyle \lim_{x \to 5} \frac{x^2 - 5x + 6}{x - 5}$ does not exist since $x - 5 \to 0$, but $x^2 - 5x + 6 \to 6$ as $x \to 5$.

12. $\displaystyle \lim_{x \to -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3} = \lim_{x \to -1} \frac{(2x+1)(x+1)}{(x-3)(x+1)} = \lim_{x \to -1} \frac{2x+1}{x-3} = \frac{2(-1)+1}{-1-3} = \frac{-1}{-4} = \frac{1}{4}$

13. $\displaystyle \lim_{t \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3} = \lim_{t \to -3} \frac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim_{t \to -3} \frac{t-3}{2t+1} = \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}$

14. $\displaystyle \lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$ does not exist since $x^2 - 3x - 4 \to 0$ but $x^2 - 4x \to 5$ as $x \to -1$.

15. $\displaystyle \lim_{h \to 0} \frac{(4+h)^2 - 16}{h} = \lim_{h \to 0} \frac{(16 + 8h + h^2) - 16}{h} = \lim_{h \to 0} \frac{8h + h^2}{h} = \lim_{h \to 0} \frac{h(8+h)}{h} = \lim_{h \to 0}(8+h) = 8 + 0 = 8$

16. $$\begin{aligned} \lim_{h \to 0} \frac{(2+h)^3 - 8}{h} &= \lim_{h \to 0} \frac{\left(8 + 12h + 6h^2 + h^3\right) - 8}{h} = \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} \\ &= \lim_{h \to 0}\left(12 + 6h + h^2\right) = 12 + 0 + 0 = 12 \end{aligned}$$

17. By the formula for the sum of cubes, we have

$$\lim_{x \to -2} \frac{x+2}{x^3+8} = \lim_{x \to -2} \frac{x+2}{(x+2)(x^2 - 2x + 4)} = \lim_{x \to -2} \frac{1}{x^2 - 2x + 4} = \frac{1}{4+4+4} = \frac{1}{12}.$$

18. $$\begin{aligned} \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \lim_{h \to 0} \frac{(1+h) - 1}{h\left(\sqrt{1+h} + 1\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{1+h} + 1\right)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2} \end{aligned}$$

19. $\displaystyle \lim_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} = \lim_{x \to -4} \frac{\frac{x+4}{4x}}{4 + x} = \lim_{x \to -4} \frac{x+4}{4x(4+x)} = \lim_{x \to -4} \frac{1}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}$

20. $$\begin{aligned} \lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1} &= \lim_{x \to -1} \frac{(x+1)^2}{(x^2+1)(x^2-1)} = \lim_{x \to -1} \frac{(x+1)^2}{(x^2+1)(x+1)(x-1)} \\ &= \lim_{x \to -1} \frac{x+1}{(x^2+1)(x-1)} = \frac{0}{2(-2)} = 0 \end{aligned}$$

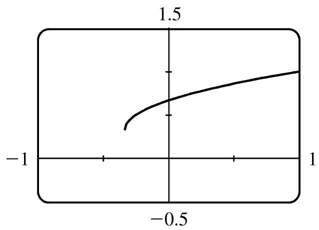
21. $$\begin{aligned} \lim_{x \to 16} \frac{4 - \sqrt{x}}{16x - x^2} &= \lim_{x \to 16} \frac{(4 - \sqrt{x})(4 + \sqrt{x})}{(16x - x^2)(4 + \sqrt{x})} = \lim_{x \to 16} \frac{16 - x}{x(16 - x)(4 + \sqrt{x})} \\ &= \lim_{x \to 16} \frac{1}{x(4 + \sqrt{x})} = \frac{1}{16\left(4 + \sqrt{16}\right)} = \frac{1}{16(8)} = \frac{1}{128} \end{aligned}$$

22. $\lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)=\lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t(t+1)}\right)=\lim\limits_{t\to 0}\dfrac{t+1-1}{t(t+1)}=\lim\limits_{t\to 0}\dfrac{1}{t+1}=\dfrac{1}{0+1}=1$

23. $$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$
$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)}=-\frac{1}{2}$$

24. $$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$
$$=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$
$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{16+9}+5}=\frac{-8}{5+5}=-\frac{4}{5}$$

25. (a)




$\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+3x}-1}\approx\dfrac{2}{3}$

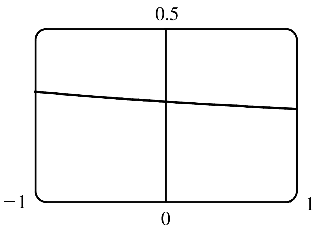
(b)

| $x$ | $f(x)$ |
|---|---|
| $-0.001$ | $0.6661663$ |
| $-0.0001$ | $0.6666167$ |
| $-0.00001$ | $0.6666617$ |
| $-0.000001$ | $0.6666662$ |
| $0.000001$ | $0.6666672$ |
| $0.00001$ | $0.6666717$ |
| $0.0001$ | $0.6667167$ |
| $0.001$ | $0.6671663$ |

The limit appears to be $\dfrac{2}{3}$.

(c) $$\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$$
$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right)\qquad\text{[Limit Law 3]}$$
$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right]\qquad\text{[1 and 11]}$$
$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right)\qquad\text{[1, 3, and 7]}$$
$$=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)\qquad\text{[7 and 8]}$$
$$=\frac{1}{3}(1+1)=\frac{2}{3}$$

26. (a)




$\lim\limits_{x\to 0} \dfrac{\sqrt{3+x}-\sqrt{3}}{x} \approx 0.29$

(b)

| $x$ | $f(x)$ |
|---|---|
| $-0.001$ | $0.2886992$ |
| $-0.0001$ | $0.2886775$ |
| $-0.00001$ | $0.2886754$ |
| $-0.000001$ | $0.2886752$ |
| $0.000001$ | $0.2886751$ |
| $0.00001$ | $0.2886749$ |
| $0.0001$ | $0.2886727$ |
| $0.001$ | $0.2886511$ |

The limit appears to be approximately $0.2887$.

(c) $$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right) = \lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$

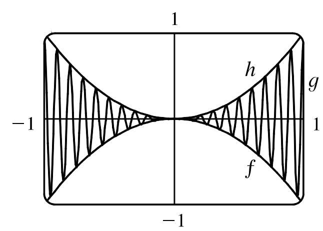
$$= \frac{\lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0}\sqrt{3+x}+\lim\limits_{x\to 0}\sqrt{3}} \qquad \text{[Limit Laws 5 and 1]}$$

$$= \frac{1}{\sqrt{\lim\limits_{x\to 0}(3+x)}+\sqrt{3}} \qquad \text{[7 and 11]}$$

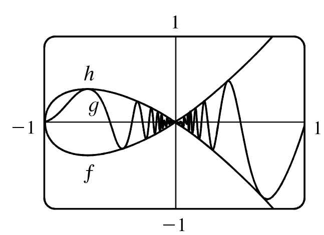
$$= \frac{1}{\sqrt{3+0}+\sqrt{3}} \qquad \text{[1, 7, and 8]}$$

$$= \frac{1}{2\sqrt{3}}$$

27. Let $f(x) = -x^2$, $g(x) = x^2\cos 20\pi x$ and $h(x) = x^2$. Then $-1 \le \cos 20\pi x \le 1 \;\Rightarrow\; -x^2 \le x^2\cos 20\pi x \le x^2 \;\Rightarrow\; f(x) \le g(x) \le h(x)$. So since $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x) = 0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0} g(x) = 0$.




28. Let $f(x) = -\sqrt{x^3+x^2}$, $g(x) = \sqrt{x^3+x^2}\,\sin(\pi/x)$, and $h(x) = \sqrt{x^3+x^2}$. Then $-1 \le \sin(\pi/x) \le 1 \;\Rightarrow\; -\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin(\pi/x) \le \sqrt{x^3+x^2} \;\Rightarrow\; f(x) \le g(x) \le h(x)$. So since $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x) = 0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0} g(x) = 0$.




29. We have $\lim\limits_{x\to 4}(4x-9) = 4(4)-9 = 7$ and $\lim\limits_{x\to 4}\left(x^2-4x+7\right) = 4^2-4(4)+7 = 7$. Since $4x-9 \le f(x) \le x^2-4x+7$ for $x \ge 0$, $\lim\limits_{x\to 4} f(x) = 7$ by the Squeeze Theorem.

30. We have $\lim\limits_{x\to 1}(2x) = 2(1) = 2$ and $\lim\limits_{x\to 1}(x^4-x^2+2) = 1^4-1^2+2 = 2$. Since $2x \le g(x) \le x^4-x^2+2$ for all $x$, $\lim\limits_{x\to 1} g(x) = 2$ by the Squeeze Theorem.

31. $-1 \le \cos(2/x) \le 1 \;\Rightarrow\; -x^4 \le x^4\cos(2/x) \le x^4$. Since $\lim\limits_{x\to 0}\left(-x^4\right) = 0$ and $\lim\limits_{x\to 0} x^4 = 0$, we have $\lim\limits_{x\to 0}\left[x^4\cos(2/x)\right] = 0$ by the Squeeze Theorem.

32. (a) $\lim\limits_{t\to 0} m(t) = \lim\limits_{t\to 0} \left[\frac{1}{2}e^{-t}(\sin t - \cos t) + \frac{1}{2}\right] = \frac{1}{2}(1)(0-1) + \frac{1}{2} = 0$

This indicates that the concentration of mRNA at $t = 0$ is zero.

(b) $\lim\limits_{t\to\infty} m(t) = \lim\limits_{t\to\infty} \frac{1}{2}e^{-t}(\sin t - \cos t) + \frac{1}{2} = \frac{1}{2}\left(\lim\limits_{t\to\infty} e^{-t}\sin t - \lim\limits_{t\to\infty} e^{-t}\cos t\right) + \frac{1}{2}$ The Product Law for limits cannot be used since $\lim_{t\to\infty}\sin t$ and $\lim_{t\to\infty}\cos t$ does not exist. We can use the Squeeze theorem instead. $-1 \le \sin t \le 1 \quad\Rightarrow\quad -e^{-t} \le e^{-t}\sin t \le e^{-t}$. Since $\lim\limits_{t\to\infty}\left(\pm e^{-t}\right) = 0$, we have $\lim\limits_{t\to\infty} e^{-t}\sin t = 0$ by the Squeeze Theorem. Replacing $\sin t$ with $\cos t$ in the above argument, we similarly find that $\lim\limits_{t\to\infty} e^{-t}\cos t = 0$. Therefore, $\lim\limits_{t\to\infty} m(t) = \frac{1}{2}(0-0) + \frac{1}{2} = \frac{1}{2}$. This indicates that the concentration of mRNA in the long-term is $0.5$.

33. $|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \ge 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Thus, $\lim\limits_{x\to 3^+}(2x + |x-3|) = \lim\limits_{x\to 3^+}(2x + x - 3) = \lim\limits_{x\to 3^+}(3x-3) = 3(3) - 3 = 6$ and $\lim\limits_{x\to 3^-}(2x + |x-3|) = \lim\limits_{x\to 3^-}(2x + 3 - x) = \lim\limits_{x\to 3^-}(x+3) = 3 + 3 = 6$. Since the left and right limits are equal, $\lim\limits_{x\to 3}(2x + |x-3|) = 6$.

34. $|x+6| = \begin{cases} x+6 & \text{if } x+6 \ge 0 \\ -(x+6) & \text{if } x+6 < 0 \end{cases} = \begin{cases} x+6 & \text{if } x \ge -6 \\ -(x+6) & \text{if } x < -6 \end{cases}$

We'll look at the one-sided limits.

$$\lim_{x\to -6^+} \frac{2x+12}{|x+6|} = \lim_{x\to -6^+} \frac{2(x+6)}{x+6} = 2 \quad \text{and} \quad \lim_{x\to -6^-} \frac{2x+12}{|x+6|} = \lim_{x\to -6^-} \frac{2(x+6)}{-(x+6)} = -2$$

The left and right limits are different, so $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$ does not exist.

35. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to 0^-}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^-}\left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x\to 0^-}\dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

36. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to -2}\dfrac{2-|x|}{2+x} = \lim\limits_{x\to -2}\dfrac{2-(-x)}{2+x} = \lim\limits_{x\to -2}\dfrac{2+x}{2+x} = \lim\limits_{x\to -2} 1 = 1$.

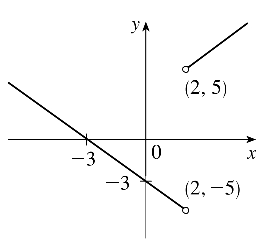
37. (a) (i) 
$$\begin{aligned}\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{x^2+x-6}{|x-2|} &= \lim_{x\to 2^+} \frac{(x+3)(x-2)}{|x-2|} \\ &= \lim_{x\to 2^+} \frac{(x+3)(x-2)}{x-2} \quad [\text{since } x-2>0 \text{ if } x\to 2^+] \\ &= \lim_{x\to 2^+}(x+3) = 5\end{aligned}$$

(ii) The solution is similar to the solution in part (i), but now $|x-2| = 2-x$ since $x - 2 < 0$ if $x \to 2^-$.

Thus, $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} -(x+3) = -5$.

(b) Since the right-hand and left-hand limits of $g$ at $x = 2$ are not equal, $\lim\limits_{x\to 2} g(x)$ does not exist.

(c)

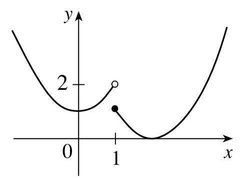


38. (a) $f(x) = \begin{cases} x^2+1 & \text{if } x<1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+1) = 1^2+1 = 2, \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-2)^2 = (-1)^2 = 1$

(b) Since the right-hand and left-hand limits of $f$ at $x=1$ are not equal, $\lim_{x\to 1} f(x)$ does not exist.

(c)




39. $$\begin{aligned} \lim_{x\to 0} \frac{\sin 3x}{x} &= \lim_{x\to 0} \frac{3\sin 3x}{3x} && \text{[multiply numerator and denominator by 3]} \\ &= 3 \lim_{3x\to 0} \frac{\sin 3x}{3x} && [\text{as } x\to 0,\ 3x\to 0] \\ &= 3 \lim_{\theta\to 0} \frac{\sin\theta}{\theta} && [\text{let } \theta = 3x] \\ &= 3(1) && \text{[Equation 6]} \\ &= 3 \end{aligned}$$

40. $\lim_{x\to 0} \frac{\sin 4x}{\sin 6x} = \lim_{x\to 0} \left(\frac{\sin 4x}{x}\cdot\frac{x}{\sin 6x}\right) = \lim_{x\to 0}\frac{4\sin 4x}{4x}\cdot\lim_{x\to 0}\frac{6x}{6\sin 6x} = 4\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{1}{6}\lim_{x\to 0}\frac{6x}{\sin 6x} = 4(1)\cdot\frac{1}{6}(1) = \frac{2}{3}$

41. $$\begin{aligned} \lim_{t\to 0}\frac{\tan 6t}{\sin 2t} &= \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t} \\ &= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3 \end{aligned}$$

42. $$\begin{aligned} \lim_{t\to 0}\frac{\sin^2 3t}{t^2} &= \lim_{t\to 0}\left(\frac{\sin 3t}{t}\cdot\frac{\sin 3t}{t}\right) = \lim_{t\to 0}\frac{\sin 3t}{t}\cdot\lim_{t\to 0}\frac{\sin 3t}{t} \\ &= \left(\lim_{t\to 0}\frac{\sin 3t}{t}\right)^2 = \left(3\lim_{t\to 0}\frac{\sin 3t}{3t}\right)^2 = (3\cdot 1)^2 = 9 \end{aligned}$$

43. Divide numerator and denominator by $\theta$. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta} = \lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}} = \frac{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}}{1+\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta}} = \frac{1}{1+1\cdot 1} = \frac{1}{2}$$

44. $$\lim_{x\to 0} x\cot x = \lim_{x\to 0} x\cdot\frac{\cos x}{\sin x} = \lim_{x\to 0}\frac{x\cos x}{\sin x} = \lim_{x\to 0}\frac{\dfrac{x\cos x}{x}}{\dfrac{\sin x}{x}} = \lim_{x\to 0}\frac{\dfrac{\cos x}{1}}{\dfrac{\sin x}{x}} = \frac{\lim\limits_{x\to 0}\cos x}{\lim\limits_{x\to 0}\dfrac{\sin x}{x}} = \frac{1}{1} = 1$$

45. (a) Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned} \lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) = a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \cdots + a_n\lim_{x\to a} x^n \\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a) \end{aligned}$$

Thus, for any polynomial $p$, $\lim_{x\to a} p(x) = p(a)$.

(b) Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a)\neq 0$. Then

$$\lim_{x\to a} r(x) = \lim_{x\to a}\frac{p(x)}{q(x)} = \frac{\lim\limits_{x\to a} p(x)}{\lim\limits_{x\to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[by part (a)]} \quad = r(a).$$

46. $\lim\limits_{v \to c^-} \left( L_0 \sqrt{1 - \dfrac{v^2}{c^2}} \right) = L_0\sqrt{1-1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since $L$ is not defined for $v > c$.

47. $\lim\limits_{h \to 0} \sin(a+h) = \lim\limits_{h \to 0} (\sin a \cos h + \cos a \sin h) = \lim\limits_{h \to 0} (\sin a \cos h) + \lim\limits_{h \to 0} (\cos a \sin h)$

$$= \left(\lim_{h \to 0} \sin a\right)\left(\lim_{h \to 0} \cos h\right) + \left(\lim_{h \to 0} \cos a\right)\left(\lim_{h \to 0} \sin h\right) = (\sin a)(1) + (\cos a)(0) = \sin a$$

48. As in the previous exercise, we must show that $\lim\limits_{h \to 0} \cos(a+h) = \cos a$ to prove that the cosine function has the Direct Substitution Property.

$$\begin{aligned} \lim_{h \to 0} \cos(a+h) &= \lim_{h \to 0} (\cos a \cos h - \sin a \sin h) = \lim_{h \to 0} (\cos a \cos h) - \lim_{h \to 0} (\sin a \sin h) \\ &= \left(\lim_{h \to 0} \cos a\right)\left(\lim_{h \to 0} \cos h\right) - \left(\lim_{h \to 0} \sin a\right)\left(\lim_{h \to 0} \sin h\right) = (\cos a)(1) - (\sin a)(0) = \cos a \end{aligned}$$

49. $\lim\limits_{x \to 1} [f(x) - 8] = \lim\limits_{x \to 1} \left[ \dfrac{f(x)-8}{x-1} \cdot (x-1) \right] = \lim\limits_{x \to 1} \dfrac{f(x)-8}{x-1} \cdot \lim\limits_{x \to 1} (x-1) = 10 \cdot 0 = 0$.

Thus, $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \{[f(x)-8]+8\} = \lim\limits_{x \to 1} [f(x)-8] + \lim\limits_{x \to 1} 8 = 0 + 8 = 8$.

*Note:* The value of $\lim\limits_{x \to 1} \dfrac{f(x)-8}{x-1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim\limits_{x \to 1} \dfrac{f(x)-8}{x-1}$ exists.

50. (a) $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \left[ \dfrac{f(x)}{x^2} \cdot x^2 \right] = \lim\limits_{x \to 0} \dfrac{f(x)}{x^2} \cdot \lim\limits_{x \to 0} x^2 = 5 \cdot 0 = 0$

(b) $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = \lim\limits_{x \to 0} \left[ \dfrac{f(x)}{x^2} \cdot x \right] = \lim\limits_{x \to 0} \dfrac{f(x)}{x^2} \cdot \lim\limits_{x \to 0} x = 5 \cdot 0 = 0$

51. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim\limits_{x \to -2} (3x^2 + ax + a + 3) = 0 \quad \Leftrightarrow$ $3(-2)^2 + a(-2) + a + 3 = 0 \quad \Leftrightarrow \quad 12 - 2a + a + 3 = 0 \quad \Leftrightarrow \quad a = 15$. With $a = 15$, the limit becomes $\lim\limits_{x \to -2} \dfrac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim\limits_{x \to -2} \dfrac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim\limits_{x \to -2} \dfrac{3(x+3)}{x-1} = \dfrac{3(-2+3)}{-2-1} = \dfrac{3}{-3} = -1$.

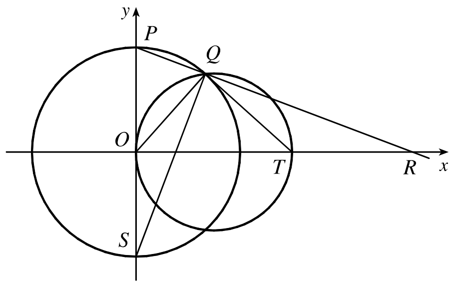
52. *Solution 1:* First, we find the coordinates of $P$ and $Q$ as functions of $r$. Then we can find the equation of the line determined by these two points, and thus find the $x$-intercept (the point $R$), and take the limit as $r \to 0$. The coordinates of $P$ are $(0, r)$. The point $Q$ is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating $y$ from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad \Leftrightarrow \quad r^2 = 1 + 2x - 1 \quad \Leftrightarrow \quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the $y$-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad \Leftrightarrow \quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad \Leftrightarrow \quad y = r\sqrt{1 - \frac{1}{4}r^2}$ (the positive $y$-value). So the coordinates of $Q$ are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining $P$ and $Q$ is thus $y - r = \dfrac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the $x$-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)$$

Now we take the limit as $r \to 0^+$: $\lim\limits_{r\to 0^+} x = \lim\limits_{r\to 0^+} 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right) = \lim\limits_{r\to 0^+} 2\left(\sqrt{1}+1\right) = 4$.

So the limiting position of $R$ is the point $(4,0)$.

*Solution 2:* We add a few lines to the diagram, as shown. Note that $\angle PQS = 90°$ (subtended by diameter $PS$). So $\angle SQR = 90° = \angle OQT$ (subtended by diameter $OT$). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90° - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle $C_2$ shrinks, the point $Q$ plainly approaches the origin, so the point $R$ must approach a point twice as far from the origin as $T$, that is, the point $(4,0)$, as above.




## 2.5 Continuity

1. From Definition 1, $\lim\limits_{x\to 4} f(x) = f(4)$.

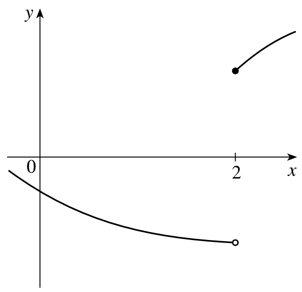
2. The graph of $f$ has no hole, jump, or vertical asymptote.

3. (a) $f$ is discontinuous at $-4$ since $f(-4)$ is not defined and at $-2$, $2$, and $4$ since the limit does not exist (the left and right limits are not the same).

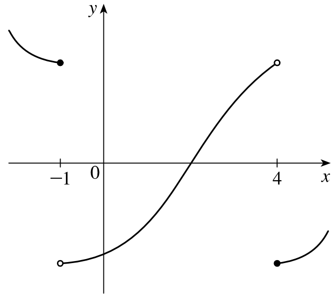
   (b) $f$ is continuous from the left at $-2$ since $\lim\limits_{x\to -2^-} f(x) = f(-2)$. $f$ is continuous from the right at $2$ and $4$ since $\lim\limits_{x\to 2^+} f(x) = f(2)$ and $\lim\limits_{x\to 4^+} f(x) = f(4)$. It is continuous from neither side at $-4$ since $f(-4)$ is undefined.

4. $g$ is continuous on $[-4,-2)$, $(-2,2)$, $[2,4)$, $(4,6)$, and $(6,8)$.

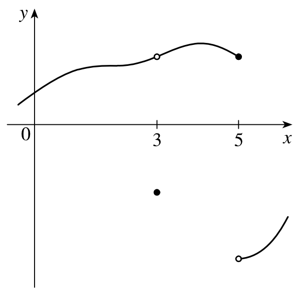
5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim\limits_{x\to 2^+} f(x) = f(2)$.



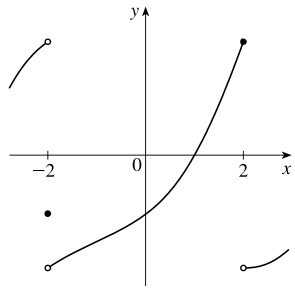

6. The graph of $y = f(x)$ must have discontinuities at $x = -1$ and $x = 4$. It must show that $\lim\limits_{x\to -1^-} f(x) = f(-1)$ and $\lim\limits_{x\to 4^+} f(x) = f(4)$.



7. The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = 3$ and a jump discontinuity at $x = 5$.



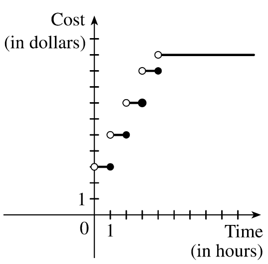

8. The graph of $y = f(x)$ must have a discontinuity at $x = -2$ with $\lim\limits_{x \to -2^-} f(x) \neq f(-2)$ and $\lim\limits_{x \to -2^+} f(x) \neq f(-2)$. It must also show that $\lim\limits_{x \to 2^-} f(x) = f(2)$ and $\lim\limits_{x \to 2^+} f(x) \neq f(2)$.




9. (a) C has discontinuities at 12, 24, and 36 hours since the limit does not exist at these points.
   (b) C has jump discontinuities at the values of $t$ listed in part (a) because the function jumps from one value to another at these points.

10. There are jump discontinuities at 1, 1.7, 3, and 3.5. They occur because the left and right side limits are different at each of these points, so the limit does not exist. For example, $\lim_{t \to 1^-} P(t) = 26$ and $\lim_{t \to 1^+} P(t) = 24$, so $\lim_{t \to 1} P(t)$ does not exist and $P(t)$ is discontinuous at $t = 1$.

11. (a)




   (b) There are discontinuities at times $t = 1$, 2, 3, and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

12. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.
   (b) Discontinuous; the population size increases or decreases in whole number increments.
   (c) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.
   (d) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.
   (e) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

13. Since $f$ and $g$ are continuous functions,

$$\begin{aligned}\lim_{x\to 3}\,[2f(x)-g(x)] &= 2\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x) && \text{[by Limit Laws 2 and 3]}\\ &= 2f(3)-g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x=3]\\ &= 2\cdot 5 - g(3) = 10 - g(3)\end{aligned}$$

Since it is given that $\lim\limits_{x\to 3}\,[2f(x)-g(x)] = 4$, we have $10-g(3)=4$, so $g(3)=6$.

14. $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\left(3x^4-5x+\sqrt[3]{x^2+4}\right) = 3\lim\limits_{x\to 2} x^4 - 5\lim\limits_{x\to 2} x + \sqrt[3]{\lim\limits_{x\to 2}(x^2+4)}$

$$= 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 48-10+2 = 40 = f(2)$$

By the definition of continuity, $f$ is continuous at $a=2$.

15. $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1}\left(x+2x^3\right)^4 = \left(\lim\limits_{x\to -1} x + 2\lim\limits_{x\to -1} x^3\right)^4 = \left[-1+2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1)$.

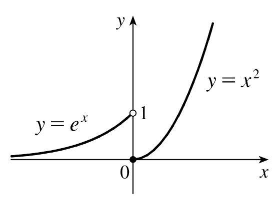
By the definition of continuity, $f$ is continuous at $a=-1$.

16. For $a>2$, we have

$$\begin{aligned}\lim_{x\to a} f(x) = \lim_{x\to a}\frac{2x+3}{x-2} &= \frac{\lim\limits_{x\to a}(2x+3)}{\lim\limits_{x\to a}(x-2)} && \text{[Limit Law 5]}\\ &= \frac{2\lim\limits_{x\to a} x + \lim\limits_{x\to a} 3}{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 2} && \text{[1, 2, and 3]}\\ &= \frac{2a+3}{a-2} && \text{[7 and 8]}\\ &= f(a)\end{aligned}$$

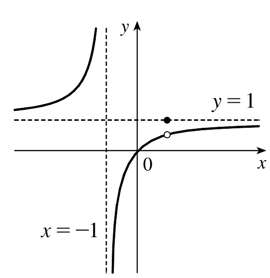
Thus, $f$ is continuous at $x=a$ for every $a$ in $(2,\infty)$; that is, $f$ is continuous on $(2,\infty)$.

17. $f(x) = \begin{cases} e^x & \text{if } x<0\\ x^2 & \text{if } x\geq 0\end{cases}$




The left-hand limit of $f$ at $a=0$ is $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} e^x = 1$. The right-hand limit of $f$ at $a=0$ is $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2 = 0$. Since these limits are not equal, $\lim\limits_{x\to 0} f(x)$ does not exist and $f$ is discontinuous at 0.

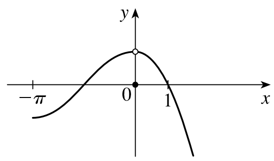
18. $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x\neq 1\\ 1 & \text{if } x=1\end{cases}$




$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}\dfrac{x^2-x}{x^2-1} = \lim\limits_{x\to 1}\dfrac{x(x-1)}{(x+1)(x-1)} = \lim\limits_{x\to 1}\dfrac{x}{x+1} = \dfrac{1}{2}$,

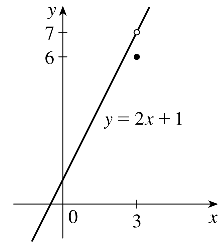
but $f(1)=1$, so $f$ is discontinous at 1.

19. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$




$\lim\limits_{x \to 0} f(x) = 1$, but $f(0) = 0 \neq 1$, so $f$ is discontinuous at 0.

20. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$




$\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{2x^2 - 5x - 3}{x - 3} = \lim\limits_{x \to 3} \dfrac{(2x+1)(x-3)}{x-3} = \lim\limits_{x \to 3} (2x + 1) = 7$,

but $f(3) = 6$, so $f$ is discontinuous at 3.

21. By Theorem 5, the polynomials $x^2$ and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x - 1}$ is continuous on its domain, $\left[\frac{1}{2}, \infty\right)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x - 1}$ is continuous on $\left[\frac{1}{2}, \infty\right)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1 + x^3$ are continuous on $\mathbb{R}$. By part 4 of Theorem 4, the product $G(x) = \sqrt[3]{x}\,(1 + x^3)$ is continuous on its domain, $\mathbb{R}$.

23. By Theorem 7, the exponential function $e^{-5t}$ and the trigonometric function $\cos 2\pi t$ are continuous on $(-\infty, \infty)$. By part 4 of Theorem 4, $L(t) = e^{-5t} \cos 2\pi t$ is continuous on $(-\infty, \infty)$.

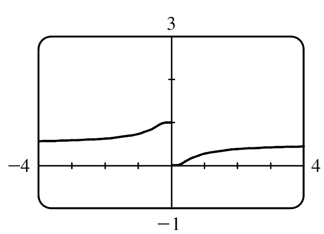
24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By part 5 of Theorem 4, $h(x) = \dfrac{\sin x}{x + 1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

25. By Theorem 5, the polynomial $t^4 - 1$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\ln x$ is continuous on its domain, $(0, \infty)$. By Theorem 9, $\ln(t^4 - 1)$ is continuous on its domain, which is

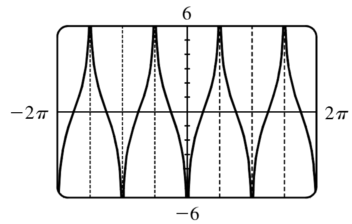
$$\{t \mid t^4 - 1 > 0\} = \{t \mid t^4 > 1\} = \{t \mid |t| > 1\} = (-\infty, -1) \cup (1, \infty)$$

26. The sine and cosine functions are continuous everywhere by Theorem 7, so $F(x) = \sin(\cos(\sin x))$, which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

27. The function $y = \dfrac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



28. The function $y = \tan^2 x$ is discontinuous at $x = \frac{\pi}{2} + \pi k$, where $k$ is any integer. The function $y = \ln(\tan^2 x)$ is also discontinuous where $\tan^2 x$ is 0, that is, at $x = \pi k$. So $y = \ln(\tan^2 x)$ is discontinuous at $x = \frac{\pi}{2} n$, $n$ any integer.




29. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x+5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \dfrac{5 + \sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0, \infty)$. Since $f$ is continuous at $x = 4$, $\lim\limits_{x \to 4} f(x) = f(4) = \frac{7}{3}$.

30. Because $x$ is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x \to \pi} f(x) = f(\pi) = \sin(\pi + \sin \pi) = \sin \pi = 0$.

31. Because $x^2 - x$ is continuous on $\mathbb{R}$, the composite function $f(x) = e^{x^2 - x}$ is continuous on $\mathbb{R}$, so $\lim\limits_{x \to 1} f(x) = f(1) = e^{1-1} = e^0 = 1$.

32. Because $p^2 - 2p$ and $p^2 - 2$ are polynomials, they are both continuous on $\mathbb{R}$. Hence, the quotient $f(p) = \dfrac{p^2 - 2p}{p^2 - 2}$ is continuous on $\mathbb{R}$ except when $p^2 - 2 = 0 \quad \Rightarrow \quad p = \pm\sqrt{2}$. Since $f$ is continuous at $p = 1/2$,

$$\lim_{p \to 1/2} f(p) = f(1/2) = \frac{\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2 - 2} = \frac{3}{7}.$$

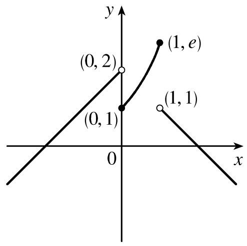
33. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial $x^2$ on $(-\infty, 1)$, $f$ is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, $f$ is continuous on $(1, \infty)$. At $x = 1$, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \sqrt{x} = 1$. Thus, $\lim\limits_{x \to 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, $f$ is continuous at $x = 1$. We conclude that $f$ is continuous on $(-\infty, \infty)$.

34. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, $f$ is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x \to (\pi/4)^-} f(x) = \lim\limits_{x \to (\pi/4)^-} \sin x = \sin \frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x \to (\pi/4)^+} f(x) = \lim\limits_{x \to (\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x \to (\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, $f$ is continuous at $\pi/4$, so $f$ is continuous on $(-\infty, \infty)$.

35. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$




$f$ is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x+2) = 2$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} e^x = 1$, so $f$ is discontinuous at 0. Since $f(0) = 1$, $f$ is continuous from the right at 0. Also $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} e^x = e$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2-x) = 1$, so $f$ is discontinuous at 1. Since $f(1) = e$, $f$ is continuous from the left at 1.

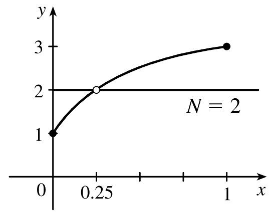
36. By Theorem 5, each piece of $F$ is continuous on its domain. We need to check for continuity at $r = R$.

$\lim\limits_{r\to R^-} F(r) = \lim\limits_{r\to R^-} \dfrac{GMr}{R^3} = \dfrac{GM}{R^2}$ and $\lim\limits_{r\to R^+} F(r) = \lim\limits_{r\to R^+} \dfrac{GM}{r^2} = \dfrac{GM}{R^2}$, so $\lim\limits_{r\to R} F(r) = \dfrac{GM}{R^2}$. Since $F(R) = \dfrac{GM}{R^2}$, $F$ is continuous at $R$. Therefore, $F$ is a continuous function of $r$.

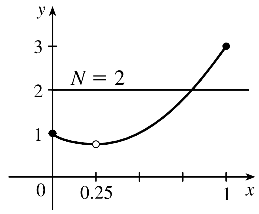
37. $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \ge 2 \end{cases}$

$f$ is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(cx^2 + 2x\right) = 4c + 4$ and $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \left(x^3 - cx\right) = 8 - 2c$. So $f$ is continuous $\Leftrightarrow$ $4c + 4 = 8 - 2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for $f$ to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

38.




$f$ does not satisfy the conclusion of the Intermediate Value Theorem.




$f$ does satisfy the conclusion of the Intermediate Value Theorem.

39. $f(x) = x^2 + 10\sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number $c$ in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number $c$ in $(-32, -31)$ such that $f(c) = 1000$.

40. Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function $f$ on the closed interval $[2, 3]$, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number $c$ in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \ge 6$. But $f(3) \ne 6$ because the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) > 6$.

41. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number $c$ in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

42. $f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number $c$ in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$.

43. The equation $e^x = 3 - 2x$ is equivalent to the equation $e^x + 2x - 3 = 0$. $f(x) = e^x + 2x - 3$ is continuous on the interval $[0, 1]$, $f(0) = -2$, and $f(1) = e - 1 \approx 1.72$. Since $-2 < 0 < e - 1$, there is a number $c$ in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + 2x - 3 = 0$, or $e^x = 3 - 2x$, in the interval $(0, 1)$.

44. The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^2 + x$ is continuous on the interval $[1, 2]$, $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number $c$ in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval $(1, 2)$.

45. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number $c$ in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

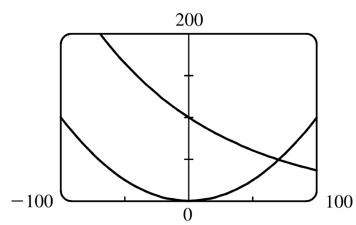
(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between $0.86$ and $0.87$, that is, in the interval $(0.86, 0.87)$.

46. (a) $f(x) = \ln x - 3 + 2x$ is continuous on the interval $[1, 2]$, $f(1) = -1 < 0$, and $f(2) = \ln 2 + 1 \approx 1.7 > 0$. Since $-1 < 0 < 1.7$, there is a number $c$ in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - 3 + 2x = 0$, or $\ln x = 3 - 2x$, in the interval $(1, 2)$.

(b) $f(1.34) \approx -0.03 < 0$ and $f(1.35) \approx 0.0001 > 0$, so there is a root between $1.34$ and $1.35$, that is, in the interval $(1.34, 1.35)$.

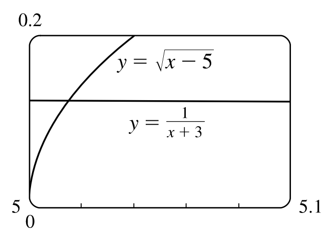
47. (a) Let $f(x) = 100e^{-x/100} - 0.01x^2$. Then $f(0) = 100 > 0$ and $f(100) = 100e^{-1} - 100 \approx -63.2 < 0$. So by the Intermediate Value Theorem, there is a number $c$ in $(0, 100)$ such that $f(c) = 0$. This implies that $100e^{-c/100} = 0.01c^2$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 70.347$, correct to three decimal places.




48. (a) Let $f(x) = \sqrt{x - 5} - \dfrac{1}{x + 3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and $f$ is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number $c$ in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c + 3} = \sqrt{c - 5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



49. If there is such a number, it satisfies the equation $x^3 + 1 = x \quad \Leftrightarrow \quad x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number $c$ between $-2$ and $-1$ such that $f(c) = 0$, so that $c = c^3 + 1$.

50. $f(x) = x^4 \sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \le \sin(1/x) \le 1$, we have $-x^4 \le x^4 \sin(1/x) \le x^4$. Because $\lim\limits_{x \to 0}(-x^4) = 0$ and $\lim\limits_{x \to 0} x^4 = 0$, the Squeeze Theorem gives us $\lim\limits_{x \to 0}(x^4 \sin(1/x)) = 0$, which equals $f(0)$. Thus, $f$ is continuous at 0 and, hence, on $(-\infty, \infty)$.

51. Define $u(t)$ to be the monk's distance from the monastery, as a function of time $t$ (in hours), on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let $D$ be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time $t_0$ between 0 and 12 such that $(u - d)(t_0) = 0 \quad \Leftrightarrow$ $u(t_0) = d(t_0)$. So at time $t_0$ after 7:00 AM, the monk will be at the same place on both days.

## 2 Review

### TRUE-FALSE QUIZ

1. True. If $\lim\limits_{n \to \infty} a_n = L$, then as $n \to \infty$, $2n + 1 \to \infty$, so $a_{2n+1} \to L$.

2. True. $0.99999\ldots = 0.9 + 0.9(0.1)^1 + 0.9(0.1)^2 + 0.9(0.1)^3 + \cdots = \sum\limits_{n=1}^{\infty} (0.9)(0.1)^{n-1} = \dfrac{0.9}{1 - 0.1} = 1$ by the formula for the sum of a geometric series $[S = a/(1 - r)]$ with ratio $r$ satisfying $|r| < 1$.

3. False. Limit Law 2 applies only if the individual limits exist (these don't).

4. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

5. True. Limit Law 5 applies.

6. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as $x$ approaches 5. (The denominator approaches 0 and the numerator doesn't.)

7. False. Consider $\lim\limits_{x \to 5} \dfrac{x(x - 5)}{x - 5}$ or $\lim\limits_{x \to 5} \dfrac{\sin(x - 5)}{x - 5}$. The first limit exists and is equal to 5. By Equation 6 in Section 2.4, we know that the latter limit exists (and it is equal to 1).

8. False. Consider $\lim\limits_{x \to 6} [f(x)g(x)] = \lim\limits_{x \to 6} \left[ (x - 6) \dfrac{1}{x - 6} \right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

9. True. A polynomial is continuous everywhere, so $\lim\limits_{x \to b} p(x)$ exists and is equal to $p(b)$.

10. False. Consider $\lim\limits_{x \to 0} [f(x) - g(x)] = \lim\limits_{x \to 0} \left( \dfrac{1}{x^2} - \dfrac{1}{x^4} \right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches $\infty$.

11. True. For example, the function $f(x) = \dfrac{\sqrt{4x^2+1}}{x-5}$ has two different horizontal asymptotes since $\lim\limits_{x\to\infty} f(x) = 2$ and $\lim\limits_{x\to-\infty} f(x) = -2$. The horizontal asymptotes are $y = 2$ and $y = -2$.

12. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim\limits_{x\to\infty} f(x) \neq \pm\infty$ and $f$ has no horizontal asymptote.

13. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

14. False. The function $f$ must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0,3]$ with $f(c) = 0$.

15. True. Use Theorem 2.5.7 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

16. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

## EXERCISES

1. $\left\{\dfrac{2+n^3}{1+2n^3}\right\}$ converges since $\lim\limits_{n\to\infty} \dfrac{2+n^3}{1+2n^3} = \lim\limits_{n\to\infty} \dfrac{2/n^3+1}{1/n^3+2} = \dfrac{1}{2}$.

2. $a_n = \dfrac{9^{n+1}}{10^n} = 9 \cdot \left(\frac{9}{10}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 9 \lim\limits_{n\to\infty} \left(\frac{9}{10}\right)^n = 9 \cdot 0 = 0$ by (11.1.9).

3. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^3}{1+n^2} = \lim\limits_{n\to\infty} \dfrac{n}{1/n^2+1} = \infty$, so the sequence diverges.

4. $a_n = (-2)^n = (-1)^n \cdot 2^n$. As $n$ increases $2^n$ increases and $(-1)^n$ alternates between positive and negative values. Hence $\lim\limits_{n\to\infty} (-2)^n$ does not exist, so the sequence is divergent.

5. $a_{n+1} = \frac{1}{3}a_n + 3$, $a_1 = 1$, $a_2 \approx 3.3333$, $a_3 \approx 4.1111$, $a_4 \approx 4.3704$, $a_5 \approx 4.4568$, $a_6 \approx 4.4856$, $a_7 \approx 4.4952$, $a_8 \approx 4.4984$

| $n$ | $a_n$ | $n$ | $a_n$ |
|---|---|---|---|
| 1 | 1.0000 | 5 | 4.4568 |
| 2 | 3.3333 | 6 | 4.4856 |
| 3 | 4.1111 | 7 | 4.4952 |
| 4 | 4.3704 | 8 | 4.4984 |

The sequence appears to converge to 4.5. Assume the limit exists so that $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = a$, then $a_{n+1} = \frac{1}{3}a_n + 3 \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \left(\frac{1}{3}a_n + 3\right) \Rightarrow a = \frac{1}{3}a + 3 \Rightarrow a = \frac{9}{2} = 4.5$. This agrees with the value estimated from the data table.

6. (a) The concentration of the drug in the body after the first injection is 0.25 mg/mL. After the second injection, there is 0.25 mg/mL plus 20% of the concentration from the first injection, that is, $[0.25 + 0.25(0.20)] = 0.3$ mg/mL. After the third injection, the concentration is $[0.25 + 0.3(0.20)] = 0.31$ mg/mL, and after the fourth injection it is $[0.25 + 0.31(0.20)] = 0.312$ mg/mL.

(b) The drug concentration is $0.2C_n$ just before the $n^{\text{th}} + 1$ injection, after which the concentration increases by 0.25 mg/mL. Hence $C_{n+1} = 0.2C_n + 0.25$.

(c) From Formula (6) in §2.1, the solution to $C_{n+1} = 0.2C_n + 0.25$, $C_0 = 0$ mg/mL is

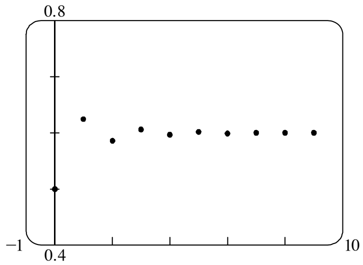
$$C_n = (0.2)(0) + 0.25\left(\frac{1-0.2^n}{1-0.2}\right) = \frac{0.25}{0.8}(1-0.2^n) = \frac{5}{16}(1-0.2^n)$$

(d) The limiting value of the concentration is

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \tfrac{5}{16}(1-0.2^n) = \tfrac{5}{16}\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} 0.2^n\right) = \tfrac{5}{16}(1-0) = \tfrac{5}{16} = 0.3125 \text{ mg/mL}.$$

7. $1.2345345345\ldots = 1.2 + 0.0\overline{345} = \dfrac{12}{10} + \dfrac{345/10{,}000}{1 - 1/1000} = \dfrac{12}{10} + \dfrac{345}{9990} = \dfrac{4111}{3330}$

8. (a)



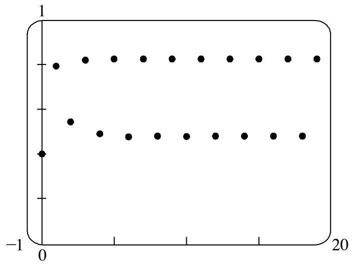

Computer software was used to plot the first 10 points of the recursion equation $x_{t+1} = 2.5x_t(1 - x_t)$, $x_0 = 0.5$. The sequence appears to converge to a value of $0.6$. Assume the limit exists so that $\lim\limits_{t\to\infty} x_{n+1} = \lim\limits_{t\to\infty} x_t = x$, then

$x_{t+1} = 2.5x_t(1 - x_t) \quad \Rightarrow \quad \lim\limits_{t\to\infty} x_{t+1} = \lim\limits_{t\to\infty} 2.5x_t(1 - x_t) \quad \Rightarrow$

$x = 2.5x(1 - x) \quad \Rightarrow \quad x\,(1.5 - 2.5x) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1.5/2.5 = 0.6$.

This agrees with the value estimated from the plot.

(b)




Computer software was used to plot the first 20 points of the recursion equation $x_{t+1} = 3.3x_t(1 - x_t)$, $x_0 = 0.4$. The sequence does not appear to converge to a fixed value of $x_t$. Instead, the terms oscillate between values near $0.48$ and $0.82$.

9. (a) (i) $\lim\limits_{x\to 2^+} f(x) = 3$ (ii) $\lim\limits_{x\to -3^+} f(x) = 0$

(iii) $\lim\limits_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is $-2$.)

(iv) $\lim\limits_{x\to 4} f(x) = 2$

(v) $\lim\limits_{x\to 0} f(x) = \infty$ (vi) $\lim\limits_{x\to 2^-} f(x) = -\infty$

(vii) $\lim\limits_{x\to\infty} f(x) = 4$ (viii) $\lim\limits_{x\to -\infty} f(x) = -1$

(b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.

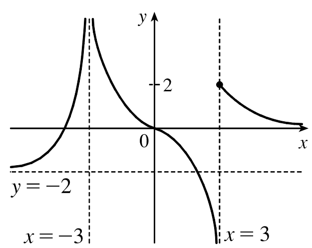
(c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(d) $f$ is discontinuous at $x = -3$, $0$, $2$, and $4$. The discontinuities are jump, infinite, infinite, and removable, respectively.

10. $\lim\limits_{x\to -\infty} f(x) = -2$, $\lim\limits_{x\to \infty} f(x) = 0$, $\lim\limits_{x\to -3} f(x) = \infty$,

$\lim\limits_{x\to 3^-} f(x) = -\infty$, $\lim\limits_{x\to 3^+} f(x) = 2$,

$f$ is continuous from the right at 3




11. $\lim\limits_{x\to\infty} \dfrac{1 - x}{2 + 5x} = \lim\limits_{x\to\infty} \dfrac{1/x - 1}{2/x + 5} = \lim\limits_{x\to\infty} \dfrac{0 - 1}{0 + 5} = -\dfrac{1}{5}$

12. $\lim\limits_{t\to\infty} 3^{-2t} = \lim\limits_{t\to\infty} \dfrac{1}{3^{2t}} = 0$ since $2t \to \infty$ as $t \to \infty$.

13. Since the exponential function is continuous, $\lim\limits_{x\to 1} e^{x^3 - x} = e^{1-1} = e^0 = 1$.

14. Since rational functions are continuous, $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3} = \dfrac{3^2 - 9}{3^2 + 2(3) - 3} = \dfrac{0}{12} = 0$.

15. $\lim\limits_{x\to-3}\dfrac{x^2-9}{x^2+2x-3}=\lim\limits_{x\to-3}\dfrac{(x+3)(x-3)}{(x+3)(x-1)}=\lim\limits_{x\to-3}\dfrac{x-3}{x-1}=\dfrac{-3-3}{-3-1}=\dfrac{-6}{-4}=\dfrac{3}{2}$

16. $\lim\limits_{x\to1^+}\dfrac{x^2-9}{x^2+2x-3}=-\infty$ since $x^2+2x-3\to0^+$ as $x\to1^+$ and $\dfrac{x^2-9}{x^2+2x-3}<0$ for $1<x<3$.

17. $\lim\limits_{h\to0}\dfrac{(h-1)^3+1}{h}=\lim\limits_{h\to0}\dfrac{\left(h^3-3h^2+3h-1\right)+1}{h}=\lim\limits_{h\to0}\dfrac{h^3-3h^2+3h}{h}=\lim\limits_{h\to0}\left(h^2-3h+3\right)=3$

*Another solution:* Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned}\lim_{h\to0}\frac{(h-1)^3+1}{h}&=\lim_{h\to0}\frac{(h-1)^3+1^3}{h}=\lim_{h\to0}\frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h}\\&=\lim_{h\to0}\left[(h-1)^2-h+2\right]=1-0+2=3\end{aligned}$$

18. $\lim\limits_{t\to2}\dfrac{t^2-4}{t^3-8}=\lim\limits_{t\to2}\dfrac{(t+2)(t-2)}{(t-2)(t^2+2t+4)}=\lim\limits_{t\to2}\dfrac{t+2}{t^2+2t+4}=\dfrac{2+2}{4+4+4}=\dfrac{4}{12}=\dfrac{1}{3}$

19. $\lim\limits_{r\to9}\dfrac{\sqrt{r}}{(r-9)^4}=\infty$ since $(r-9)^4\to0^+$ as $r\to9$ and $\dfrac{\sqrt{r}}{(r-9)^4}>0$ for $r\neq9$.

20. $\lim\limits_{v\to4^+}\dfrac{4-v}{|4-v|}=\lim\limits_{v\to4^+}\dfrac{4-v}{-(4-v)}=\lim\limits_{v\to4^+}\dfrac{1}{-1}=-1$

21. $\lim\limits_{u\to1}\dfrac{u^4-1}{u^3+5u^2-6u}=\lim\limits_{u\to1}\dfrac{(u^2+1)(u^2-1)}{u(u^2+5u-6)}=\lim\limits_{u\to1}\dfrac{(u^2+1)(u+1)(u-1)}{u(u+6)(u-1)}=\lim\limits_{u\to1}\dfrac{(u^2+1)(u+1)}{u(u+6)}=\dfrac{2(2)}{1(7)}=\dfrac{4}{7}$

22.
$$\begin{aligned}\lim_{x\to3}\frac{\sqrt{x+6}-x}{x^3-3x^2}&=\lim_{x\to3}\left[\frac{\sqrt{x+6}-x}{x^2(x-3)}\cdot\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}\right]=\lim_{x\to3}\frac{(\sqrt{x+6})^2-x^2}{x^2(x-3)(\sqrt{x+6}+x)}\\&=\lim_{x\to3}\frac{x+6-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to3}\frac{-(x^2-x-6)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to3}\frac{-(x-3)(x+2)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}\\&=\lim_{x\to3}\frac{-(x+2)}{x^2\left(\sqrt{x+6}+x\right)}=-\frac{5}{9(3+3)}=-\frac{5}{54}\end{aligned}$$

23. Let $t=\sin x$. Then as $x\to\pi^-$, $\sin x\to0^+$, so $t\to0^+$. Thus, $\lim\limits_{x\to\pi^-}\ln(\sin x)=\lim\limits_{t\to0^+}\ln t=-\infty$.

24. $\lim\limits_{x\to-\infty}\dfrac{1-2x^2-x^4}{5+x-3x^4}=\lim\limits_{x\to-\infty}\dfrac{(1-2x^2-x^4)/x^4}{(5+x-3x^4)/x^4}=\lim\limits_{x\to-\infty}\dfrac{1/x^4-2/x^2-1}{5/x^4+1/x^3-3}=\dfrac{0-0-1}{0+0-3}=\dfrac{-1}{-3}=\dfrac{1}{3}$

25. Since $x$ is positive, $\sqrt{x^2}=|x|=x$. Thus,

$$\lim_{x\to\infty}\frac{\sqrt{x^2-9}}{2x-6}=\lim_{x\to\infty}\frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x}=\lim_{x\to\infty}\frac{\sqrt{1-9/x^2}}{2-6/x}=\frac{\sqrt{1-0}}{2-0}=\frac{1}{2}$$

26. Let $t=x-x^2=x(1-x)$. Then as $x\to\infty$, $t\to-\infty$, and $\lim\limits_{x\to\infty}e^{x-x^2}=\lim\limits_{t\to-\infty}e^t=0$.

27.
$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)&=\lim_{x\to\infty}\left[\frac{\sqrt{x^2+4x+1}-x}{1}\cdot\frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}\right]=\lim_{x\to\infty}\frac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}\\&=\lim_{x\to\infty}\frac{(4x+1)/x}{(\sqrt{x^2+4x+1}+x)/x}\qquad\left[\text{divide by } x=\sqrt{x^2}\text{ for } x>0\right]\\&=\lim_{x\to\infty}\frac{4+1/x}{\sqrt{1+4/x+1/x^2}+1}=\frac{4+0}{\sqrt{1+0+0}+1}=\frac{4}{2}=2\end{aligned}$$

28. $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}+\dfrac{1}{x^2-3x+2}\right)=\lim\limits_{x\to 1}\left[\dfrac{1}{x-1}+\dfrac{1}{(x-1)(x-2)}\right]=\lim\limits_{x\to 1}\left[\dfrac{x-2}{(x-1)(x-2)}+\dfrac{1}{(x-1)(x-2)}\right]$

$$=\lim_{x\to 1}\left[\frac{x-1}{(x-1)(x-2)}\right]=\lim_{x\to 1}\frac{1}{x-2}=\frac{1}{1-2}=-1$$

29. $\lim\limits_{[S]\to\infty} v=\lim\limits_{[S]\to\infty}\dfrac{0.50[S]}{3.0\times 10^{-4}+[S]}=\lim\limits_{[S]\to\infty}\dfrac{0.50}{3.0\times 10^{-4}/[S]+1}=\dfrac{0.50}{0+1}=0.50$. As the concentration grows larger the enzymatic reaction rate will approach $0.50$.

30. Let $f(x)=-x^2$, $g(x)=x^2\cos(1/x^2)$ and $h(x)=x^2$. Then since $\left|\cos(1/x^2)\right|\le 1$ for $x\neq 0$, we have $f(x)\le g(x)\le h(x)$ for $x\neq 0$, and so $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} h(x)=0 \quad\Rightarrow\quad \lim\limits_{x\to 0} g(x)=0$ by the Squeeze Theorem.

31. (a) $f(x)=\sqrt{-x}$ if $x<0$, $f(x)=3-x$ if $0\le x<3$, $f(x)=(x-3)^2$ if $x>3$.

(i) $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(3-x)=3$

(ii) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\sqrt{-x}=0$

(iii) Because of (i) and (ii), $\lim\limits_{x\to 0} f(x)$ does not exist.

(iv) $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(3-x)=0$

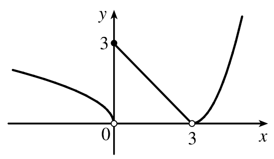
(v) $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(x-3)^2=0$

(vi) Because of (iv) and (v), $\lim\limits_{x\to 3} f(x)=0$.

(b) $f$ is discontinuous at 0 since $\lim\limits_{x\to 0} f(x)$ does not exist.

$f$ is discontinuous at 3 since $f(3)$ does not exist.

(c)




32. (a) $x^2-9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0,\infty)$ by Theorem 6 in Section 2.5, so the composition $\sqrt{x^2-9}$ is continuous on $\{x\mid x^2-9\ge 0\}=(-\infty,-3]\cup[3,\infty)$ by Theorem 8. Note that $x^2-2\neq 0$ on this set and so the quotient function $g(x)=\dfrac{\sqrt{x^2-9}}{x^2-2}$ is continuous on its domain, $(-\infty,-3]\cup[3,\infty)$ by Theorem 4.

(b) $\sin x$ and $e^x$ are continuous on $\mathbb{R}$ by Theorem 6 in Section 2.5. Since $e^x$ is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 8 in Section 2.5. Lastly, $x$ is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 4 in Section 2.5.

33. $f(x)=2x^3+x^2+2$ is a polynomial, so it is continuous on $[-2,-1]$ and $f(-2)=-10<0<1=f(-1)$. So by the Intermediate Value Theorem there is a number $c$ in $(-2,-1)$ such that $f(c)=0$, that is, the equation $2x^3+x^2+2=0$ has a root in $(-2,-1)$.

34. $f(x)=e^{-x^2}-x$ is continuous on $\mathbb{R}$ so it is continuous on $[0,1]$. $f(0)=1>0>1/e-1=f(1)$. So by the Intermediate Value Theorem, there is a number $c$ in $(0,1)$ such that $f(c)=0$. Thus, $e^{-x^2}-x=0$, or $e^{-x^2}=x$, has a root in $(0,1)$.

## CASE STUDY 2a Hosts, Parasites, and Time-Travel

1. The functions $q(t)$ and $p(t)$ describing the genotype frequencies of the host and parasite are both transformations of the function $\cos(t)$. Thus, they are oscillatory functions that exhibit periodic or repeating behavior. Biologically, we expect the parasite genotype will evolve to infect the host genotype that is most prevalent in the population. As this happens, the host genotypes will evolve to avoid infection. In turn, the parasite genotype frequency will evolve (for survival) towards the new prevalent host genotype frequency. This cat-and-mouse game causes a cycling of the host and parasite genotype frequencies that is described by the periodic functions $q(t)$ and $p(t)$. E.g. if the frequency of type A is high, the parasite will evolve toward a high frequency of type B. The host population will then evolve leading to a lower frequency of genotype A (to avoid infection), and in turn, the parasite population will evolve toward a lower frequency of genotype B (for survival).

2. $M_q$ and $M_p$ represent vertical stretch factors to the parent function $\cos(t)$. They are the amplitudes of oscillation for the frequencies of genotype A and B respectively. Therefore, an increase in $M$ results in a higher maximum frequency and a lower minimum frequency over time for the respective genotype.

3. The constant $c$ represents a horizontal compression factor to the parent function $\cos(t)$. This gives a period of oscillation of $\dfrac{2\pi}{c}$ for both $q(t)$ and $p(t)$. Therefore, increasing $c$ results in a smaller period of oscillation for *both* the host and parasite frequencies.
   *Biological explanation:* Suppose uninfected hosts have a large reproductive advantage over infected hosts so that $c$ is large. The uninfected host population will grow rapidly in size compared to the infected host population. Consequently, the parasite with the genotype capable of infecting the growing uninfected host population will also increase rapidly, since more hosts will be available to the parasite for infection. As this occurs, the uninfected hosts become infected and now the formerly infected hosts will have the reproductive advantage. Thus, the frequencies of the host and parasite genotypes will cycle rapidly back and forth. That is, the period of oscillation will be small.

4. The constants $\phi_p$ and $\phi_q$ affect the horizontal translation of the parent function $\cos(t)$. They are the phase shifts that determine the time at which the genotype frequencies reach a maximum.

5. The constants $\phi_p$ and $\phi_q$ affect the horizontal position of the periodic functions $p(t)$ and $q(t)$, so the difference $\phi^* = \phi_p - \phi_q$ measures the time lag between the cycles of $q(t)$ and $p(t)$. Hence, this quantity is a measure of the length of time it takes for the frequency of the parasite genotype to "respond" to the frequency of the host genotype.

6. Consider the general form of equations 2a and 2b given by $f_{\text{ave}}(\tau) = \frac{1}{2} + M\cos\left(c\tau - \phi\right)\dfrac{2\sin\left(\frac{1}{2}cW\right)}{cW}$. Evaluating $f_{\text{ave}}(\tau)$ in the limit $W \to 0$ gives
   $$\lim_{W\to 0}\left(\tfrac{1}{2} + M\cos\left(c\tau-\phi\right)\frac{2\sin\left(\frac{1}{2}cW\right)}{cW}\right) = \tfrac{1}{2} + M\cos\left(c\tau-\phi\right)\lim_{W\to 0}\frac{\sin\left(\frac{1}{2}cW\right)}{\frac{1}{2}cW} = \tfrac{1}{2} + M\cos\left(c\tau-\phi\right)\lim_{x\to 0}\frac{\sin(x)}{x}$$
   where $x = \frac{1}{2}cW$. The last limit was investigated in Example 2.3.4 where it was shown that $\lim_{x\to 0}\sin(x)/x = 1$. So we have $\lim\limits_{W\to 0} f_{\text{ave}}(\tau) = \frac{1}{2} + M\cos\left(c\tau - \phi_q\right)$. This is the same form as equations (1a) and (1b). Thus, when extracting and mixing a very small layer of sediment ($W \to 0$), the average frequency of the host and parasite genotypes are the same as the instantaneous frequencies $q(\tau)$ and $p(\tau)$.

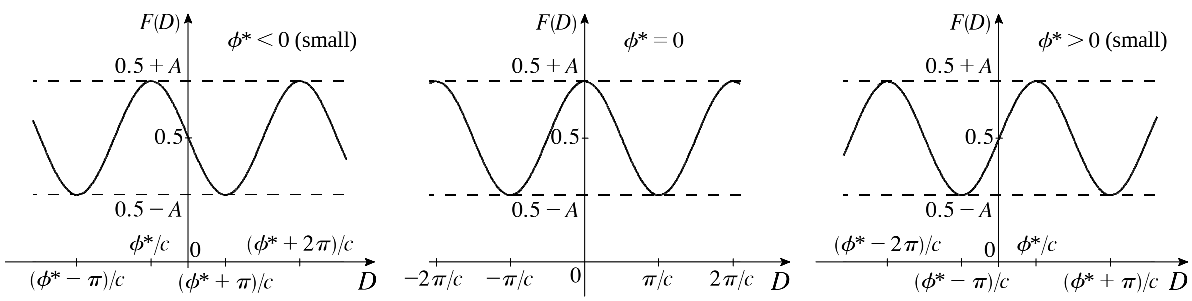
To determine the limit of $f_{\text{ave}}(\tau)$ as $W \to \infty$, first observe that $\lim\limits_{W\to\infty} \dfrac{\sin\left(\frac{1}{2}cW\right)}{cW} \leq \lim\limits_{W\to\infty} \dfrac{1}{cW} = 0$ since $\sin x \leq 1$. Also $\lim\limits_{W\to\infty} \dfrac{\sin\left(\frac{1}{2}cW\right)}{cW} \geq \lim\limits_{W\to\infty} \dfrac{-1}{cW} = 0$ since $\sin x \geq -1$. Thus, by the Squeeze Theorem we have $\lim\limits_{W\to\infty} \dfrac{\sin\left(\frac{1}{2}cW\right)}{cW} = 0$ so that $\lim\limits_{W\to\infty} f_{\text{ave}}(\tau) = \frac{1}{2}$.

In terms of the biology, extracting and mixing an extremely small width of sediment will capture parasites and hosts from a very short period of time (nearly an instant). In contrast, extracting and mixing a very large layer of sediment will homogenize the host and parasite frequencies across a large period of time. Hence, the frequencies at different points in time can no longer be differentiated so we expect a constant average frequency of host and parasite genotypes.

7. The graph of $F(D) = \frac{1}{2} + M_p M_q \cos\left[c\left(D - \dfrac{\phi^*}{c}\right)\right] \dfrac{4\sin^2\left(\frac{1}{2}cW\right)}{c^2W^2}$ is obtained by applying the following transformations to the graph of $\cos(D)$:

- Horizontal compression by a factor of $c$. Thus, the period is $2\pi/c$.
- Vertical stretch by a factor $M_p M_q \dfrac{4\sin^2\left(\frac{1}{2}cW\right)}{c^2W^2}$. Thus, the amplitude is $A = M_p M_q \dfrac{4\sin^2\left(\frac{1}{2}cW\right)}{c^2W^2}$.
- Horizontal translation $\phi^*/c$ units to the right if positive, or left if negative
- Vertical translation $\frac{1}{2}$ units up

These properties are illustrated in the sketches of $F(D)$ below for $\phi^* = 0$, $\phi^*$ small positive, and $\phi^*$ small negative. We have assumed that $c$ is sufficiently large, so that if $\phi^*$ is close to zero, then $\phi^*/c$ is also close to zero.




8. The experimental data in Figure 3 shows an increase in the fraction of hosts infected as the sample points move from the past to the future. The graphs from Problem 7 illustrate the variety of situations that can arise given different phase lags $\phi^*$. In the figure corresponding to $\phi^* > 0$, observe that $F(D)$ is an increasing function in the interval $\left[\dfrac{\phi^* - \pi}{c}, \dfrac{\phi^*}{c}\right]$. That is, the fraction of hosts infected increases as the sample points move from the past to the future for relatively small values of $cD$. This is the same pattern observed in Figure 3. Thus, we require that the phase lag $\phi^*$ be small and positive in order to observe the experimental trend in Figure 3.

*Biological interpretation:* When $\phi^* > 0$ there is a phase or time lag between the oscillations in frequency of the host and parasite populations. In the experiment, parasites from the past had not yet evolved to infect the present hosts, so fewer hosts were infected by parasites from the past. Parasites from the future *had* evolved to infect hosts from the present, so a greater number of hosts were infected by parasites from the future.

9. The experimental results depicted in Figure 4 show a decrease in the fraction of infected hosts when challenged with parasites from both the past and future. Examining the figures from Problem 6, we observe that this experimental result is achievable when $\phi^* = 0$. In this case, $F(D)$ is increasing on the interval $\left[-\frac{\pi}{c}, 0\right]$ and decreasing on the interval $\left[0, \frac{\pi}{c}\right]$. So if we start at $D = 0$ and decrease $D$ by a small amount (move into the past), the fraction of infected hosts will decrease. Similarly, if we start at $D = 0$ and increase $D$ by a small amount (move into the future), the fraction of infected hosts will also decrease. This is the same pattern observed in Figure 4. Thus, we require that the phase lag $\phi^*$ be zero in order to observe the experimental trend in Figure 4.

   *Biological interpretation:* When $\phi^* = 0$ there is no phase lag between the frequency of the host and parasite genotypes. That is, the frequency of the parasite genotype oscillates in a synchronous manner with the frequency of the host genotype. In a sense, the parasites' genotype evolves with the host in real-time to maximize the number of infected hosts. Thus, parasites from the past will infect fewer hosts than contemporary parasites since the contemporary parasites have already evolved to infect contemporary hosts. Similarly, parasites from the future will infect fewer hosts than contemporary parasites since the future parasites have evolved to infect future hosts.