# Exercises for Chapter 2

*In this problem set, interpret all discrete sequences as starting from n* = 0: {*X*(0), *X*(1), *X*(2), …}.

**1.** A coin is flipped each second. When it is heads, the random variable *X*(*n*) takes the value +1; when it is tails then *X*(*n*) = -1. What are the mean and autocorrelation of {*X*(0), *X*(1), *X*(2), …}?

**Solution** ; (independent) except

**2.** The random process {*X*(0), *X*(1), *X*(2), …} is created by a spinning dial that selects numbers

*X*(*n*) = [1, 2, or 3] with equal probability. What are the mean, autocorrelation, and autocovariance for this process?

**Solution**  ;

;

except

; (independent)

except

**3.** There are 3 signal generators in a box. The first one puts out the sequence {*X*(0), *X*(1), *X*(2), …} = {1 1 1 1 1 ….}. The second puts out {2 0 2 0 2 0 2 ...}. The third puts out {0 2 0 2 0 2 ...}. One of these generators is selected at random (equally likely). What are the mean μ(*n*), autocorrelation *Rx*(*m,n*), and autocovariance *Cx*(*m,n*) of the resulting sequence for *m*=3 and *n* =4?

**Solution**  ( ;

;

**4.** The random process *X*(*n*) is determined by a single flip of a fair coin. If the coin shows heads then the process is a series of ones: {*X*(0), *X*(1), *X*(2), …} = {1 1 1 1 1 1 1 1 1 ….}. If the coin shows tails then *X*(*n*) = 1/(*n*+1): {*X*(0), *X*(1), *X*(2), …} = {1 1/2 1/3 1/4 1/5 ….}. What are the mean and the autocorrelation?

**Solution**  ;

**5.** The random process *X*(*n*) is created by starting with the *deterministic* sequence *Y*(*n*) = (-1)*n*:

{*Y*(0), *Y*(1), *Y*(2), …} = {1 -1 1 -1 1 -1 1 -1 …}, and adding ±1 to each term according to the outcome of a coin flip (+1 for heads, -1 for tails). So if the sequence of (independent) coin flips turned out to be HHTTHT…, the random process outcomes {*X*(0), *X*(1), *X*(2), …} would be

(1+1) (-1+1) (1-1) (-1-1) (1+1) (-1-1) ….

or 2 0 0 -2 2 -2 …..

What are the mean, autocorrelation, and autocovariance?

**Solution**  ;

except

(independent)

except

**6.** The random process {*X*(*n*)} is created by starting with the *nonrandom* repeating sequence

{1 , 2 , 3 , 1 , 2 , 3 , 1 , 2 , 3 , …} and adding to each term the outcome of a throw of a die (each integer 1 through 6 is equally likely). So if the sequence of (independent) throws turned out to be {4 2 3 4 1 5 …}, the random process outcomes would be {*X*(0), *X*(1), *X*(2), …} = {5 4 6 5 3 8 ….}. What are the following autocorrelations: *RX*(0, 0), *RX*(1, 1), *RX*(2, 2), *RX*(0, 1), *RX*(1, 0), *RX*(0, 2) ?

**Solution**  ;

;

;

(MATLAB® code: sum(sum([2:1:7]'\*[3:1:8]))/36 )

(MATLAB® code: sum(sum([2:1:7]'\*[4:1:9]))/36 )

**7.**  A sequence *X*(*n*) = {*X*(0), *X*(1), *X*(2), …} of coin flips is recorded in the following manner. For trials #0, 2, 4, 6, 8, ... and all even-numbered trials, the outcome of the experiment is 0 if the coin reads heads, and 1 if the coin reads tails. On odd-numbered trials #1, 3, 5, 7, 9, ... the outcome is 1 if the coin reads heads, and 0 if the coin reads tails. For example:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Trial | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Coin | H | H | H | T | H | T | T | H | T | T |
| *X* | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |

What are the mean, autocorrelation, and autocovariance of *X*?

**Solution**  **;**

,

except ;

(independence) ,

except

**8.** Suppose the random process ***X***(*t*) = *eat* is a family of exponentials depending on the value of a random variable *a* with pdf *fa*(*a*) . Express the mean, the autocorrelation, and the pdf *fX*(*t*)(*x*) of *X*(*t*) in terms of the pdf *fa*(*a*) of *a.*

**Solution**  ; ;

Let ; then ("derived distributions")

**9.** A fair coin is flipped one time. Define the process *X*(*t*) as follows: *X*(*t*) = cos π*t* if *heads* shows, *X*(*t*) = *t* if *tails* shows. Find *E*{*X*(*t*)} for *t =* 0.25, *t* = 0.5, and *t* = 1 .

**Solution**

**10.** Express the mean and autocorrelation of *Y*(*t*) = *X*(*t*) + 5 in terms of those of *X*(*t*).

**Solution**

**11**. Let *p*(*t*) be a periodic square wave as illustrated in Figure 2.16*c*. Suppose a random process is created according to *X*(*t*) = *p*(*t* − τ), where τis a random variable uniformly distributed over (0, *T*). Find .

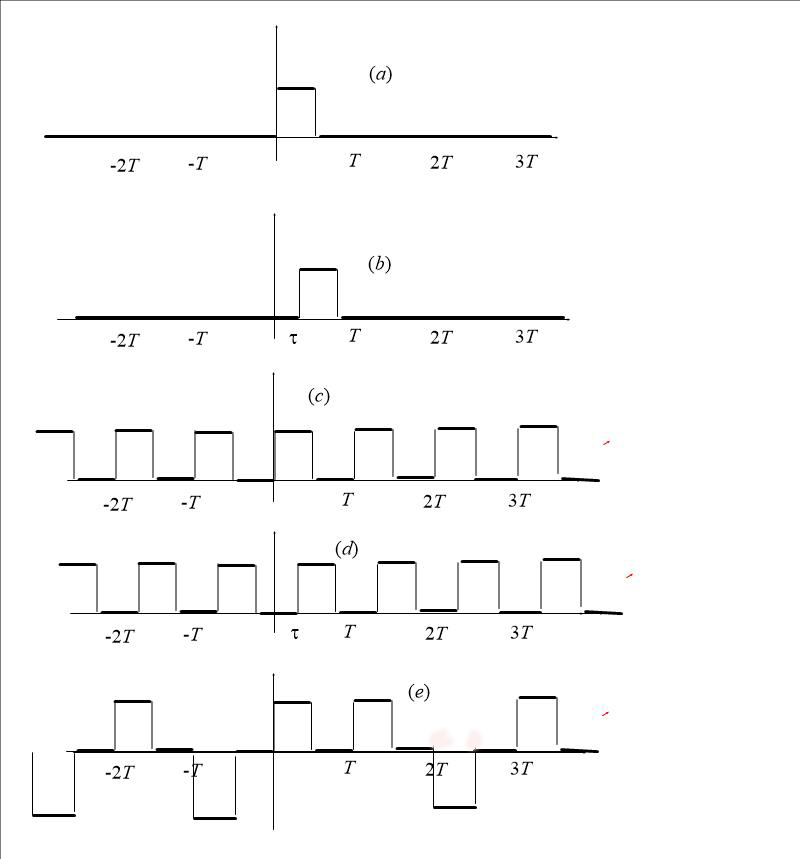
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Figure 2.16 (*a*) Pulse (*b*) Delayed pulse (*c*) Pulse train

(*d*) Delayed pulse train

(*e*) Pulse code modulation for *An* = … -1 1 -1 1 1 -1 1 …

(*All pulses have unit height*)

**Solution**

; ;

Over a time interval *T*, if |*t*2-*t*1| < *T*/2, the +1 portion of the pulse train will cover both *t*1 and *t*2 for *T*/2 - |*t*2-*t*1| sec.; if *T*/2 < |*t*2-*t*1| < *T*, the +1 portion of the train cannot cover both *t*1 and *t*2. If |*t*2-*t*1| > *T*, apply this to |*t*2-*t*1| *mod* *T*.

RX(*t*1, *t*2) = RX(|*t*2-*t*1|)

|*t*2-*t*1|

1/2

-*T*

*T*

0

**12.** Prove the second-moment identities (2.3) in Section 2.2.

**Solution**



**13.** Let and be independent processes.

**a.** Determine the autocorrelation functions of

and

in terms the moments of *X* and *Y*.

**b.** Determine the cross-correlation function of and .

**c.** How do these formulas simplify if *X* and *Y* have identical means and autocorrelations?

**Solution**  **(a)**

;

**(b)**

**(c)**  ;

**14.** Suppose the energy collected each day by a solar panel is normal *N*(μ,σ) and independent of the energies collected on other days, and that the collector is ideal (100% efficient, lossless). What are the mean and autocorrelation of the *accumulated* energy from day #1 to day #*n*?

**Solution**  ;

contains *n*2 terms *X*(*p*) *X*(*q*); but if |*j-k*|<*n*, then *n* - |*j-k*| have equal indices (*p*=*q*). Therefore

if |*j-k*|<*n*,

otherwise

**15.**  Find the mean and autocorrelation for the random process , when *c* is uniformly distributed in the interval (0, 1).

**Solution**  .

if 0 < *t* < 1; 0 otherwise.

. (Clearly this is also 0 unless .)

**16.** Suppose {*X*(0), *X*(1), *X*(2), …} is a discrete zero-mean random process, and that for each *n*, *X*(*n*) is independent of all other *X*(*j*) except *X*(*n*±1). The autocorrelation is given by *RX*(*n,n*) = 2, *RX*­(*n,n*-1) = *RX*­(*n,n*+1) = 1. What are the mean and autocorrelation of the "moving average" sequence ?

**Solution**  ;

A typical autocorrelation calculation is :

,

otherwise

**17.** If is a function only of *t*1 - *t*2, what is the autocorrelation of

(for constant *T*)?

**Solution**

**18.** (*Pulse-code-modulation*) Let *An* be a random sequence determined by coin flips, +1 for heads and -1 for tails, and let *p*(*t*) be the rectangular pulse train depicted in Figure 2.16*c*.

*X*(*t*) is the random process formed by modulating the pulse train by the sequence {*An*}: . A possible outcome is depicted in Figure 2.16*e*.

**a.** What are the mean and autocovariance of *X*(*t*)?

**b.** What are the mean and autocovariance of a randomly delayed version of

*X*(*t*): , where is uniformly distributed over the interval [0,*T*] and is statistically independent of the {*An*}? (See Figure 2.16*d*.)

**Solution (a)**

if *t*1 and *t*2 lie in the first half of the same pulse,

if *t*1 and *t*2 lie in the first half of different pulses (independence!),

otherwise

**(b)** as before. Reasoning as in (a), if |*t*1-*t*2| > *T*/2. Otherwise, then the probability that *t*1 and *t*2 lie in the first half of the same pulse equals [*T*/2 - |*t*1-*t*2|] / T ; *then*

**19.** Rework Problem 18 if the {*An*} are independent but take values +*a* with probability *p* and +*b* with probability (1-*p*).

**Solution**  (See #18 for discussion.)

**(a)**

if *t*1 and *t*2 lie in the first half of the same pulse,

if *t*1 and *t*2 lie in the first half of different pulses (independence!), otherwise

**(b)**

Reasoning as in (**a**), if |*t*1-*t*2| > *T*/2. Otherwise, then the probability that *t*1 and *t*2 lie in the first half of the same pulse equals [*T*/2 - |*t*1-*t*2|] / T ; *then*

*In communications engineering it is common to combine message signals with sinusoids* () *to facilitate their transmission. The solutions to Problems* 20-22 *are facilitated by the trigonometric identities*

**20.** A random process is given by , where *A* and *B* are independent zero mean random variables.

**a.** Find the mean *μX*(*t*).

**b.** Find the autocorrelation function  .

**c.** Under what conditions on the variances of *A* and *B* does  depend only on (*t*1-*t*2)?

**Solution (a)**

**(b)**  **,**

**(c)** if .

**21.** (*Quadrature amplitude modulation*) Consider the signals and , where *X*(*t*) and *Y*(*t*) are zero-mean independent processes with identical autocorrelation functions . Determine  and  in terms of . Show that if  depends only on (*t*1-*t*2), then so does . (How about ?)

**Solution** Of course .

**, ,**

**,**

**, ,**

If **, ,**  then

**,**  depends only on (but not **,**  ).

**22.** (*Binary phase shift keying*) The 1 coin flip sequence {*An*} described in Problem 18 is encoded for transmission as a phase shift in a sine wave:

,

with (see Figure 2.17). What are the mean and autocorrelation of *X*(*t*)? Be careful to distinguish between the cases when *t*1 and *t*2 lie in different, or the same, intervals.

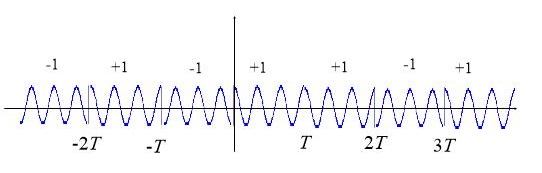


Figure 2.17 Binary phase shift keying for *An* = … -1 1 -1 1 1 -1 1 …

**Solution** Note that ,

so ;

**,**  ,

otherwise

**23.** Assume *X*(*t*) is a zero-mean Gaussian process with autocorrelation function  .

**a.** What is the probability that the value of *X*(7) lies between -1 and 2?

**b.** If *X*(6) is measured and found to have the value 2, what is the probability that the value of *X*(7) lies between -1 and 2?

**c.** What is the probability that *X*(6) and *X*(7) both lie between -1 and 2? (Consult Section 1.10 for MATLAB codes for numerical integration.)

**Solution**

**(a)** Therefore *X*(7) is *N*(0,1) and *p*(-1 < *X*(7) < 2) = 0.8186 .

If *x* = *X*(7) and *y* = *X*(6), then 

with .

**(b)** = *N*()

0.8362 .

**(c)** 0.7161

**24.** Suppose that *X*(*t*) is a zero-mean Gaussian process with autocorrelation function  .

**a.** What is the standard deviation of *X*(1.5)?

**b.** Write out the formula for the joint probability density of and .

**c.** Write out the formula for the marginal probability density of .

**d.** If *X*(1.5) is determined to be 3, what are the (conditional) mean and standard deviation of *X*(2)?

**e.** Write out the formula for the conditional probability density for , given that *X*(*t*1) = 3 .

**f.** *X*(3) and *X*(4) are independent, but *X*(1) and *X*(1.5) are not. Explain.

**Solution** As in #23, ; **(a)** ;

**(b)**  with *X* = *X*(*t*1), *Y* = *X*(*t*2),

.

**(c)**

**(d)** , .

**(e)**

**(f)** ; uncorrelated zero-mean Gaussian variables are independent.

**25.** If *X*(*t*) is a Gaussian random process whose first order pdf has constant mean = 0 and whose autocorrelation is given by , what is its third-order joint pdf ?

**Solution**  ; ;

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;

;



**26.** Prove the formula (2.9), Section 2.3, for the least mean square error.

**Solution** Observe that from the extended second moment identity as expressed in "Summary of Important Equations for Bivariate Random Variables", Section 1.11, .

Therefore (equations (2.7, 2.8))



which can be rearranged as



after the cancellations are tabulated.