Exercises for Chapter 2

In this problem set, interpret all discrete sequences as starting from n = 0: {$X(0)$, $X(1)$, $X(2)$, …}.

1. A coin is flipped each second. When it is heads, the random variable $X(n)$ takes the value +1; when it is tails then $X(n)$ = -1. What are the mean and autocorrelation of {$X(0)$, $X(1)$, $X(2)$, …}?

Solution $\mu_X(n) = 0$; $R_X(n_1, n_2) = C_X(n_1, n_2) = 0$ (independent) except $R_X(n_1, n_1) = C_X(n_1, n_1) = 1$

2. The random process {$X(0)$, $X(1)$, $X(2)$, …} is created by a spinning dial that selects numbers

$X(n)$ = [1, 2, or 3] with equal probability. What are the mean, autocorrelation, and autocovariance for this process?

Solution $\mu_X(n) = \frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times 3 = 2$;

$R_X(n_1, n_2) = \frac{1}{9} \times 1 \times 1 + \frac{1}{9} \times 1 \times 2 + \frac{1}{9} \times 1 \times 3 + \frac{1}{9} \times 2 \times 1 + \cdots = 4$;

except $R_X(n_1, n_1) = \frac{1}{3} \times 1 \times 1 + \frac{1}{3} \times 2 \times 2 + \frac{1}{3} \times 3 \times 3 = \frac{14}{3}$

$C_X(n_1, n_2) = \frac{1}{9} \times (1-2) \times (1-2) + \frac{1}{9} \times (1-2) \times (2-2) + \cdots = 0$; (independent)

except $C_X(n_1, n_1) = \frac{1}{3} \times (1-2) \times (1-2) + \frac{1}{3} \times 0 \times 0 + \frac{1}{3} \times 1 \times 1 = \frac{2}{3}$

3. There are 3 signal generators in a box. The first one puts out the sequence {$X(0)$, $X(1)$, $X(2)$, …} = {1 1 1 1 1 ….}. The second puts out {2 0 2 0 2 0 2 ...}. The third puts out {0 2 0 2 0 2 ...}. One of these generators is selected at random (equally likely). What are the mean μ(n), autocorrelation $R_x(m,n)$, and autocovariance $C_x(m,n)$ of the resulting sequence for m=3 and n =4?

Solution $\mu_X(4) = \frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times 0 = 1$ $\left(\mu_X(3) = \frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 1\right)$;

$R_X(3,4) = \frac{1}{3} \times 1 \times 1 + \frac{1}{3} \times 0 \times 2 + \frac{1}{3} \times 2 \times 0 = \frac{1}{3}$;

$C_X(3,4) = \frac{1}{3} \times \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{1}{3}\right) + \frac{1}{3} \times \left(0 - \frac{1}{3}\right) \times \left(2 - \frac{1}{3}\right) + \frac{1}{3} \times \left(2 - \frac{1}{3}\right) \times \left(0 - \frac{1}{3}\right) = -\frac{2}{3}$

4. The random process $X(n)$ is determined by a single flip of a fair coin. If the coin shows heads then the process is a series of ones: {$X(0)$, $X(1)$, $X(2)$, …} = {1 1 1 1 1 1 1 1 1 ….}. If the coin shows tails then $X(n) = 1/(n+1)$: $\{X(0), X(1), X(2), \dots\} = \{1\ 1/2\ 1/3\ 1/4\ 1/5\ \dots\}$. What are the mean and the autocorrelation?

Solution $\mu_X(n) = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{n+1} = \frac{n+2}{2(n+1)}$;

$$R_X(n_1, n_2) = \frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times \frac{1}{n_1+1} \frac{1}{n_2+1}$$

5. The random process $X(n)$ is created by starting with the *deterministic* sequence $Y(n) = (-1)^n$:

$\{Y(0), Y(1), Y(2), \dots\} = \{1\ \ -1\ \ 1\ \ -1\ \ 1\ \ -1\ \ 1\ \ -1\ \ \dots\}$, and adding ± 1 to each term according to the outcome of a coin flip (+1 for heads, -1 for tails). So if the sequence of (independent) coin flips turned out to be HHTTHT…, the random process outcomes $\{X(0), X(1), X(2), \dots\}$ would be

(1+1) (-1+1) (1-1) (-1-1) (1+1) (-1-1) ….

or 2 0 0 -2 2 -2 …..

What are the mean, autocorrelation, and autocovariance?

Solution $\mu_X(n) = (-1)^n + \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = (-1)^n$;

$$R_X(n_1, n_2) = \frac{1}{4} \times [(-1)^{n_1} + 1]\,[(-1)^{n_2} + 1] + \frac{1}{4} \times [(-1)^{n_1} - 1]\,[(-1)^{n_2} - 1]$$

$$+\frac{1}{4} \times [(-1)^{n_1} - 1]\,[(-1)^{n_2} + 1] + \frac{1}{4} \times [(-1)^{n_1} + 1]\,[(-1)^{n_2} - 1]$$

$$= (-1)^{n_1+n_2}$$

except $R_X(n_1, n_1) = (-1)^{n_1+n_1} + 1 = 2$

$$C_X(n_1, n_2) = \frac{1}{4} \times [+1]\,[+1] + \frac{1}{4} \times [-1]\,[-1] + \frac{1}{4} \times [-1]\,[+1] + \frac{1}{4} \times [+1]\,[-1] = 0$$

(independent)

except $C_X(n_1, n_1) = \frac{1}{2} \times (1)^2 + \frac{1}{2} \times (-1)^2 = 1$

6. The random process $\{X(n)\}$ is created by starting with the *nonrandom* repeating sequence

$\{1, 2, 3, 1, 2, 3, 1, 2, 3, \dots\}$ and adding to each term the outcome of a throw of a die (each integer 1 through 6 is equally likely). So if the sequence of (independent) throws turned out to be $\{4\ 2\ 3\ 4\ 1\ 5\ \dots\}$, the random process outcomes would be $\{X(0), X(1), X(2), \dots\} = \{5\ 4\ 6\ 5\ 3\ 8\ \dots\}$. What are the following autocorrelations: $R_X(0, 0)$, $R_X(1, 1)$, $R_X(2, 2)$, $R_X(0, 1)$, $R_X(1, 0)$, $R_X(0, 2)$?

Solution $R_X(0,0) = \frac{1}{6} \times (1+1)^2 + \frac{1}{6} \times (1+2)^2 + \dots + \frac{1}{6} \times (1+6)^2 = 23\frac{1}{6}$;

$R_X(1,1) = \frac{1}{6} \times (2+1)^2 + \frac{1}{6} \times (2+2)^2 + \cdots + \frac{1}{6} \times (2+6)^2 + \cdots = 33\frac{1}{6}$;

$R_X(2,2) = \frac{1}{6} \times (3+1)^2 + \frac{1}{6} \times (3+2)^2 + \cdots + \frac{1}{6} \times (3+6)^2 + \cdots = 45\frac{1}{6}$;

$$R_X(0,1) = \frac{1}{36} \times (1+1)(2+1) + \frac{1}{36} \times (1+1)(2+2) + \cdots + \frac{1}{36} \times (1+1)(2+6)$$
$$+ \frac{1}{36} \times (1+2)(2+1) + \frac{1}{36} \times (1+2)(2+2) + \cdots = 24.75 = R_X(1,0)$$

(MATLAB® code: sum(sum([2:1:7]'*[3:1:8]))/36)

$$R_X(0,2) = \frac{1}{36} \times (1+1)(3+1) + \frac{1}{36} \times (1+1)(3+2) + \cdots + \frac{1}{36} \times (1+1)(3+6)$$
$$+ \frac{1}{36} \times (1+2)(3+1) + \frac{1}{36} \times (1+2)(3+2) + \cdots = 29.25$$

(MATLAB® code: sum(sum([2:1:7]'*[4:1:9]))/36)

7. A sequence $X(n) = \{X(0), X(1), X(2), \ldots\}$ of coin flips is recorded in the following manner. For trials #0, 2, 4, 6, 8, ... and all even-numbered trials, the outcome of the experiment is 0 if the coin reads heads, and 1 if the coin reads tails. On odd-numbered trials #1, 3, 5, 7, 9, ... the outcome is 1 if the coin reads heads, and 0 if the coin reads tails. For example:

Trial	0	1	2	3	4	5	6	7	8	9
Coin	H	H	H	T	H	T	T	H	T	T
X	0	1	0	0	0	0	1	1	1	0

What are the mean, autocorrelation, and autocovariance of X?

Solution $\mu_X(n) = 1/2$ **;**

$R_X(n_1,n_2) = \frac{1}{4} \times 0 \times 0 + \frac{1}{4} \times 0 \times 1 + \frac{1}{4} \times 1 \times 0 + \frac{1}{4} \times 1 \times 1 = \frac{1}{4}$,

except $R_X(n_1,n_1) = \frac{1}{2} \times 0 \times 0 + \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$;

$$C_X(n_1,n_2) = \frac{1}{4} \times \left(0 - \frac{1}{2}\right) \times \left(0 - \frac{1}{2}\right) + \frac{1}{4} \times \left(0 - \frac{1}{2}\right) \times \left(1 - \frac{1}{2}\right)$$
$$+ \frac{1}{4} \times (1 - \tfrac{1}{2}) \times (0 - \tfrac{1}{2}) + \frac{1}{4} \times (1 - \tfrac{1}{2}) \times (1 - \tfrac{1}{2}) = 0 \quad \text{(independence)},$$

except $C_X(n_1,n_1) = \frac{1}{2} \times (0 - \frac{1}{2}) \times (0 - \frac{1}{2}) + \frac{1}{2} \times (1 - \frac{1}{2}) \times (1 - \frac{1}{2}) = \frac{1}{4}$

8. Suppose the random process $\boldsymbol{X}(t) = e^{at}$ is a family of exponentials depending on the value of a random variable a with pdf $f_a(a)$. Express the mean, the autocorrelation, and the pdf $f_{X(t)}(x)$ of $X(t)$ in terms of the pdf $f_a(a)$ of a.

Solution $\mu_X(t) = \int f_a(a)\, e^{at}\, da$; $R_X(t_1, t_2) = \int f_a(a)\, e^{a(t_1+t_2)}\, da$;

Let $x = e^{at}$; then $f_{X(t)}(x) = \frac{f_a(a)}{|dx/da|} = \frac{f_a(\frac{\ln x}{t})}{t\, e^{at}} = \frac{f_a(\frac{\ln x}{t})}{t\, x}$ ("derived distributions")

9. A fair coin is flipped one time. Define the process $X(t)$ as follows: $X(t) = \cos \pi t$ if *heads* shows, $X(t) = t$ if *tails* shows. Find $E\{X(t)\}$ for $t = 0.25$, $t = 0.5$, and $t = 1$.

Solution $E\{X(t)\} = \frac{1}{2}\cos \pi t + \frac{1}{2} t$

10. Express the mean and autocorrelation of $Y(t) = X(t) + 5$ in terms of those of $X(t)$.

Solution $E\{Y(t)\} = \mu_{X(t)} + 5$;

$R_Y(t_1, t_2) = E\{[X(t_1) + 5][X(t_2) + 5]\} = R_X(t_1, t_2) + 5(\mu_{X(t_1)}\mu_{X(t_2)}) + 25$

11. Let $p(t)$ be a periodic square wave as illustrated in Figure 2.16*c*. Suppose a random process is created according to $X(t) = p(t - \tau)$, where τ is a random variable uniformly distributed over $(0, T)$. Find $f_{X(t)}(x), \mu_{X(t)}, and\ R_{X(t)}(t_1, t_2)$.

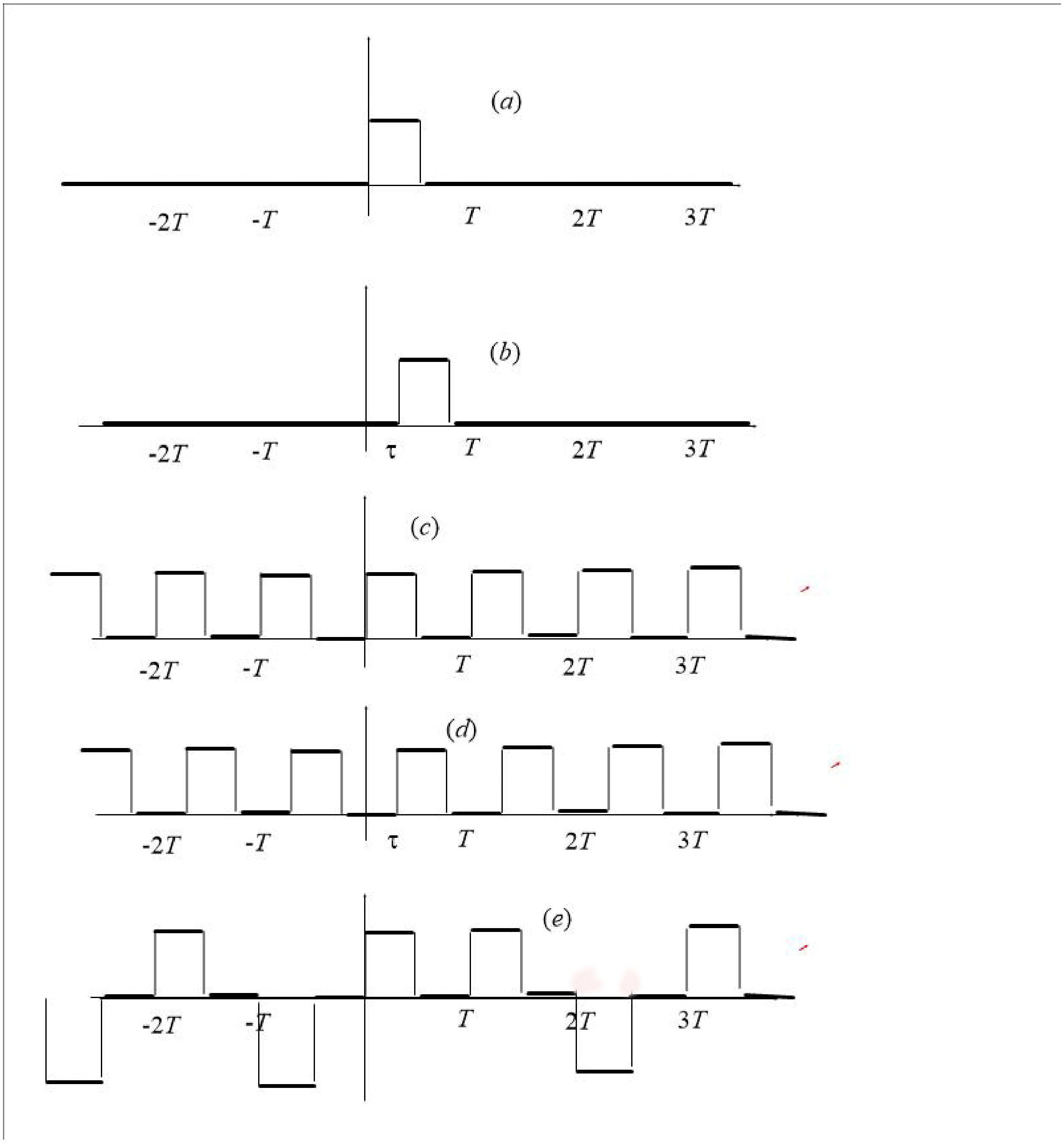


Figure 2.16 (a) Pulse $p(t)$ (b) Delayed pulse $p(t-\tau)$ (c) Pulse train $\sum_{n=-\infty}^{\infty} p(t-nT)$

(d) Delayed pulse train $\sum_{n=-\infty}^{\infty} p(t-nT-\tau)$

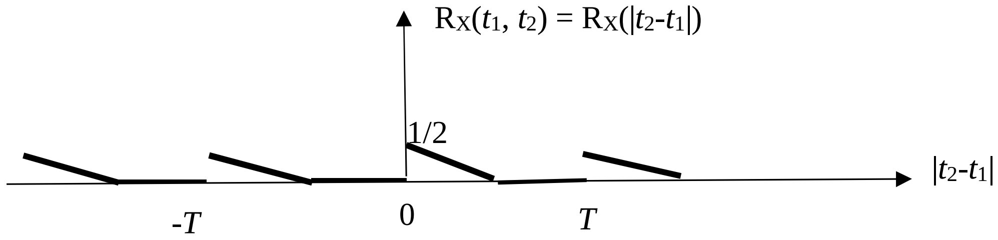
(e) Pulse code modulation for A_n = … -1 1 -1 1 1 -1 1 …

(*All pulses have unit height*)

Solution

$f_{X(t)=p(t-\tau)}(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1)$; $\mu_X(t) = 0.5$;

Over a time interval T, if $|t_2-t_1| < T/2$, the +1 portion of the pulse train will cover both t_1 and t_2 for $T/2 - |t_2-t_1|$ sec.; if $T/2 < |t_2-t_1| < T$, the +1 portion of the train cannot cover both t_1 and t_2. If $|t_2-t_1| > T$, apply this to $|t_2-t_1|$ *mod T*.



12. Prove the second-moment identities (2.3) in Section 2.2.

Solution

$$\begin{aligned} C_{XY}(t_1,t_2) &= E\{[X(t_1)-\mu_X(t_1)][Y(t_2)-\mu_Y(t_2)]\} \\ &= E\{X(t_1)Y(t_2)\} - E\{X(t_1)\mu_Y(t_2)\} - E\{\mu_X(t_1)Y(t_2)\} + \mu_X(t_1)\mu_Y(t_2) \\ &= R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2) \end{aligned}$$

13. Let $X(t)$ and $Y(t)$ be independent processes.

a. Determine the autocorrelation functions of

$$Z_1(t) = X(t) + Y(t) \text{ and } Z_2(t) = X(t) - Y(t)$$

in terms the moments of X and Y.

b. Determine the cross-correlation function of $Z_1(t)$ and $Z_2(t)$.

c. How do these formulas simplify if X and Y have identical means and autocorrelations?

Solution **(a)** $E\{Z_1(t_1)Z_1(t_2)\} = E\{[X(t_1) + Y(t_1)][X(t_2) + Y(t_2)]\}$

$$= R_X(t_1,t_2) + R_{XY}(t_1,t_2) + R_{YX}(t_1,t_2) + R_Y(t_1,t_2)\ ;$$

$$E\{Z_2(t_1)Z_2(t_2)\} = R_X(t_1,t_2) - R_{XY}(t_1,t_2) - R_{YX}(t_1,t_2) + R_Y(t_1,t_2)$$

(b) $E\{Z_1(t_1)Z_2(t_2)\} = E\{[X(t_1) + Y(t_1)][X(t_2) - Y(t_2)]\}$

$$= R_X(t_1,t_2) - R_{XY}(t_1,t_2) + R_{YX}(t_1,t_2) - R_Y(t_1,t_2)$$

(c) $E\{Z_1(t_1)Z_1(t_2)\} = 2R_X(t_1,t_2) + R_{XY}(t_1,t_2) + R_{YX}(t_1,t_2)$;

$E\{Z_2(t_1)Z_2(t_2)\} = 2R_X(t_1,t_2) - R_{XY}(t_1,t_2) - R_{YX}(t_1,t_2)$

$E\{Z_1(t_1)Z_2(t_2)\} = -R_{XY}(t_1,t_2) + R_{YX}(t_1,t_2)$

14. Suppose the energy collected each day by a solar panel is normal $N(\mu,\sigma)$ and independent of the energies collected on other days, and that the collector is ideal (100% efficient, lossless). What are the mean and autocorrelation of the *accumulated* energy from day #1 to day #n?

Solution $E\{X(j) + X(j+1) + \cdots X(j+n-1)\} = n\mu$;

$E\{X(j)X(k)\} = \mu^2 + \sigma^2$ if $j = k$, μ^2 otherwise

$[X(j) + X(j+1) + \cdots X(j+n-1)][X(k) + X(k+1) + \cdots X(k+n-1)]$ contains n^2 terms $X(p)\,X(q)$; but if $|j\text{-}k|<n$, then $n - |j\text{-}k|$ have equal indices (p=q). Therefore

$E\{[X(j) + X(j+1) + \cdots X(j+n-1)][X(k) + X(k+1) + \cdots X(k+n-1)]\}$

$= (n - |j-k|)(\mu^2 + \sigma^2) + (n^2 - n + |j-k|)\mu^2 = n^2\mu^2 + (n - |j-k|)\sigma^2$ if $|j\text{-}k|<n$,

$= n^2\mu^2$ otherwise

15. Find the mean and autocorrelation for the random process $X(t) = \delta(t-c)$, when c is uniformly distributed in the interval (0, 1).

Solution $f_C(c) = 1$ if $0 < c < 1$, 0 otherwise.

$E\{\delta(t-c)\} = \int_0^1 \delta(t-c)\ f_C(c)\ dc = f_C(t) = 1$ if $0 < t < 1$; 0 otherwise.

$$E\{\delta(t_1-c)\delta(t_2-c)\} = \int_0^1 \delta(t_1-c)\delta(t_2-c)\ f_C(c)\ dc = f_C(t_2)\delta(t_1-t_2)$$

$= \delta(t_1-t_2)$ if $0 < t_2 < 1$, 0 otherwise. (Clearly this is also 0 unless $0 < t_1 < 1$.)

16. Suppose $\{X(0), X(1), X(2), \ldots\}$ is a discrete zero-mean random process, and that for each n, $X(n)$ is independent of all other $X(j)$ except $X(n\pm1)$. The autocorrelation is given by $R_X(n,n) = 2$, $R_X(n,n\text{-}1) = R_X(n,n\text{+}1) = 1$. What are the mean and autocorrelation of the "moving average" sequence $Y(n) = \dfrac{X(n) + X(n-1) + X(n-2)}{3}$?

Solution $E(Y(n)\} = \frac{0+0+0}{3}$;

A typical autocorrelation calculation is $R_Y(n, n+1)$:

$$E\{Y(n)Y(n+1)\} = (\frac{1}{3})^2 E\{[X(n) + X(n-1) + X(n-2)][X(n+1) + X(n) + X(n-1)]\}$$

$$= \frac{1}{9}E\{X(n)X(n+1) + X(n)X(n) + X(n)X(n-1) + X(n-1)X(n+1) + X(n-1)X(n)$$

$$+X(n-1)X(n-1) + X(n-2)X(n+1) + X(n-2)X(n) + X(n-2)X(n-1)]\}$$

$$= \frac{1}{9}\{1+2+1+0+1+2+0+0+1\} = \frac{8}{9}$$

$R_Y(n, n\pm 3) = \frac{1}{9}$, $R_Y(n, n\pm 2) = \frac{4}{9}$, $R_Y(n, n\pm 1) = \frac{8}{9}$, $R_Y(n,n) = \frac{10}{9}$,

$R_Y(n,m) = 0$ otherwise

17. If $R_X(t_1, t_2)$ is a function only of t_1 - t_2, what is the autocorrelation of

$Y(t) \equiv X(t) - X(t-T)$ (for constant T)?

Solution $E\{(Y(t_1, t_2)\} = E\{[X(t_1) - X(t_1 - T)][X(t_2) - X(t_2 - T]\}$

$$= 2R_X(t_1 - t_2) - R_X(t_1 - t_2 + T) - R_X(t_1 - t_2 - T)\}$$

18. (*Pulse-code-modulation*) Let A_n be a random sequence determined by coin flips, +1 for heads and -1 for tails, and let $p(t)$ be the rectangular pulse train depicted in Figure 2.16*c*.

$X(t)$ is the random process formed by modulating the pulse train by the sequence $\{A_n\}$: $X(t) = \sum_{n=-\infty}^{\infty} A_n\, p(t - nT)$. A possible outcome is depicted in Figure 2.16*e*.

a. What are the mean and autocovariance of $X(t)$?

b. What are the mean and autocovariance of a randomly delayed version of

$X(t)$: $Y(t) = X(t-\tau) = \sum_{n=-\infty}^{\infty} A_n\, p(t - nT - \tau)$, where τ is uniformly distributed over the interval $[0,T]$ and is statistically independent of the $\{A_n\}$? (See Figure 2.16*d*.)

Solution **(a)** $\mu_X(t) = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$ if $[t \bmod T] < \frac{T}{2}$, and 0 otherwise as well.

$C_X(t_1, t_2) = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$ if t_1 and t_2 lie in the first half of the same pulse,

$= \frac{1}{4}(1)(1) + \frac{1}{4}(1)(-1) + \frac{1}{4}(-1)(1) + \frac{1}{4}(1)(1) = 0$ if t_1 and t_2 lie in the first half of different pulses (independence!),

$= 0$ otherwise

(b) $\mu_X(t) = 0$ as before. Reasoning as in (a), $C_X(t_1, t_2) = 0$ if $|t_1\text{-}t_2| > T/2$. Otherwise, then the probability that t_1 and t_2 lie in the first half of the same pulse equals $[T/2 - |t_1\text{-}t_2|] / \mathrm{T}$; *then*

$$C_X(t_1, t_2) = \frac{\frac{T}{2}-|t_1-t_2|}{T}\frac{1}{2}(1)^2 + \frac{\frac{T}{2}-|t_1-t_2|}{T}\frac{1}{2}(-1)^2 = \frac{\frac{T}{2}-|t_1-t_2|}{T} = \frac{1-\frac{|t_1-t_2|}{T/2}}{2}$$

19. Rework Problem 18 if the $\{A_n\}$ are independent but take values $+a$ with probability p and $+b$ with probability $(1\text{-}p)$.

Solution (See #18 for discussion.)

(a) $\mu_X(t) = pa + (1-p)b$ if $[t \bmod T] < \frac{T}{2}$, and 0 otherwise.

$C_X(t_1, t_2) = p(a - [pa + (1-p)b])^2 + (1-p)(b - [pa + (1-p)b])^2$ if t_1 and t_2 lie in the first half of the same pulse,

$= 0$ if t_1 and t_2 lie in the first half of different pulses (independence!),
$= 0$ otherwise

(b) $\mu_X(t) = a \times p \times \text{Prob}(t \text{ lies in the first half of the pulse})$

$+ b \times (1-p) \times \text{Prob}(t \text{ lies in the first half of the pulse})$

$+ 0 \times \text{Prob}(t \text{ lies in the second half of the pulse})$

$= \frac{pa+(1-p)b}{2}$

Reasoning as in (**a**), $C_X(t_1, t_2) = 0$ if $|t_1\text{-}t_2| > T/2$. Otherwise, then the probability that t_1 and t_2 lie in the first half of the same pulse equals $[T/2 - |t_1\text{-}t_2|] / \mathrm{T}$; *then*

$$C_X(t_1, t_2) = \frac{\frac{T}{2}-|t_1-t_2|}{T}\{p(a - [pa + (1-p)b])^2 + (1-p)(b - [pa + (1-p)b])^2\}$$

In communications engineering it is common to combine message signals with sinusoids $(\cos \omega t, \sin \omega t)$ *to facilitate their transmission. The solutions to Problems* 20-22 *are facilitated by the trigonometric identities*

$$\cos A \ \cos B = \frac{\cos(A-B)+\cos(A+B)}{2}$$

$$\sin A \ \sin B = \frac{\cos(A-B)-\cos(A+B)}{2}$$

$$\sin A \ \cos B = \frac{\sin(A-B)+\sin(A+B)}{2}$$

20. A random process is given by $X(t) = A\cos\omega t + B\sin\omega t$, where A and B are independent zero mean random variables.

a. Find the mean $\mu_X(t)$.

b. Find the autocorrelation function $R_X(t_1, t_2)$.

c. Under what conditions on the variances of A and B does $R_X(t_1, t_2)$ depend only on (t_1-t_2)?

Solution (a) $\mu_X(t) = 0 \times \cos\omega t + 0 \times \sin\omega t = 0$

(b) $R_X(t_1, t_2) = E\{[A\cos\omega t_1 + B\sin\omega t_1][A\cos\omega t_2 + B\sin\omega t_2]\}$

$$= \cos\omega t_1 \cos\omega t_2\, E\{A^2\} + \sin\omega t_1 \sin\omega t_2\, E\{B^2\} + [\cdots]E\{AB\}$$

$$(= 0)$$

$$= \cos\omega t_1 \cos\omega t_2\, \sigma_A^2 + \sin\omega t_1 \sin\omega t_2\, \sigma_B^2$$

(c) $= \cos\omega(t_1 - t_2)\, \sigma_A^2$ if $\sigma_A = \sigma_B$.

21. (*Quadrature amplitude modulation*) Consider the signals $Z_1(t) = X(t)\cos\omega t$ and $Z_2(t) = X(t)\cos\omega t + Y(t)\sin\omega t$, where $X(t)$ and $Y(t)$ are zero-mean independent processes with identical autocorrelation functions $R_X \equiv R_Y$. Determine $R_{Z_1}(t_1, t_2)$ and $R_{Z_2}(t_1, t_2)$ in terms of $R_X(t_1, t_2)$. Show that if $R_X(t_1, t_2)$ depends only on (t_1-t_2), then so does $R_{Z_2}(t_1, t_2)$. (How about $R_{Z_1}(t_1, t_2)$?)

Solution Of course $\mu_{Z_1}(t) = \mu_{Z_1}(t) = 0$.

$$R_{Z_1}(t_1, t_2) = E\{X(t_1)\cos\omega t_1\, X(t_2)\cos\omega t_2\} = R_X(t_1, t_2)\cos\omega t_1 \cos\omega t_2$$

$$R_{Z_2}(t_1, t_2) = E\{[X(t_1)\cos\omega t_1 + Y(t_1)\sin\omega t_1]\,[X(t_2)\cos\omega t_2 + Y(t_2)\sin\omega t_2]\}$$

$$= R_X(t_1, t_2)\cos\omega t_1 \cos\omega t_2 + R_Y(t_1, t_2)\sin\omega t_1 \sin\omega t_2 + [\cdots R_{XY} = 0 \cdots]$$

If $R_X(t_1, t_2) = R_Y(t_1, t_2) = R_X(t_1 - t_2)$ then

$R_{Z_2}(t_1, t_2) = R_X(t_1 - t_2)\cos\omega(t_1\text{-}t_2)$ depends only on t_1-t_2 (but not $R_{Z_1}(t_1, t_2)$).

22. (*Binary phase shift keying*) The ± 1 coin flip sequence $\{A_n\}$ described in Problem 18 is encoded for transmission as a $\pm\pi/2$ phase shift in a sine wave:

$$X(t) = \sin(\omega t + A_n \pi/2) = \begin{cases} \cos \omega t & \text{if } A_n = +1 \\ -\cos \omega t & \text{if } A_n = -1 \end{cases} \quad \text{for } nT \le t < (n+1)T \; ,$$

with $T \gg 2\pi/\omega$ (see Figure 2.17). What are the mean and autocorrelation of $X(t)$? Be careful to distinguish between the cases when t_1 and t_2 lie in different, or the same, intervals.

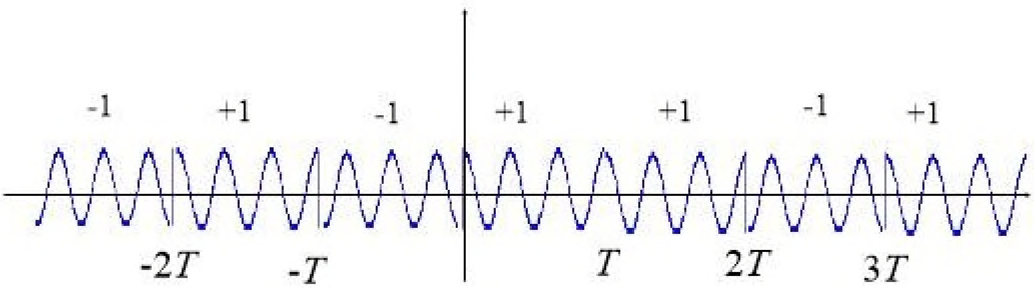


Figure 2.17 Binary phase shift keying for $A_n = \ldots$ -1 1 -1 1 1 -1 1 …

Solution Note that $X(t) = A_n \cos \omega t$ for $nT \le t < (n+1)T$,

so $\mu_X(t) = E\{A\} \cos \omega t = 0$;

$$R_X(t_1, t_2) = E\{A_n \cos \omega t_1 \, A_n \cos \omega t_2\} = (1) \cos \omega t_1 \cos \omega t_2 \quad \text{for } nT \le t_1, t_2 < (n+1)T \; ,$$

$$E\{A_n A_{m \ne n}\} (\cdots) = 0 \quad \text{otherwise}$$

23. Assume $X(t)$ is a zero-mean Gaussian process with autocorrelation function $R_X(t_1, t_2) = e^{-|t_1 - t_2|/2}$.

a. What is the probability that the value of $X(7)$ lies between -1 and 2?

b. If $X(6)$ is measured and found to have the value 2, what is the probability that the value of $X(7)$ lies between -1 and 2?

c. What is the probability that $X(6)$ and $X(7)$ both lie between -1 and 2? (Consult Section 1.10 for MATLAB codes for numerical integration.)

Solution $E\{X(t)\} = 0$; $E\{X(t)^2\} = R_x(t,t) = e^0 = 1$.

(a) Therefore $X(7)$ is $N(0,1)$ and $p(-1 < X(7) < 2) = 0.8186$.

If $x = X(7)$ and $y = X(6)$, then $f_{XY}(x,y) = e^{-\frac{1}{2(1-\rho^2)}[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}]} \Big/ 2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}$

with $\mu_X = \mu_Y = 0,\ \sigma_X = \sigma_Y = 1, \rho = \frac{R_X(7-6)}{1\times 1} = e^{-1/2}$.

(b) $f_{X|Y}(x|y=2) = N(\mu_X + \rho\dfrac{\sigma_X}{\sigma_Y}(y-\mu_Y), \sigma_X\sqrt{1-\rho^2})$

$$= N(0 + e^{-1/2}(1)(2-0), (1)\sqrt{1-e^{-2/2}})$$

$p(-1 < X(7) < 2 \mid X(6) = 2) = 0.8362$.

(c) $p(-1 < X(7), X(6) < 2) = 0.7161$

24. Suppose that $X(t)$ is a zero-mean Gaussian process with autocorrelation function $R_X(t_1,t_2) = \dfrac{\sin\pi(t_1-t_2)}{t_1-t_2}$.

a. What is the standard deviation of $X(1.5)$?

b. Write out the formula for the joint probability density $f_{X(t_1)X(t_2)}(x_1,x_2)$ of $X(t_1)$ and $X(t_2)$.

c. Write out the formula for the marginal probability density $f_{X(t_1)}(x)$ of $X(t_1)$.

d. If $X(1.5)$ is determined to be 3, what are the (conditional) mean and standard deviation of $X(2)$?

e. Write out the formula for the conditional probability density $f_{X(t_2)|X(t_1)}(x_2|x_1)$ for $X(t_2)$, given that $X(t_1) = 3$.

f. $X(3)$ and $X(4)$ are independent, but $X(1)$ and $X(1.5)$ are not. Explain.

Solution As in #23, $E\{X(t)\} = 0$; $E\{X(t)^2\} = R_x(t,t) = \dfrac{\sin\pi(0)}{0} = \pi$; **(a)** $\sigma_X(t) \equiv \sqrt{\pi}$;

(b) $f_{XY}(x,y) = e^{-\frac{1}{2(1-\rho^2)}[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}]} \Big/ 2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}$ with $X = X(t_1)$, $Y = X(t_2)$,

$\mu_X = \mu_Y = 0,\ \sigma_X = \sigma_Y = \sqrt{\pi}, \rho = \frac{R_X(t_1-t_2)}{\pi} = \frac{1}{\pi}\frac{\sin\pi(t_1-t_2)}{t_1-t_2}$.

(c) $N(0,\sqrt{\pi})$

(d) $\mu_{X(2)|\,X(1.5)=3} = 0 + \frac{\sin .5\pi}{.5}(1)(3-0)$, $\sigma_{X(2)|\,X(1.5)=3} = \sqrt{\pi}\sqrt{1-(\frac{\sin .5\pi}{.5})^2}$.

(e) $N(\frac{\sin .5\pi}{.5}3, \sqrt{\pi}\sqrt{1-(\frac{\sin .5\pi}{.5})^2})$

(f) $R_x(3{,}4) = \frac{\sin(1)\pi}{(1)} = 0$, $R_x(1, 1.5) = \frac{\sin(.5)\pi}{.5} = 2$; uncorrelated zero-mean Gaussian variables are independent.

25. If $X(t)$ is a Gaussian random process whose first order pdf $f_{X(t)}(x)$ has constant mean = 0 and whose autocorrelation is given by $R_X(t_1,t_2) = e^{-|t_1-t_2|}$, what is its third-order joint pdf $f_{X(t_1),X(t_2),X(t_3)}(x_1, x_2, x_3)$?

Solution $\boldsymbol{x} = [X(t_1)\ \ X(t_2)\ \ X(t_3)]$; $\boldsymbol{\mu} = E\{[X(t_1)\ \ X(t_2)\ \ X(t_3)]\} = [0\ 0\ 0]$;

$\boldsymbol{\sigma} = \left[\sqrt{R_X(t_1,t_1)}\ \ \sqrt{R_X(t_2,t_2)}\ \ \sqrt{R_X(t_3,t_3)}\right] = [1\ 1\ 1]$;

$\rho_{X(t_j),X(t_k)} = e^{-|t_j-t_k|}/1\times 1$;

$$\mathbf{Cov} = \begin{bmatrix} 1 & e^{-|t_2-t_1|} & e^{-|t_3-t_1|} \\ e^{-|t_2-t_1|} & 1 & e^{-|t_3-t_2|} \\ e^{-|t_3-t_1|} & e^{-|t_3-t_2|} & 1 \end{bmatrix};$$

$$f_{\mathbf{X}}(\mathbf{x}) = e^{-\frac{1}{2}[\mathbf{x}-\mu][\mathbf{Cov}]^{-1}[\mathbf{x}-\mu]^T} \Big/ (2\pi)^{3/2}\sqrt{\det(\mathbf{Cov})}$$

26. Prove the formula (2.9), Section 2.3, for the least mean square error.

Solution Observe that from the extended second moment identity as expressed in "Summary of Important Equations for Bivariate Random Variables", Section 1.11, $E\{BN\} = \mu_N\mu_B + \rho\sigma_B\sigma_N$.

Therefore (equations (2.7, 2.8))

$$E\{[B-(\alpha+\beta N)]^2\} = E\{B^2+\alpha^2+\beta^2 N^2-2\alpha B-2\beta BN+2\alpha\beta N\}$$

$$=\sigma_B^2+\mu_B^2+[\mu_B-\rho\frac{\sigma_B}{\sigma_N}\mu_N]^2+(\rho\frac{\sigma_B}{\sigma_N})^2(\mu_N^2+\sigma_N^2)-2[\mu_B-\rho\frac{\sigma_B}{\sigma_N}\mu_N]\mu_B$$

$$-2\rho\frac{\sigma_B}{\sigma_N}[\mu_N\mu_B+\rho\sigma_B\sigma_N]+2[\mu_B-\rho\frac{\sigma_B}{\sigma_N}\mu_N]\rho\frac{\sigma_B}{\sigma_N}\mu_N$$

which can be rearranged as

$$\sigma_B^2+\{[\mu_B-\rho\frac{\sigma_B}{\sigma_N}\mu_N]-\mu_B\}^2+(\rho\frac{\sigma_B}{\sigma_N})^2(\mu_N^2+\sigma_N^2)$$

$$-2\rho\frac{\sigma_B}{\sigma_N}[\mu_N\mu_B+\rho\sigma_B\sigma_N]+2[\mu_B-\rho\frac{\sigma_B}{\sigma_N}\mu_N]\rho\frac{\sigma_B}{\sigma_N}\mu_N$$

$$=\sigma_B^2-\rho^2\sigma_B^2$$

after the cancellations are tabulated.