Review of Perturbative Field Theory

2.1 PROBLEMS

2.1 From (A.7), the Hamiltonian for a free complex scalar field is

$$H = \int d^3\vec{x} \left[\left(\frac{\partial \phi^\dagger}{\partial t}\right)\left(\frac{\partial \phi}{\partial t}\right) + (\vec{\nabla}\phi^\dagger)\cdot(\vec{\nabla}\phi) + m^2 \phi^\dagger \phi \right].$$

Show that

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} E_p \left[a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p}) \right].$$

Assume that H is normal ordered.

Solution: From (2.93) and using

$$\int d^3\vec{x}\; e^{\pm i(\vec{p}\pm\vec{p}\,')\cdot\vec{x}} = (2\pi)^3\, \delta^3\left(\vec{p}\pm\vec{p}\,'\right)$$

we find

$$\begin{aligned} H = &\int \frac{d^3\vec{p}}{(2\pi)^3 (2E_p)^2} E_p^2 \left[a_p^\dagger a_p + b_p^\dagger b_p - e^{2iE_pt}\; a_p^\dagger b_{-p}^\dagger - e^{-2iE_pt}\; b_p a_{-p} \right] \\ &+ \left(\vec{p}^{\,2} + m^2\right) \left[a_p^\dagger a_p + b_p^\dagger b_p + e^{2iE_pt}\; a_p^\dagger b_{-p}^\dagger + e^{-2iE_pt}\; b_p a_{-p} \right] \\ = &\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} E_p \left[a_p^\dagger a_p + b_p^\dagger b_p \right]. \end{aligned}$$

2.2 The CM differential and total cross sections for the elastic scattering of a Hermitian scalar of mass m with potential (2.22) is given in (2.59) and (2.61). Choose $m = 0.5$ GeV, $\lambda = 0.3$, and $\kappa = 1.2$ GeV. Plot the CM differential cross section $d\sigma/d\cos\theta$ in units of $\text{fm}^2 = 10^{-26}\ \text{cm}^2$ as a function of $\cos\theta$ for $s = 5m^2$, $6m^2$, and $7m^2$, and plot the total cross section as a function of $\sqrt{s}$ for $2m < \sqrt{s} < 3m$. Use any convenient plotting program.

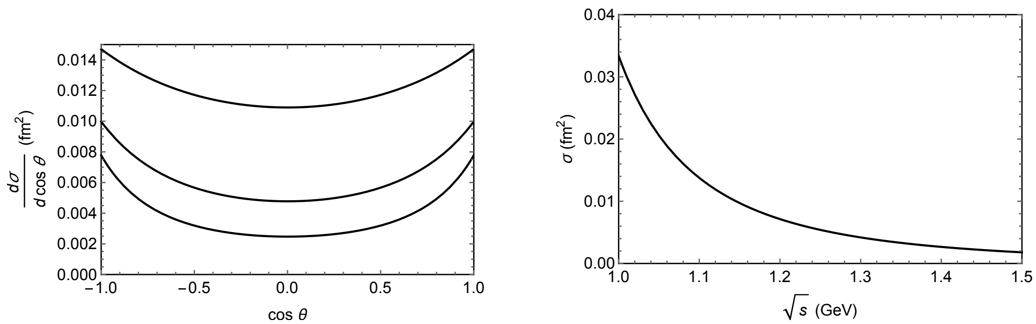
Solution: We have

$$t = -2p^2(1-\cos\theta), \qquad u = -2p^2(1+\cos\theta) \textit{ with } p = \sqrt{s-4m^2}/2.$$

Also,

$$\frac{d\sigma}{d\cos\theta} = \frac{|M_{fi}|^2}{32\pi s}, \qquad \sigma = \frac{1}{2}\int_{-1}^{+1} \frac{d\sigma}{d\cos\theta}\, d\cos\theta.$$

These must be multiplied by $(\hbar c)^2 = (0.197\ GeV\text{-}fm)^2$*, to convert from* GeV^{-2} *to* fm^2*.*



Left: $d\sigma/d\cos\theta$ *in units of* fm^2 *as a function of* $\cos\theta$ *for* $s = 5m^2,\ 6m^2,$ *and* $7m^2$ *(from upper to lower). Right:* $\sigma(s)$ *in units of* fm^2 *as a function of* $\sqrt{s}$

2.3 Derive (2.64) for the special case $m_1 = m_3 = 0$ and $m_2 = m_4$ by Lorentz transforming (2.57). Hint: use the fact that σ, s, and t are invariant.

Solution: In the lab, using (2.42) and (2.46),

$$p_1 = \frac{s - m_2^2}{2m_2}, \qquad t = -2p_1 p_3 \left(1 - \cos\theta_3\right)$$

and

$$\frac{p_3}{p_1} = \frac{1}{1 + \frac{p_1}{m_2}\left(1 - \cos\theta_3\right)},$$

while in the CM

$$p_{cm} = \frac{s - m_2^2}{2\sqrt{s}}, \qquad t = -2p_{cm}^2 \left(1 - \cos\theta\right).$$

But σ *is invariant, so*

$$\begin{aligned} \frac{d\sigma}{d\cos\theta_3} &= \frac{d\sigma}{d\cos\theta}\ \frac{d\cos\theta}{d\cos\theta_3} \\ &= \frac{|M_{fi}|^2}{32\pi s}\frac{p_1^2}{p_{cm}^2}\ \frac{d}{d\cos\theta_3}\left[\frac{p_3}{p_1}\left(\cos\theta_3 - 1\right)\right] = \frac{|M_{fi}|^2}{32\pi s}\ \frac{s}{m_2^2}\ \left(\frac{p_3}{p_1}\right)^2, \end{aligned}$$

from which the result follows easily.

2.4 Consider the process $\pi^+(\vec{p}_1)\pi^-(\vec{p}_2) \to \pi^+(\vec{p}_3)\pi^-(\vec{p}_4)$, with $p_1 \neq p_3$ and $p_2 \neq p_4$, in the theory of a complex scalar field with Lagrangian density given in (2.88) with

$$V_I = \frac{\lambda}{4}(\phi^\dagger\phi)^2.$$

As shown in Appendix B, the tree-level amplitude M_{fi} is given by

$$(2\pi)^4 \delta^4 (p_3 + p_4 - p_1 - p_2) M_{fi} = \int d^4x \, \langle \vec{p}_3 \vec{p}_4 | - iV_I(\phi_0(x), \phi_0^\dagger(x)) | \vec{p}_1 \vec{p}_2 \rangle.$$

Calculate this explicitly using the free-field expression for ϕ_0, and show that $M_{fi} = -i\lambda$.

Solution: This is similar to the calculation for the Hermitian scalar field in Appendix B. Here, the relevant matrix element is

$$-i\frac{\lambda}{4} \int d^4x \, \langle 0 | a_3 b_4 (\phi_0^\dagger \phi_0)^2 a_1^\dagger b_2^\dagger | 0 \rangle,$$

where the free-field expressions for ϕ_0 and $\phi_0^\dagger$ are given in (2.93). Similar to (B.7) one can move the a's and b's from the fields to the right, and the $a^\dagger$'s and $b^\dagger$'s to the left. The only surviving terms involve the commutators with the $a_3, b_4, a_1^\dagger$, and $b_2^\dagger$. The $\delta^3(\vec{p})$ functions eliminate the momentum integrals, and the $(2\pi)^3 2E_p$ factors from the commutators cancel the denominators in the fields. There are two ways to associate the a's from the ϕ_0's with $a_1^\dagger$, and two ways to associate the $a^\dagger$'s from the $\phi_0^\dagger$'s with a_3, so one obtains

$$-i\lambda \int d^4x \, e^{i(p_3+p_4-p_1-p_2)} = -i\lambda (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2).$$

2.5 Consider the Lagrangian density in (2.84) for 3 non-identical Hermitian fields. Calculate the lowest order differential cross section in the center of mass for $\phi_1(\vec{p}_1)\phi_2(\vec{p}_2) \to \phi_1(\vec{p}_3)\phi_2(\vec{p}_4)$ as a function of s and the CM scattering angle θ for the special case $m_1 = m_2 \neq m_3$.

Solution: There are two diagrams, involving ϕ_3 exchange in the s and u channels. The scattering amplitude is

$$M_{fi} = (-i\kappa)^2 \left[\frac{i}{s - m_3^2} + \frac{i}{u - m_3^2} \right]$$

and

$$\frac{d\sigma}{d\cos\theta} = \frac{|M_{fi}|^2}{32\pi s},$$

where (for equal external masses) $p_i = p_f \equiv p = \sqrt{s - 4m_1^2}/2$ and $u = -2p^2(1 + \cos\theta)$.

2.6 Suppose that the interaction potential for a complex scalar field were

$$V_I = \sigma_4 \left(\phi^4 + \phi^{\dagger 4} \right)$$

(rather than $\lambda(\phi^\dagger \phi)^2/4$), which is *not* $U(1)$ invariant. Show that charge is not conserved and calculate the lowest order amplitude for $\pi^+(\vec{p}_1)\, \pi^+(\vec{p}_2) \to \pi^-(\vec{p}_3)\, \pi^-(\vec{p}_4)$.

Solution: Both operators in ϕ_0 lower charge by one unit, while those in $\phi_0^\dagger$ raise it by one. Thus, the first term lowers the charge by 4. The relevant matrix element is

$$-i\sigma_4 \langle 0 | b_3 b_4 \, \phi_0^4 \, a_1^\dagger a_2^\dagger | 0 \rangle \sim -i\sigma_4 \langle 0 | b_3 b_4 \, (a + b^\dagger)^4 \, a_1^\dagger a_2^\dagger | 0 \rangle.$$

There are 4! *ways to associate the four ϕ_0 fields with the external states, so $M_{fi} = -24i\sigma_4$.*

2.7 Consider charged pion QED with a *massive* photon, i.e., add the term $\frac{1}{2}M_A^2 A_\mu A^\mu$ to the Lagrangian density in (2.133) (with ($V_I = 0$)). Assume $M_A > 2m$. (The photon mass term is not gauge invariant, but the model still makes sense at tree level.)
(a) Calculate the decay rate for $\gamma \to \pi^+\pi^-$ in the photon rest frame at tree level for an unpolarized massive photon.
(b) Calculate the π^+ angular distribution $d\Gamma/d\cos\theta$ for a polarized photon, where θ is the angle between the photon polarization direction in the rest frame and the π^+ direction.
(c) Show that one recovers the result in (a) when $d\Gamma/d\cos\theta$ is integrated over $\cos\theta$.

Solution: (a) Define momenta by $\gamma(\vec{k}\,) \to \pi^+(\vec{p}_1)\,\pi^-(\vec{p}_2)$. *In the rest frame,* $k = p_1 + p_2 = (M_A, \vec{0}\,)$, $p_1 = (E_f, \vec{p}_f)$, *and* $p_2 = (E_f, -\vec{p}_f)$, *where* $E_f = M_A/2$ *and* $p_f \equiv |\vec{p}_f| = \sqrt{M_A^2 - 4m^2}/2$. *Just as for a massless photon, the tree-level amplitude is* $M_{fi} = -ie(p_1 - p_2)\cdot\epsilon(\vec{k}, \lambda)$. *Thus, the spin-averaged squared amplitude is*

$$|\bar{M}_{fi}|^2 \equiv \frac{1}{3}\sum_{\lambda=1}^{3}|M_{fi}|^2 = \frac{e^2}{3}(p_1 - p_2)^\mu\,(p_1 - p_2)^\nu \sum_{\lambda=1}^{3}\epsilon(\vec{k}, \lambda)_\mu\,\epsilon(\vec{k}, \lambda)^*_\nu.$$

For a massive vector,

$$\sum_{\lambda=1}^{3}\epsilon(\vec{k}, \lambda)_\mu\,\epsilon(\vec{k}, \lambda)^*_\nu = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_A^2}.$$

But $k\cdot(p_1 - p_2) = 0$, *and* $(p_1 - p_2)^2 = 4m^2 - M_A^2$, *so*

$$\bar{\Gamma} = \frac{p_f}{8\pi M_A^2}|\bar{M}_{fi}|^2 = \frac{\alpha M_A}{12}\left(1 - \frac{4m^2}{M_A^2}\right)^{3/2}.$$

(b)

$$\frac{d\Gamma}{d\cos\theta} = \frac{p_f}{16\pi M_A^2}|M_{fi}|^2, \qquad |M_{fi}|^2 = e^2\,(p_1 - p_2)^\mu\,(p_1 - p_2)^\nu\epsilon_\mu\,\epsilon^*_\nu.$$

Choose axes so that

$$\epsilon = (0, 0, 0, 1), \qquad p_{1,2} = (E_f, \pm p_f\sin\theta, 0, \pm p_f\cos\theta).$$

Then, $(p_1 - p_2)\cdot\epsilon = -2p_f\cos\theta$ *and*

$$\frac{d\Gamma}{d\cos\theta} = \frac{\alpha M_A}{8}\left(1 - \frac{4m^2}{M_A^2}\right)^{3/2}\cos^2\theta.$$

(c) Integrating $\int_{-1}^{+1} d\cos\theta \frac{d\Gamma}{d\cos\theta}$ *recovers* $\bar{\Gamma}$ *from (a), i.e., the decay rate is independent of the spin direction.*

2.8 Consider $\pi^+(\vec{p}_1)\pi^-(\vec{p}_2) \to \pi^+(\vec{p}_3)\pi^-(\vec{p}_4)$ in *massive scalar electrodynamics*, i.e., with

$$V_I = g\phi^\dagger\phi A,$$

where A, the analog of the electromagnetic field, is a Hermitian spin-0 field with mass $\mu \neq 0$. The analog of the charge is g, which now has dimensions of mass.
(a) Find expressions for the differential and total cross sections in the CM to lowest non-trivial order, in terms of s, m, μ, g, and $\cos\theta$.
(b) Define the dimensionless variable $x = s/m^2 \geq 4$, and specialize to the values $m = g = 1$

GeV, $\mu = 0.5$ GeV. Plot $d\sigma/d\cos\theta$ vs $\cos\theta$ in units of 1 $\text{fm}^2 = 10^{-26}$ cm^2 for $x = 4, 4.2$, and 4.4. Use any other plotting program.
(c) Plot σ in units of 1 fm^2 vs x for the same parameter values and the range $4 \leq x \leq 5$.

Solution: (a) The $\pi\pi A$ vertex is $-ig \propto -iV_I$. The scalar photon can be exchanged in the s or t channels, so

$$M_{fi} = (-ig)^2 \left(\frac{i}{s-\mu^2} + \frac{i}{t-\mu^2} \right),$$

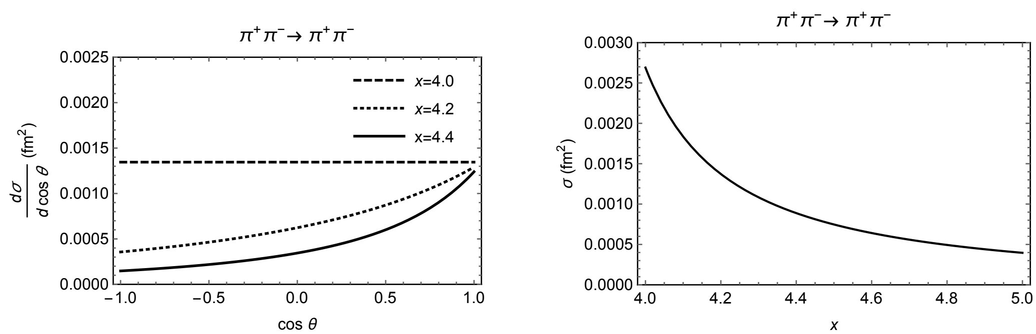
where

$$t = -2p^2(1-\cos\theta), \qquad p = \frac{\sqrt{s-4m^2}}{2}, \qquad E = \frac{\sqrt{s}}{2},$$

and θ is the CM scattering angle. Then

$$\frac{d\sigma}{d\cos\theta} = \frac{|M_{fi}|^2}{32\pi s}, \qquad \sigma = \int_{-1}^{+1} \frac{d\sigma}{d\cos\theta} d\cos\theta.$$

(b,c) Plugging in the numerical values, and multiplying by $(\hbar c)^2 = (0.197\,GeV\text{-}fm)^2$ to convert the units from GeV^{-2} to fm^2 one can evaluate and plot these functions numerically.



Left: $d\sigma/d\cos\theta$ in units of fm^2 for $x = 4, 4.2$, and 4.4. Right: σ in units of fm^2 vs x.

2.9 Prove directly from the defining relations that

$$\sigma^{\mu\nu}\gamma^5 = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}.$$

Solution: Using (2.155), $\sigma^{\mu\nu}\gamma^5$ must be of the form $C\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$. Obtain $C = -i/2$ from example, e.g.,

$$\sigma^{01}\gamma^5 = -\gamma_2\gamma_3 = -\frac{i}{2}\epsilon^{01\rho\sigma}\sigma_{\rho\sigma}.$$

2.10 Prove the *Gordon decomposition* formulas

$$\begin{aligned}
2m\,(\bar{u}_2\gamma^\mu u_1) &= \bar{u}_2(p_2+p_1)^\mu u_1 + i\bar{u}_2\sigma^{\mu\nu}(p_2-p_1)_\nu u_1 \\
2m\,(\bar{u}_2\gamma^\mu\gamma^5 u_1) &= \bar{u}_2(p_2-p_1)^\mu\gamma^5 u_1 + i\bar{u}_2\sigma^{\mu\nu}(p_2+p_1)_\nu\gamma^5 u_1 \\
0 &= \bar{u}_2(p_2-p_1)^\mu u_1 + i\bar{u}_2\sigma^{\mu\nu}(p_2+p_1)_\nu u_1 \\
0 &= \bar{u}_2(p_2+p_1)^\mu\gamma^5 u_1 + i\bar{u}_2\sigma^{\mu\nu}(p_2-p_1)_\nu\gamma^5 u_1,
\end{aligned}$$

where $u_{1,2}$ are two Dirac u spinors for a particle of mass m.

Solution: Using (2.161) and (2.170) ,

$$\begin{aligned} i\bar{u}_2\sigma^{\mu\nu}(p_2-p_1)_\nu u_1 &= -\frac{1}{2}\bar{u}_2\left[\gamma^\mu(\not{p}_2-\not{p}_1)-(\not{p}_2-\not{p}_1)\gamma^\mu\right]u_1 \\ &= 2m\left(\bar{u}_2\gamma^\mu u_1\right)-\bar{u}_2(p_2+p_1)^\mu u_1, \end{aligned}$$

and similarly for other identities.

2.11 Prove the identity

$$\gamma^\mu\gamma^\nu\gamma^\rho = \gamma^\mu g^{\nu\rho}+\gamma^\rho g^{\mu\nu}-\gamma^\nu g^{\mu\rho}+i\epsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma^5.$$

Solution: Any 4×4 matrix can be expanded in terms of $I,\gamma^5,\gamma^\sigma,\gamma^\sigma\gamma^5$, and $\sigma^{\tau\sigma}$. In this case, only γ^σ and $\gamma^\sigma\gamma^5$ will contribute, since we have an odd number of γ matrices. Thus

$$\gamma^\mu\gamma^\nu\gamma^\rho = A_\sigma\gamma^\sigma+B_\sigma\gamma^\sigma\gamma^5 = A^\sigma\gamma_\sigma+B^\sigma\gamma_\sigma\gamma^5,$$

where A and B depend on μ,ν, and ρ. Clearly,

$$A^\sigma=\frac{1}{4}Tr(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho),\qquad B^\sigma=-\frac{1}{4}Tr(\gamma^\sigma\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho).$$

The result follows using (2.172).

2.12 Show by explicit construction that the Pauli-Dirac and chiral representations are related by a unitary transformation, i.e., that there exists a unitary matrix U such that

$$U\gamma^\mu_{PD}U^\dagger=\gamma^\mu_{ch},\qquad Uu_{PD}(\vec{p},s)=u_{ch}(\vec{p},s),\qquad Uv_{PD}(\vec{p},s)=-v_{ch}(\vec{p},s).$$

(The extra sign in the v-spinor transformation is due to a sign convention.)

Solution: U must commute with γ^i but not γ^0, so it should be of form

$$\alpha\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}+\beta\begin{pmatrix} 0 & I\\ -I & 0\end{pmatrix}\rightarrow\frac{1}{\sqrt{2}}\begin{pmatrix} I & -I\\ I & I\end{pmatrix},$$

where the explicit form is obained by the transformation of γ^0. The correct transformation of the spinors follows by inspection.

2.13 The angular momentum operator for a Dirac field can be written as

$$J^i=\int d^3\vec{x}\,\psi^\dagger(x)\frac{1}{4}\epsilon^{ijk}\sigma_{jk}\psi(x)+\text{orbital},$$

where normal ordering is implied. Show, in the free field limit, that J^3 has the expected behavior

$$J^3|\psi\,(\vec{p},s_{1,2})\rangle=\pm\frac{1}{2}|\psi\,(\vec{p},s_{1,2})\rangle,\qquad J^3|\psi^c\,(\vec{p},s_{1,2})\rangle=\pm\frac{1}{2}|\psi^c\,(\vec{p},s_{1,2})\rangle,$$

where ψ and ψ^c represent particle and antiparticle states, $\vec{p}$ is in the $\hat{z}$ direction (so that the orbital terms do not enter), and $s = s_1$ or s_2 represent spins in the $\pm\hat{z}$ direction.

Solution: One has

$$\frac{1}{4}\epsilon^{ijk}\sigma_{jk} = \frac{1}{2}\Sigma^i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

Using the expression (2.159) for the free fermion field

$$\begin{aligned}\psi(x)|\psi(\vec{p},s)\rangle &= \int \frac{d^3\vec{p}\,'}{(2\pi)^3 2E_{p'}}\; e^{-ip'\cdot x}\; u(\vec{p}\,',s')\;\; a(\vec{p}\,',s')\;\; a^\dagger(\vec{p},s)\,|0\rangle\\ &= e^{-ip\cdot x}\;\, u(\vec{p},s)\,|0\rangle,\end{aligned}$$

so that

$$\begin{aligned}J^3|\psi(\vec{p},s)\rangle &= \sum_{s''}\frac{1}{2E_p}u^\dagger(\vec{p},s'')\,\frac{1}{2}\Sigma^3 u(\vec{p},s)\,|\psi(\vec{p},s'')\rangle\\ &= \sum_{s''}\frac{1}{2}\left(\frac{E_p+m}{2E_p}\right)\phi^\dagger_{s''}\left(\sigma^3+\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\;\sigma^3\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\phi_s|\psi(\vec{p},s'')\rangle\\ &= \pm\frac{1}{2}\sum_{s''}\delta_{s'',s}\left(\frac{E_p+m}{2E_p}\right)\left(1+\frac{\vec{p}^{\,2}}{(E_p+m)^2}\right)|\psi(\vec{p},s'')\rangle\\ &= \pm\frac{1}{2}|\psi(\vec{p},s)\rangle,\end{aligned}$$

where the $\pm$ corresponds to $s_{1,2}$, i.e., $\sigma^3\phi_{s_{1,2}} = \pm\phi_{s_{1,2}}$. The calculation is similar for ψ^c except for two compensating signs:

$$J^3|\psi^c(\vec{p},s)\rangle = -\sum_{s'}\frac{1}{2E_p}\;\, v^\dagger(\vec{p},s)\,\frac{1}{2}\Sigma^3 v(\vec{p},s')\,|\psi^c(\vec{p},s')\rangle,$$

where the $-$ sign is because b and $b^\dagger$ had to be anticommuted. But this is

$$\begin{aligned}&= -\frac{1}{2}\sum_{s'}\left(\frac{E_p+m}{2E_p}\right)\chi^\dagger_s\left(\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\;\sigma^3\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}+\sigma^3\right)\chi_{s'}|\psi^c(\vec{p},s')\rangle\\ &= \pm\frac{1}{2}|\psi^c(\vec{p},s)\rangle,\end{aligned}$$

where the second $-$ sign is from $\sigma^3\chi_{s_{1,2}} = \mp\chi_{s_{1,2}}$.

2.14 Derive the results in (2.181) for the helicity projections in the massless limit.

Solution: Write $p = \left(E, p\hat{\beta}\right)$ where $p = \beta E$. Then

$$\begin{aligned}ms_\pm &= \pm E\left(\beta,\hat{\beta}\right) = \pm\left(p, E\hat{\beta}\right)\\ &= \pm\left(E,p\hat{\beta}\right)\pm\left(p-E,[E-p]\,\hat{\beta}\right)\sim\pm\left[p-\frac{m^2}{2p^2}\,(p,-\vec{p})\right].\end{aligned}$$

Therefore,

$$\begin{aligned}(\not{p}+m)\,\frac{1+\gamma_5\;\not{s}}{2} &= \frac{1}{2}\left(\not{p}+m+\gamma_5 m\;\not{s}_\pm-\gamma_5\;\not{p}\;\not{s}_\pm\right)\\ &= \frac{1}{2}\left(\not{p}\pm\gamma_5\;\not{p}\right)+\mathcal{O}(m)\\ &= \not{p}P_{L,R} = P_{R,L}\;\not{p}.\end{aligned}$$

Similarly,

$$(\not p - m)\,\frac{1+\gamma_5 \not s_\pm}{2} = \not p P_{R,L} = P_{L,R} \not p.$$

2.15 Weak charged current transitions involve the chiral spinors $u_L(\vec{p}, s)$ and $v_L(\vec{p}, s)$ defined in (2.203). Show that in the relativistic limit such transitions mainly involve negative helicity particles or positive helicity antiparticles, and estimate the suppression factor for transitions involving the "wrong" helicity.

Solution: Expressions for the helicity spinors in the chiral representation are given in (2.193). For $E \gg m$, $\lambda_+ \to m/E$ *and* $\lambda_- \to 2$. *Then, applying* $P_L = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$,

$$\begin{aligned} P_L u(+) &\to \frac{m}{2E}\sqrt{2E}\begin{pmatrix} \phi_+ \\ 0 \end{pmatrix}, \qquad & P_L u(-) &\to \sqrt{2E}\begin{pmatrix} \phi_- \\ 0 \end{pmatrix} \\ P_L v(+) &\to \sqrt{2E}\begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}, \qquad & P_L v(-) &\to \frac{m}{2E}\sqrt{2E}\begin{pmatrix} \chi_- \\ 0 \end{pmatrix}, \end{aligned}$$

so the rates for the wrong helicity are suppressed by $m^2/4E^2$.

2.16 Show in two ways that $|\bar{u}(\vec{p}_2, +)u(\vec{p}_1, -)|^2 = 2p_1 \cdot p_2$, where $u(\vec{p}, \pm)$ are the helicity spinors for a massless fermion: (a) directly from the form of the spinors in the chiral representation, (b) using trace techniques.

Solution: (a) From (2.195) and Table 2.1,

$$\begin{aligned} \bar{u}(\vec{p}_2, +)u(\vec{p}_1, -) &= \sqrt{2E_2}\sqrt{2E_1}\phi_+(2)^\dagger \phi_-(1) \\ &= \sqrt{2E_2}\sqrt{2E_1}\left[-\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}e^{-i\varphi_1} + \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}e^{-i\varphi_2}\right], \end{aligned}$$

so that

$$\begin{aligned} |\bar{u}(\vec{p}_2, +)u(\vec{p}_1, -)|^2 =& 2E_2\, 2E_1 \left[\cos^2\frac{\theta_2}{2}\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2}\cos^2\frac{\theta_1}{2}\right. \\ & \left. -2\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\cos(\varphi_1 - \varphi_2)\right] \\ =& 2E_1E_2\left[1 - \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2)\right] \\ =& 2p_1 \cdot p_2. \end{aligned}$$

(b) From (2.181)

$$|\bar{u}(\vec{p}_2, +)u(\vec{p}_1, -)|^2 = Tr(P_L \not p_1 P_R \not p_2) = 2p_1 \cdot p_2.$$

2.17 Prove the Fierz identity in (2.216).

Solution: Expand

$$w_{2L}\bar{w}_{3L} = P_L w_{2L}\bar{w}_{3L} = aP_L + P_L c_\nu \gamma^\nu + P_L e_{\rho\tau}\sigma^{\rho\tau}.$$

Only the c^ν survives when inserted back in the l.h.s. of (2.216), yielding

$$(\bar{w}_{1L}\gamma^\mu\gamma^\nu\gamma_\mu w_{4L})\, c_\nu = -2\,(\bar{w}_{1L}\gamma^\nu w_{4L})\, c_\nu.$$

But

$$c_\nu = \frac{1}{2}\, Tr\,(\gamma_\nu w_{2L}\bar{w}_{3L}) = \frac{1}{2}\,(\bar{w}_{3L}\gamma_\nu w_{2L})\,.$$

(The last step would yield an extra minus sign for anticommuting fields.)

2.18 Suppose a fermion ψ of mass m interacts with a Hermitian scalar ϕ of mass μ with

$$\mathcal{L}_I = h\bar{\psi}\psi\phi,$$

where h is small.
(a) Calculate the spin-averaged differential cross section for $\psi(\vec{p}_1)\psi(\vec{p}_2) \to \psi(\vec{p}_3)\psi(\vec{p}_4)$ in the CM in terms of the invariants s, t, and u.
(b) Specialize to $m = \mu = 0$. Show that the scattering is isotropic in that limit and calculate the total cross section.

Solution: (a)

$$M = (ih)^2\, i\left[\frac{\bar{u}_3 u_1 \bar{u}_4 u_2}{t-\mu^2} - \frac{\bar{u}_4 u_1 \bar{u}_3 u_2}{u-\mu^2}\right],$$

so that

$$\frac{d\bar{\sigma}}{d\cos\theta} = \frac{1}{32\pi s}|\bar{M}|^2$$

with

$$\begin{aligned}
|\bar{M}|^2 =& \frac{1}{4}\sum |M|^2 \\
=& \frac{h^4}{4}\left[\left(\frac{1}{t-\mu^2}\right)^2 [Tr\,(\not{p}_1+m)\,(\not{p}_3+m)]\,[Tr\,(\not{p}_2+m)\,(\not{p}_4+m)]\right. \\
&+\left(\frac{1}{u-\mu^2}\right)^2 [Tr\,(p_1+m)\,(\not{p}_4+m)]\,[Tr\,(\not{p}_2+m)\,(\not{p}_3+m)] \\
&\left.-\left(\frac{1}{t-\mu^2}\right)\left(\frac{1}{u-\mu^2}\right)\left[Tr\,(\not{p}_2+m)\,(\not{p}_3+m)\,(\not{p}_1+m)\,(\not{p}_4+m) + (3\leftrightarrow 4)\right]\right] \\
=& h^4\left[\left(\frac{1}{t-\mu^2}\right)^2 \left(4m^2-t\right)^2 + \left(\frac{1}{u-\mu^2}\right)^2 \left(4m^2-u\right)^2\right. \\
&\left.-\frac{1}{2}\left(\frac{1}{t-\mu^2}\right)\left(\frac{1}{u-\mu^2}\right)\left[\left(4m^2-u\right)^2 + \left(4m^2-t\right)^2 - \left(s-4m^2\right)^2\right]\right],
\end{aligned}$$

where

$$E = \frac{\sqrt{s}}{2}, \qquad k = \frac{\sqrt{s-4m^2}}{2}, \qquad t = -2k^2(1-\cos\theta), \qquad u = -2k^2(1+\cos\theta).$$

(b)

$$\begin{aligned}
\frac{d\bar{\sigma}}{d\cos\theta} &= \frac{1}{32\pi s}|\bar{M}|^2 = \frac{2h^4}{32\pi s}\left[1 - \frac{t^2+u^2-s^2}{4tu}\right] \\
&= \frac{h^4}{16\pi s}\left[1 - \frac{1}{4}\left(\frac{(1-\cos\theta)^2 + (1+\cos\theta)^2 - 4}{1-\cos^2\theta}\right)\right] = \frac{3h^4}{32\pi s}.
\end{aligned}$$

Therefore,

$$\bar{\sigma} = \frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{d\bar{\sigma}}{d\cos\theta} = \frac{3h^4}{32\pi s}.$$

2.19 Consider $e^-(\vec{k}_1)\,\pi^+(\vec{p}_1) \;\rightarrow\; e^-(\vec{k}_2)\,\pi^+(\vec{p}_2)$ elastic scattering. Show that the spin-averaged differential cross section in the pion rest frame is

$$\frac{d\bar{\sigma}}{d\cos\theta_L} = \frac{\pi\alpha^2\cos^2\frac{\theta_L}{2}}{2k_1^2\sin^4\frac{\theta_L}{2}\left[1+\frac{2k_1}{m_\pi}\sin^2\frac{\theta_L}{2}\right]},$$

where θ_L is the electron scattering angle and we have neglected the electron mass. Hint: use (2.224).

Solution: From (2.65) and (2.46),

$$\frac{d\bar{\sigma}}{d\cos\theta_L} = \frac{|\bar{M}|^2}{32\pi m_\pi^2}\left(\frac{k_2}{k_1}\right)^2,$$

where

$$\frac{k_2}{k_1} = \frac{1}{1+\frac{2k_1}{m_\pi}\sin^2\frac{\theta_L}{2}}.$$

But from (2.224) and using $t=-2k_1\cdot k_2$ and $p_2 = k_1+p_1-k_2$,

$$|\bar{M}|^2 = \frac{e^4}{2(k_1\cdot k_2)^2}\left[8k_1\cdot p_1 k_2\cdot p_1 - 4k_1\cdot k_2 m_\pi^2\right].$$

But

$$\begin{aligned} k_1\cdot k_2 &= k_1k_2(1-\cos\theta_L) = 2k_1k_2\sin^2\frac{\theta_L}{2} \\ k_1\cdot p_1 &= k_1 m_\pi, \qquad k_2\cdot p_1 = k_2 m_\pi, \end{aligned}$$

so that

$$|\bar{M}|^2 = \frac{16\pi^2\alpha^2 m_\pi^2}{k_1k_2}\,\frac{\cos^2\frac{\theta_L}{2}}{\sin^4\frac{\theta_L}{2}}.$$

2.20 Calculate the CM differential cross section for the process $e^-(\vec{p}_1)\mu^+(\vec{k}_1) \rightarrow e^-(\vec{p}_2)\mu^+(\vec{k}_2)$ in terms of $s = E_{CM}^2$, the CM scattering angle θ, and the muon mass m_μ. Neglect the electron mass.

Solution: Similar to (2.225),

$$M_{fi} = \frac{ie^2}{t}\bar{u}_2\gamma^\mu u_1\;\bar{v}_1\gamma_\mu v_2,$$

where the u spinors refer to $p_{1,2}$ with $m_e \sim 0$, while the v spinors refer to $k_{1,2}$ and m_μ. Then,

$$\begin{aligned} |\bar{M}_{fi}|^2 =& \frac{e^4}{4t^2}\, Tr\,(\gamma^\mu\;\not{p}_1\gamma^\nu\;\not{p}_2)\times Tr\,(\gamma_\mu(\not{k}_2-m_\mu)\gamma_\nu(\not{k}_1-m_\mu)) \\ =& \frac{8e^4}{t^2}\left[p_1\cdot k_1\,p_2\cdot k_2\;+\;p_1\cdot k_2\,p_2\cdot k_1\;-\;m_\mu^2\,p_1\cdot p_2\right], \end{aligned}$$

and

$$\frac{d\bar{\sigma}}{d\cos\theta} = \frac{1}{32\pi s}\ \frac{p_f}{p_i}|\bar{M}_{fi}|^2.$$

In the CM, the four-momenta are

$$\begin{aligned} p_1 &= (p, 0, 0, p), \qquad p_2 = (p, p\sin\theta, 0, p\cos\theta) \\ k_1 &= (E, 0, 0, -p), \qquad k_2 = (E, -p\sin\theta, 0, -p\cos\theta), \end{aligned}$$

where

$$p = p_i = p_f = \frac{s - m_\mu^2}{2\sqrt{s}}, \qquad E = \frac{s + m_\mu^2}{2\sqrt{s}}.$$

Therefore,

$$\begin{aligned} p_1 \cdot k_1 &= p_2 \cdot k_2 = p\,(E + p) = p\sqrt{s} \\ p_1 \cdot k_2 &= p_2 \cdot k_1 = p\,(E + p\cos\theta) \\ p_1 \cdot p_2 &= -t/2 = p^2\,(1 - \cos\theta), \end{aligned}$$

so that

$$\frac{d\bar{\sigma}}{d\cos\theta} = 4\pi\alpha^2 \frac{p^2}{st^2}\left[s + (E + p\cos\theta)^2 - m_\mu^2(1 - \cos\theta)\right].$$

2.21 Verify the expressions for Bhabha scattering in (2.234) and (2.235). Rewrite the final result in terms of s and $\cos\theta$.

Solution: From (2.233) and (2.208),

$$|\bar{M}_{fi}|^2 = \frac{e^4}{4}\left[\frac{\bar{u}_3\gamma_\mu v_4 \bar{v}_2\gamma^\mu u_1}{s} - \frac{\bar{u}_3\gamma_\mu u_1 \bar{v}_2\gamma^\mu v_4}{t}\right]\left[\frac{\bar{v}_4\gamma_\nu u_3 \bar{u}_1\gamma^\nu v_2}{s} - \frac{\bar{u}_1\gamma_\nu u_3 \bar{v}_4\gamma^\nu v_2}{t}\right],$$

which yields (2.234). The first term is

$$\begin{aligned} &\frac{4e^4}{s^2}\left[p_{4\mu}p_{3\nu} + p_{4\nu}p_{3\mu} - g_{\mu\nu}\,p_3\cdot p_4\right]\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}\,p_1\cdot p_2\right] \\ &\quad = \frac{8e^4}{s^2}(p_1\cdot p_4\, p_2\cdot p_3 + p_1\cdot p_3\, p_2\cdot p_4) = 32\pi^2\alpha^2\left(\frac{t^2 + u^2}{s^2}\right) \end{aligned}$$

and similarly for the second. Using (2.174), the third term is

$$-\frac{e^4}{4st}[-2\,Tr(\not{p}_2\gamma_\nu\ \not{p}_4\ \not{p}_1\gamma^\nu\ \not{p}_3)] = \frac{2e^4u^2}{st} = 32\pi^2\alpha^2\left(\frac{u^2}{st}\right),$$

and the same for the last term. For massless particles,

$$t = -\frac{s}{2}(1 - \cos\theta) = -s\sin^2(\theta/2), \qquad u = -\frac{s}{2}(1 + \cos\theta) = -s\cos^2(\theta/2),$$

leading to

$$\frac{d\bar{\sigma}}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[\frac{1 + \cos^2\theta}{2} + \frac{(1 + \cos^4\frac{\theta}{2})}{\sin^4\frac{\theta}{2}} - \frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right] = \frac{\pi\alpha^2}{2s}\left(\frac{3 + \cos^2\theta}{1 - \cos\theta}\right)^2,$$

which displays the singularity from the t-channel pole in the forward direction.

2.22 Calculate the differential cross section for unpolarized Møller scattering, $e^-e^- \to e^-e^-$, both in terms of the invariants and θ.

Solution: From the t and u-channel diagrams in Figure 2.14

$$M = ie^2 \left[\frac{\bar{u}_3 \gamma_\mu u_1 \bar{u}_4 \gamma^\mu u_2}{(p_1 - p_3)^2} - \frac{\bar{u}_4 \gamma_\mu u_1 \bar{u}_3 \gamma^\mu u_2}{(p_1 - p_4)^2} \right].$$

By a calculation essentially identical to the one for Bhabha scattering,

$$\begin{aligned}
\frac{d\bar{\sigma}}{d\cos\theta} &= \frac{\pi\alpha^2}{s} \left[\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + \frac{2s^2}{tu} \right] \\
&= \frac{\pi\alpha^2}{s} \left[\frac{(1 + \cos^4 \frac{\theta}{2})}{\sin^4 \frac{\theta}{2}} + \frac{(1 + \sin^4 \frac{\theta}{2})}{\cos^4 \frac{\theta}{2}} + \frac{2}{\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \right] \\
&= \frac{2\pi\alpha^2}{s} \left[\frac{16}{\sin^4 \theta} - \frac{8}{\sin^2 \theta} + 1 \right],
\end{aligned}$$

showing the singularities for $\theta = 0$ and π.

2.23 Calculate the spin-average differential cross section $d\bar{\sigma}/d\cos\theta$ in the center of mass for $e^-(p_1)e^+(p_2) \to \pi^-(p_3)\pi^+(p_4)$, and the total cross section $\bar{\sigma}$. Neglect the electron mass but not the pion mass. Ignore strong interaction effects. The angular distribution should be proportional to $\sin^2\theta$. Interpret this result.

Solution: The calculation is similar to $e^-\pi^+$ scattering and $e^-e^+ \to f\bar{f}$ in Section 2.8. The differential cross section is the same as in (2.226), with the kinematic variables given in (2.229) (with $m_f = m_\pi$). The amplitude due to the s channel photon diagram is

$$M_{fi} = -ie\ (p_4 - p_3)_\mu \left(\frac{-ig^{\mu\rho}}{s} \right) (+ie\ \bar{v}_2 \gamma_\rho u_1) = \frac{-ie^2}{s} (p_4 - p_3)_\mu\ \bar{v}_2 \gamma^\mu u_1,$$

so that

$$\begin{aligned}
|\bar{M}_{fi}|^2 &= \frac{1}{4} \frac{e^4}{s^2} (p_4 - p_3)_\mu (p_4 - p_3)_\nu\ Tr[\gamma^\mu\ \not{p}_1 \gamma^\nu\ \not{p}_2] \\
&= \frac{e^4}{s^2} (p_4 - p_3)_\mu (p_4 - p_3)_\nu\ (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} p_1 \cdot p_2) \\
&= \frac{1}{2} e^4 \beta_f^2 \sin^2\theta,
\end{aligned}$$

where we have used (2.229) and (2.230). Then,

$$\frac{d\bar{\sigma}}{d\cos\theta} = \frac{\pi\alpha^2\beta_f^3}{4s} \sin^2\theta, \qquad \bar{\sigma} = \frac{\pi\alpha^2\beta_f^3}{3s}.$$

Because of their vector couplings, the initial e^- and e^+ have opposite helicities in the relativistic limit. Thus, the z component of angular momentum is ± 1. The pions have no spin and there is no orbital angular momentum in the direction of motion, so they cannot move in the $\pm z$ direction. Equivalently, their angular distribution should be $\propto (|Y_1^1|^2 + |Y_1^{-1}|^2) \sim \sin^2\theta$, where Y_l^m is a spherical harmonic.

2.24 (a) Consider the Mott scattering process in which an electron of momentum $p = \beta E$ scatters from a static Coulomb potential of charge Ze,

$$A^\mu(x) = \frac{Ze}{4\pi|\vec{x}|}(1,0,0,0).$$

Show that the unpolarized (Mott) cross section for scattering angle θ is

$$\frac{d\bar{\sigma}}{d\cos\theta} = \frac{(Z\alpha)^2\pi(1-\beta^2\sin^2\frac{\theta}{2})}{2\beta^2p^2\sin^4\frac{\theta}{2}} \xrightarrow[\beta\ll 1]{} \frac{Z^2\alpha^2\pi}{2\beta^2p^2\sin^4\frac{\theta}{2}}.$$

The last formula is the the Rutherford cross section.
(b) Suppose the Coulomb potential for an electron in a nuclear field transformed as a scalar rather than as the time component of a four-vector, i.e.,

$$\mathcal{H}_I = -e\bar{\psi}(x)\psi(x)\phi(x), \qquad \phi(x) = \frac{Ze}{4\pi|\vec{x}|}.$$

Calculate the unpolarized differential cross section, and compare it with the Mott formula.

Solution: (a) This is an example of scattering from a static source, with $\mathcal{L}_I(x)$ in (2.66) given by

$$\mathcal{L}_p = Ze^2\bar{\psi}(x)\gamma^0\psi(x), \qquad \Phi(0,\vec{x}) = \frac{1}{4\pi|\vec{x}|}, \qquad \tilde{\Phi}(\vec{q}) = \frac{1}{|\vec{q}|^2}.$$

Therefore, from (2.74)

$$\begin{aligned}\frac{d\bar{\sigma}}{d\cos\theta} &= \frac{1}{8\pi}|\bar{M}_{fi}|^2 = \frac{1}{2}\sum_{s_1,s_2}\frac{2\pi(Z\alpha)^2}{|\vec{q}|^4}|\bar{u}_2\gamma^0u_1|^2 \\ &= \frac{\pi(Z\alpha)^2}{|\vec{q}|^4}Tr[\gamma^0(\not{p}_1+m)\gamma^0(\not{p}_2+m)] = \frac{\pi(Z\alpha)^2[2E^2-p_1\cdot p_2+m^2]}{4p^4\sin^4\frac{\theta}{2}},\end{aligned}$$

from which the result follows.
(b) Calculation is identical to Mott scattering, except

$$Tr[\gamma^0(\not{p}_1+m)\gamma^0(\not{p}_2+m)] = 4\left(E^2+m^2+p^2\cos\theta\right) = 8E^2\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

is replaced by

$$Tr[(\not{p}_1+m)(\not{p}_2+m)] = 4\left(E^2+m^2-p^2\cos\theta\right) = 8E^2\left(1-\beta^2\cos^2\frac{\theta}{2}\right).$$

2.25 The interaction of the Z (a massive neutral vector boson in the electroweak theory) with a fermion f is

$$\mathcal{L} = -G\bar{\psi}(x)\gamma^\mu\left(g_V - g_A\gamma^5\right)\psi(x)\,Z_\mu(x),$$

where G, g_V, and g_A are real constants. Calculate the width for $Z \to f\bar{f}$. Let M_Z and m be the Z and f masses, and set $G = 1$.

Solution: The amplitude for $Z(p_1) \to f(p_2)\bar{f}(p_3)$ is

$$M = -i\epsilon_\mu(\lambda)\bar{u}_2\gamma^\mu\left(g_V - g_A\gamma^5\right)v_3.$$

From (2.82) the differential decay rate is

$$\frac{d\Gamma}{d\cos\theta} = \frac{p_f}{16\pi M_Z^2}\ \frac{1}{3}\ \sum_\lambda |M|^2, \qquad p_f = \frac{\sqrt{M_Z^2 - 4m^2}}{2},$$

with

$$\begin{aligned}
\frac{1}{3}\ \sum_\lambda |M|^2 \equiv |\bar{M}|^2 &= \frac{1}{3}\sum_\lambda \epsilon_\mu(\lambda)^* \epsilon_\nu(\lambda) \\
&\quad \times Tr\left[\gamma^\mu \left(g_V - g_A\gamma^5\right)(\not{p}_3 - m)\,\gamma^\nu \left(g_V - g_A\gamma^5\right)(\not{p}_2 + m)\right] \\
&= \frac{4}{3}\left(-g_{\mu\nu} + \frac{p_{1\mu}p_{1\nu}}{M_Z^2}\right)\left[\left(p_3^\mu p_2^\nu + p_3^\nu p_2^\mu - g^{\mu\nu} p_2\cdot p_3\right)\left(g_V^2 + g_A^2\right)\right. \\
&\quad \left. - m^2 g^{\mu\nu}\left(g_V^2 - g_A^2\right)\ + 2ig_V g_A \epsilon^{\mu\rho\nu\sigma} p_{3\rho} p_{2\sigma}\right] \\
&= \frac{4}{3}\left(M_Z^2 - m^2\right)\left(g_V^2 + g_A^2\right) + 4m^2\left(g_V^2 - g_A^2\right),
\end{aligned}$$

where we have used $p_1 \cdot p_2 = p_1 \cdot p_3 = M_Z^2/2$ and $p_2 \cdot p_3 = (M_Z^2 - 2m^2)/2$. Therefore,

$$\Gamma = \frac{p_f}{8\pi M_Z^2}|\bar{M}|^2 \xrightarrow[m\to 0]{} \frac{M_Z}{12\pi}\left(g_V^2 + g_A^2\right).$$

2.26 The Λ is a heavy spin-$\frac{1}{2}$ hyperon that decays into $p\pi^-$ via the non-leptonic weak interactions. The decay interaction can be modeled by

$$\mathcal{L}_I = \bar{\psi}_p(g_S - g_P\gamma^5)\psi_\Lambda \phi_{\pi^+} + h.c.,$$

where g_S and g_P are complex constants that lead respectively to S and P-wave final states.
(a) Calculate the width Γ and the differential width $d\Gamma/d\cos\theta$ in the Λ rest frame for a polarized Λ, where θ is the angle between $\hat{s}_\Lambda$ and the proton momentum $\vec{p}_p$. Use trace techniques.
(b) Show that $d\Gamma/d\cos\theta$ is not reflection invariant for $\Re e\,(g_P g_S^*) \neq 0$, i.e., that it is not invariant under $\vec{p}_p \to -\vec{p}_p$, $\hat{s}_\Lambda \to \hat{s}_\Lambda$.
(c) Repeat (a), but use explicit expressions for the Λ and p spinors in the Pauli-Dirac representation. Justify the claim that g_S and g_P generate S and P-wave amplitudes.

Solution: (a,b) The decay amplitude and differential width are

$$i\bar{u}_p\left(g_S - g_P\gamma^5\right)u_\Lambda, \qquad \frac{d\Gamma}{d\cos\theta} = \frac{p_f|M|^2}{16\pi m_\Lambda^2},$$

where p_f is the final state momentum in the Λ rest frame, given by a form similar to (2.83). Summing over the proton spin,

$$\begin{aligned}
\sum_{s_p} |M|^2 &= Tr\left[\left(g_S - g_P\gamma^5\right)(\not{p}_\Lambda + m_\Lambda)\left(\frac{1+\gamma^5\ \not{s}_\Lambda}{2}\right)\left(g_S^* + g_P^*\gamma^5\right)(\not{p}_p + m_p)\right] \\
&= 2m_p m_\Lambda\left(|g_S|^2 - |g_P|^2\right) + 2p_p\cdot p_\Lambda\left(|g_S|^2 + |g_P|^2\right) \\
&\quad - 2m_\Lambda p_p\cdot s_\Lambda\left(g_P g_S^* + g_P^* g_S\right),
\end{aligned}$$

where we used $p_\Lambda \cdot s_\Lambda = 0$. But $p_p \cdot p_\Lambda = E_p m_\Lambda$ and $p_p \cdot s_\Lambda = -p_f \cos\theta$, where θ is the angle between $\vec{p}_p$ and $\hat{s}_\Lambda$. Therefore,

$$\begin{aligned}\frac{d\Gamma}{d\cos\theta} =& \frac{p_f}{8\pi m_\Lambda}\left[\left(|g_S|^2+|g_P|^2\right)E_p+\left(|g_S|^2-|g_P|^2\right)m_p\right.\\ &\left.+\,2\Re e\left(g_P g_S^*\right)p_f\cos\theta\right].\end{aligned}$$

This is clearly not *invariant under $\cos\theta \to -\cos\theta$.*
(c) The Dirac spinors in the Pauli-Dirac representation are

$$u_\Lambda = \sqrt{2m_\Lambda}\begin{pmatrix}\phi_\Lambda\\ 0\end{pmatrix}, \qquad u_p = \sqrt{E_p+m_p}\begin{pmatrix}\phi_{s_p}\\ \frac{\vec{\sigma}\cdot\vec{p}_p}{E_p+m_p}\,\phi_{s_p}\end{pmatrix}.$$

Using the expressions in (2.164) for γ^0 and γ^5,

$$M = i\sqrt{2m_\Lambda}\sqrt{E_p+m_p}\,\phi_{s_p}^\dagger\left(g_S+g_P\frac{\vec{\sigma}\cdot\vec{p}_p}{E_p+m_p}\right)\phi_\Lambda,$$

which are obviously S and P wave. Therefore,

$$\begin{aligned}\sum_{s_p}|M|^2 =& \sum_p 2m_\Lambda\left(E_p+m_p\right)\\ &\phi_\Lambda^\dagger\underbrace{\left(g_S^*+g_P^*\frac{\vec{\sigma}\cdot\vec{p}_p}{E_p+m_p}\right)}_{A^\dagger}\phi_{s_p}\phi_{s_p}^\dagger\underbrace{\left(g_S+g_P\frac{\vec{\sigma}\cdot\vec{p}_p}{E_p+m_p}\right)}_{A}\phi_\Lambda\\ =&2m_\Lambda\left(E_p+m_p\right)Tr\left(A^\dagger A\phi_\Lambda\phi_\Lambda^\dagger\right),\end{aligned}$$

where we have used $\sum_{s_p}\phi_{s_p}\phi_{s_p}^\dagger = I$. Taking

$$\phi_\Lambda = \begin{pmatrix}1\\ 0\end{pmatrix} \;\Rightarrow\; \phi_\Lambda\phi_\Lambda^\dagger = \frac{1}{2}\left(I+\sigma^3\right),$$

we have

$$\begin{aligned}\sum_{s_p}|M|^2 =&2m_\Lambda\left(E_p+m_p\right)\\ &\times\left[|g_S|^2+|g_P|^2\;\frac{p_p^2}{\left(E_p+m_p\right)^2}+\left(g_S^*g_P+g_Sg_P^*\right)\;\frac{p_p\cos\theta}{E_p+m_p}\right],\end{aligned}$$

which reproduces the result obtained from the trace calculation.

2.27 A vector resonance V_μ of mass M_V and width Γ_V couples to massless fermions a and b with the interaction in (F.12) of Appendix F. Calculate the total spin-averaged cross section for $a\bar{a} \to b\bar{b}$. Assume that the propagator in (2.129) is modified to the Breit-Wigner form

$$iD_V^{\mu\nu}(k) = i\left[\frac{-g^{\mu\nu}+\frac{k^\mu k^\nu}{M_V^2}}{k^2-M_V^2+iM_V\Gamma_V}\right],$$

and express the result in a form similar to (F.11).

Solution: The $k^\mu k^\nu/M_V^2$ *term in the propagator vanishes when contracted with the massless fermion bilinears. The total cross section calculation is then almost identical to the one for* $e^+e^- \to f\bar{f}$. *Applying the obvious modifications to (2.232) one finds*

$$\bar{\sigma}(s) = \frac{g_a^2 g_b^2}{12\pi} \frac{s}{(s - M_V^2)^2 + M_V^2 \Gamma_V^2}.$$

To put this into the form (F.11) we calculate the spin-average total width for $V \to b\bar{b}$,

$$\begin{aligned}
\bar{\Gamma}_{b\bar{b}} &= \frac{g_b^2}{16\pi M_V} \frac{1}{3} \sum_\lambda \left[\epsilon^{\mu *}(\lambda)\epsilon^\nu(\lambda)\right] Tr\, (\gamma_\nu \not{p}_{\bar{b}} \gamma_\mu \not{p}_b) \\
&= \frac{g_b^2}{12\pi M_V} \left(-g^{\mu\nu} + \frac{p_V^\mu p_V^\nu}{M_V^2} \right) \left(p_{\bar{b}\mu} p_{b\nu} + p_{b\mu} p_{\bar{b}\nu} - g_{\mu\nu} p_b \cdot p_{\bar{b}} \right) \\
&= \frac{g_b^2 M_V}{12\pi}
\end{aligned}$$

with a similar form for $\bar{\Gamma}_{a\bar{a}}$. *Therefore*

$$\bar{\sigma}(s) = \frac{12\pi (s/M_V^2) \bar{\Gamma}_{a\bar{a}} \bar{\Gamma}_{b\bar{b}}}{(s - M_V^2)^2 + M_V^2 \Gamma_V^2} \xrightarrow[s = M_V^2]{} \frac{12\pi}{M_V^2} B_{a\bar{a}} B_{b\bar{b}},$$

where the branching ratio into $b\bar{b}$ *is* $B_{b\bar{b}} = \bar{\Gamma}_{b\bar{b}}/\Gamma_V$.

2.28 Consider the interaction

$$\mathcal{L}_I = g \left(\bar{\psi}_{aL} \psi_{bR} \phi + \bar{\psi}_{bR} \psi_{aL} \phi^\dagger \right)$$

between distinct fermions ψ_a and ψ_b, where ϕ is a complex scalar and g is real. Show that the Lagrangian violates P and C, but is CP invariant.

Solution: Under space reflection

$$\mathcal{L}_I(t, \vec{x}\,) \;\to g\eta_P \left(\bar{\psi}_{aR} \psi_{bL} \phi + \bar{\psi}_{bL} \psi_{aR} \phi^\dagger \right),$$

where η_P *is a parity phase from the 3 fields, the* $(t, \vec{x}\,) \to (t, -\vec{x}\,)$ *dependence is implicit, and we have used Table 2.2. This is* $\neq \mathcal{L}_I(t, -\vec{x}\,)$ *for any* η_P. *Similarly, under charge conjugation*

$$\mathcal{L}_I \to g \left(\eta_C \bar{\psi}_{bL} \psi_{aR} \phi^\dagger + \eta_C^* \bar{\psi}_{aR} \psi_{bL} \phi \right) \neq \mathcal{L}_I.$$

However, under $CP = PC$,

$$\mathcal{L}_I(t, \vec{x}\,) \to g \left(\eta_C \eta_P \bar{\psi}_{bR} \psi_{aL} \phi^\dagger + \eta_C^* \eta_P \bar{\psi}_{aL} \psi_{bR} \phi \right),$$

which is equal to $\mathcal{L}_I(t, -\vec{x}\,)$ *for the phase choice* $\eta_C \eta_P = 1$.

2.29 Consider $e^-(\vec{k}_1)\, \pi^+(\vec{p}_1) \;\to\; e^-(\vec{k}_2)\, \pi^+(\vec{p}_2)$ scattering, as in Figure 2.15. Use the two-component formalism of Section 2.11 to calculate the amplitudes $M(-,-)$ and $M(+,+)$ for $m_e = 0$ and $m_\pi \neq 0$. Express your results in terms of α, β_π, and the CM scattering angle θ. (The two amplitudes should be equal up to a possible sign by (2.278).)

Solution: From (2.342), (2.207), and Figure 2.10,

$$M(-,-) = \left[ie \;2k\; \phi_-(k_2)^\dagger \bar{\sigma}^\mu \phi_-(k_1) \right] \left[\frac{-ig_{\mu\nu}}{q^2} \right] \left[-ie(p_1 + p_2)^\nu \right].$$

$M(+,+)$ is the same except $\phi_-(k_i) \to \phi_+(k_i)$ and $\bar\sigma^\mu \to \sigma^\mu$, where $\bar\sigma^\mu$ and σ^μ are defined in (2.167). But,

$$\begin{aligned} k_1 &= k(1,0,0,1), \qquad k_2 = k(1,\sin\theta,0,\cos\theta) \\ p_1 &= E(1,0,0,-\beta_\pi), \qquad p_2 = E(1,-\beta_\pi\sin\theta,0,-\beta_\pi\cos\theta) \\ q^2 &= -2k^2(1-\cos\theta), \qquad k = \frac{s-m_\pi^2}{2\sqrt{s}} = \beta_\pi E, \qquad E = \frac{s+m_\pi^2}{2\sqrt{s}}. \end{aligned}$$

The spinors are given in Table 2.1, i.e.,

$$\begin{aligned} \phi_-(k_1) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \phi_+(k_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \phi_-(k_2) &= \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \qquad \phi_+(k_2) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}. \end{aligned}$$

Putting everything together,

$$M(-,-) = M(+,+) = 8\pi\alpha\, i \left(\frac{1}{\beta_\pi} + 1\right) \frac{\cos\frac{\theta}{2}}{1-\cos\theta}.$$

2.30 Consider the non-relativistic limit of the matrix element (2.356) for an e^- in a static external field.
(a) Compute the limit to linear order in the momenta. Hint: use the explicit forms for the spinors in the Pauli-Dirac representation. It simplifies the calculation to rewrite $\bar u_2 \Gamma^\mu u_1$ using the Gordon decomposition.
(b) Suppose that $\Gamma^\mu(p_2,p_1)$ in (2.352) contained a term $\frac{i\sigma^{\mu\nu}}{2m} q_\nu \gamma^5 G_2(q^2)$. This violates P and T but in principle could be generated by a new interaction. Show how $G_2(0)$ is related to the electric dipole moment $\vec{d}_e$ of the electron, which is defined by the non-relativistic interaction $H_{EDM} = -\vec{d}_e \cdot \vec{E}(\vec{x})$, where $\vec{E}$ is an external electric field. Note that the Hermiticity condition (2.354) requires that G_2 is pure imaginary.

Solution: (a) Using the first Gordon identity in Problem 2.10, the r.hs. of (2.356) is

$$-e\bar u_2 \left(\frac{p_1^\mu + p_2^\mu}{2m} F_1 + \frac{i\sigma^{\mu\nu}}{2m} q_\nu \left[F_1 + F_2\right] \right) u_1 \; \tilde A_\mu(\vec{q}).$$

From the explicit forms for the γ matrices in (2.164) and for the spinors in (2.184), only the upper two components contribute to linear order, yielding

$$-e\phi_{s_2}^\dagger \left(2m\tilde A_0(\vec q) - (\vec p_1 + \vec p_2)\cdot \tilde{\vec A}(\vec q) \right) \phi_{s_1}$$

for the first term. Only the space components contribute at $\mathcal{O}(\vec q)$ in the second term, giving

$$-ie\, \phi_{s_2}^\dagger \sigma^k \phi_{s_1} \epsilon_{ijk} q^j \tilde A^i(\vec q)\, [1 + F_2(0)] = e\,[1+F_2(0)]\, \phi_{s_2}^\dagger \vec\sigma \cdot \tilde{\vec B}(\vec q) \phi_{s_1},$$

since $\epsilon_{ijk} q^j \tilde A^i(\vec q) = i\tilde B^k(\vec q)$.
(b) The $\sigma^{i0} q_0$ and $\sigma^{ij} q_j$ terms are second order in the momenta, while the $\sigma^{0i} q_i$ term reduces to

$$-eG_2(0)\phi_{s_2}^\dagger \vec\sigma \cdot \vec q \tilde A_0(\vec q) \phi_{s_1} = -ieG_2(0) \phi_{s_2}^\dagger \vec\sigma \cdot \tilde{\vec E}(\vec q) \phi_{s_1},$$

where $\tilde{\vec{E}}(\vec{q}) = -i\vec{q}\tilde{A}_0(\vec{q})$ is the Fourier transform of the electric field $\vec{E}(\vec{x}) = -\vec{\nabla}A_0(\vec{x})$. This corresponds to the covariantly normalized momentum space matrix element of H_{EDM} for

$$\vec{d}_e = \frac{ieG_2(0)}{2m}\vec{\sigma}.$$

2.31 Suppose there is a small electric charge-violating coupling between the electron and a massless left-chiral neutrino ν_L, with

$$\mathcal{L}_{e\nu} = -\delta e A_\mu \bar{\psi}_{\nu_L}\gamma^\mu\psi_e + h.c.$$

Calculate the lifetime for $e^- \to \nu_L\gamma$, and find the value of δ corresponding to the limit in Table 2.3.

Solution: The decay amplitude is

$$M = -i\delta e\epsilon^*_\mu \bar{u}_{\nu_L}\gamma^\mu u_e.$$

From (2.178) and (2.181), $u_{\nu_L}\bar{u}_{\nu_L} \to P_L\ \not{p}_\nu$ in the massless limit, with $P_L \equiv (1-\gamma^5)/2$. Therefore,

$$|\bar{M}|^2 = \frac{1}{2}\sum_{s_e,\lambda_\gamma}|M|^2 = \frac{1}{2}\delta^2 e^2(-g_{\mu\nu})\, Tr[\gamma^\mu(\not{p}_e + m_e)\gamma^\nu P_L\ \not{p}_\nu] = 2\delta^2 e^2\, p_e\cdot p_\nu,$$

with $p_e\cdot p_\nu = m_e^2/2$. Therefore,

$$\Gamma = \tau^{-1} = \frac{|\bar{M}|^2}{16\pi m_e} = \frac{1}{4}\delta^2\alpha\, m_e.$$

Using $\hbar = 6.6\times 10^{-22}$ MeV-s and $m_e \sim 0.51$ MeV from Table 1.1, the limit $\tau > 6.6\times 10^{22}$ yr $= 2.1\times 10^{36}$ s corresponds to $\delta < 5.8\times 10^{-28}$.

2.32 Consider the proton matrix element $\bar{u}(\vec{p}_2)\Gamma^\mu_Q(q)u(\vec{p}_1)$ of the electromagnetic current in (2.393), with Γ^μ_Q given by (2.397). Calculate this explicitly in the Breit frame, in which $q^0 = 0$, i.e.,

$$q = (0,0,0,\sqrt{Q^2}), \qquad p_1 = (E,0,0,-\sqrt{Q^2}/2), \qquad p_2 = (E,0,0,+\sqrt{Q^2}/2).$$

Express the time and space components in terms of the electric and magnetic form factors defined in (2.398) and interpret the results.

Solution: Using the Gordon identity,

$$\bar{u}_2\left[\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}}{2m_p}q_\nu F_2\right]u_1 = \bar{u}_2\left[\gamma^\mu(F_1+F_2) - \frac{(p_1+p_2)^\mu}{2m_p}F_2\right]u_1.$$

But $(p_1+p_2)^0 = 2E = 2\sqrt{m_p^2 + Q^2/4}$ and $(p_1+p_2)^i = 0$. Writing out the spinors and γ matrices in the Pauli-Dirac representation,

$$\begin{aligned}\bar{u}_2\Gamma^0_Q u_1 &= 2m_p\phi_2^\dagger G_E(Q^2)\phi_1\\ \bar{u}_2\Gamma^i_Q u_1 &= G_M(Q^2)\phi_2^\dagger\left[\vec{\sigma}\cdot\vec{p}_2\sigma^i + \sigma^i\vec{\sigma}\cdot\vec{p}_1\right]\phi_1,\\ &= G_M(Q^2)\phi_2^\dagger i(\vec{\sigma}\times\vec{q})^i\phi_1,\end{aligned}$$

where $\phi_{1,2}$ are the Pauli spinors and we used $\vec{p}_2 = -\vec{p}_1 = \vec{q}/2$. Then, similar to (2.356) (but for the proton with charge $+e$),

$$\begin{aligned}\langle p(2)|H|p(1)\rangle &= \langle p(2)| \int d^3\vec{x}\, e J_Q^\mu A_\mu |p(1)\rangle \\ &= 2m_p\, e\, \phi_2^\dagger \left[G_E(Q^2)\tilde{A}_0(\vec{q}) - \frac{G_M(Q^2)}{2m_p} \vec{\sigma}\cdot\tilde{\vec{B}}(\vec{q}) \right] \phi_1\end{aligned}$$

so that $G_{E,M}$ can be interpreted as the Fourier transforms of the electric and magnetic distributions of the proton.

2.33 Let $V(\vec{x}_1 - \vec{x}_2)$ be the potential between two non-identical spin-$\frac{1}{2}$ particles in non-relativistic quantum mechanics (NRQM). (V may also depend on their spin and momentum operators). One shows in time-dependent perturbation theory that the transition amplitude U_{fi} from $|i\rangle = |\vec{p}_1 s_1, \vec{p}_2 s_2\rangle$ to $|f\rangle = |\vec{p}_3 s_3, \vec{p}_4 s_4\rangle$ with $m_1 = m_3, m_2 = m_4$ is

$$U_{fi} = -i(2\pi)^4 \delta^4 (p_3 + p_4 - p_1 - p_2)\, \phi_3^\dagger \phi_4^\dagger \left(\int d^3\vec{r}\, e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) \right) \phi_1 \phi_2,$$

where ϕ_i is a two-component Pauli spinor, and V contains appropriate spin matrices. Note that these states are in our covariant normalization convention, which has an extra factor $\sqrt{2\pi)^3 2E_i} \sim \sqrt{2\pi)^3 2m_i}$ for the i^{th} external particle compared to the usual conventions of NRQM. The corresponding formula in field theory is

$$U_{fi} = (2\pi)^4 \delta^4 (p_3 + p_4 - p_1 - p_2)\, M,$$

where M is the scattering amplitude with the phase convention of Appendix B. Comparing these results, we can read off the equivalent non-relativistic potential corresponding to a given scattering amplitude. Specifically, for

$$\vec{p}_1 = -\vec{p}_2 = \vec{p} - \frac{\vec{q}}{2}, \qquad \vec{p}_3 = -\vec{p}_4 = \vec{p} + \frac{\vec{q}}{2},$$

the non-relativistic limit $\vec{p} \to 0, |\vec{q}|^2 \ll m_i^2$ yields

$$M \to -i(2m_1)(2m_2)\phi_3^\dagger \phi_4^\dagger \tilde{V}(\vec{q})\phi_1\phi_2 \equiv -i(2m_1)(2m_2)\phi_3^\dagger\phi_4^\dagger \int d^3\vec{r}\, e^{-i\vec{q}\cdot\vec{r}} V(\vec{r})\phi_1\phi_2.$$

Non-leading terms in $\vec{p}$ can be interpreted as the non-relativistic momentum operator.
(a) Calculate the potential corresponding to the effective four-fermi Hamiltonian density $\mathcal{H}_I = \lambda \bar{\psi}_1 \psi_1\ \bar{\psi}_2 \psi_2$.
(b) Consider the interaction

$$\mathcal{L}_I = \left[g_1 \bar{\psi}_1 \psi_1 + g_2 \bar{\psi}_2 \psi_2 \right] \phi$$

between two fermions and a Hermitian scalar of mass m_ϕ. Calculate the potential between ψ_1 and ψ_2 generated by t-channel ϕ exchange, and show that it is attractive for $g_1 g_2 > 0$.
(c) Calculate the potential generated by

$$\mathcal{L}_I = \left[g_1 \bar{\psi}_1 \gamma^\mu \psi_1 + g_2 \bar{\psi}_2 \gamma^\mu \psi_2 \right] V_\mu,$$

where V_μ is a spin-1 particle of mass M_V. Show that it is repulsive for $g_1 g_2 > 0$.
(d) Repeat parts (b) and (c) for the case $g_1 g_2 > 0$, but for the potential between antiparticle $\bar{1}$ and particle 2 and interpret the results. Hint: it is slightly easier to use the charge

conjugation formalism of Section 2.10.
(e) Consider the interaction in (2.269) of a π^0 with protons and neutrons. Calculate the tree-level amplitude for $p(\vec{p}_1)\, n(\vec{p}_2) \to p(\vec{p}_3)\, n(\vec{p}_4)$ by t-channel π^0 exchange, and show that it leads to the non-relativistic potential

$$V(\vec{r}) = -\frac{g_\pi^2 m_\pi^3}{16\pi m_p^2}\left[\frac{1}{3}\vec{\sigma}_p \cdot \vec{\sigma}_n \frac{e^{-x}}{x} + \frac{S}{3}\left(\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3}\right) e^{-x}\right],$$

where $m_p \sim m_n$, $\vec{\sigma}_p(\vec{\sigma}_n)$ are the Pauli matrices acting on the $p\,(n)$ spin, $x = m_\pi r$, and S is the tensor operator

$$S = 3\,\vec{\sigma}_p \cdot \hat{r}\,\vec{\sigma}_n \cdot \hat{r} - \vec{\sigma}_p \cdot \vec{\sigma}_n.$$

Solution: (a) We have

$$M = -i\langle f|\mathcal{H}_I|i\rangle = -i\lambda \bar{u}_3 u_1\ \bar{u}_4 u_2 \xrightarrow[\vec{p}=\vec{q}=0]{} -i\lambda(2m_1)(2m_2)\phi_3^\dagger\phi_1\phi_4^\dagger\phi_2,$$

where we have used the explicit expressions for the u spinors in the Pauli-Dirac representation. The absolute sign follows from explicitly writing out the fields and using

$$\langle 3\,4| = |3\,4\rangle^\dagger = \left(a_3^\dagger a_4^\dagger|0\rangle\right)^\dagger = \langle 0|a_4 a_3.$$

Thus, $\tilde{V}(\vec{q}) = \lambda$ and $V(\vec{r}) = \lambda\delta^3(\vec{r})$.
(b)

$$M = (ig_1)(ig_2)\left(\frac{i}{q^2 - m_\phi^2}\right)\bar{u}_3 u_1\ \bar{u}_4 u_2 \to \frac{ig_1 g_2}{|\vec{q}|^2 + m_\phi^2}\,(2m_1)(2m_2)$$

where the second form is the non-relativistic limit in the CM. Thus

$$\tilde{V}(\vec{q}) = -\frac{g_1 g_2}{|\vec{q}|^2 + m_\phi^2}, \qquad V(r) = -\frac{g_1 g_2}{4\pi}\frac{e^{-m_\phi r}}{r},$$

the Yukawa potential. It is attractive for $g_1 g_2 > 0$ and repulsive for $g_1 g_2 < 0$.
(c) The calculation is similar to part (b), except the propagator has $-ig_{\mu\nu}$ in the numerator. Only the $\mu = \nu = 0$ terms survive in the NR limit, yielding

$$V(r) = +\frac{g_1 g_2}{4\pi}\frac{e^{-M_V r}}{r},$$

which is repulsive for same sign charges and attractive for opposite signs as expected. For $M_V \to 0$ and $g_i = eQ_i$, we recover the Coulomb potential $V(r) = \alpha Q_1 Q_2/r$, with $\alpha = e^2/4\pi$.
(d) From (2.296) and Table 2.2, we have that $\bar{\psi}_1\psi_1 = \bar{\psi}_1^c\psi_1^c$ and $\bar{\psi}_1\gamma^\mu\psi_1 = -\bar{\psi}_1^c\gamma^\mu\psi_1^c$. Thus, the scalar exchange potential is unchanged for $1 \to \bar{1}$ (i.e., it remains attractive), while the vector exchange changes sign and becomes attractive. One can also write out the matrix elements using the original ψ_1 field. In both cases there is an extra minus sign from anticommuting the b and $b^\dagger$ operators, and another minus sign for the scalar case because of the γ^0 in $\bar{\psi}$, e.g.,

$$\langle\bar{3}|\bar{\psi}_1\psi_1|\bar{1}\rangle = -\bar{v}_1 v_3 \to +\chi_1^\dagger\chi_3 = \phi_3^\dagger\phi_1, \qquad \langle\bar{3}|\bar{\psi}_1\gamma^0\psi_1|\bar{1}\rangle = -\bar{v}_1\gamma^0 v_3 \to -\chi_1^\dagger\chi_3 = -\phi_3^\dagger\phi_1.$$

(e) The p *and* n *vertices are respectively* $i\mathcal{L} = \mp g_\pi \gamma^5$, *so*

$$M = -\frac{ig_\pi^2}{q^2 - m_\pi^2} \bar{u}_3 \gamma^5 u_1 \bar{u}_4 \gamma^5 u_2.$$

But $\gamma^0\gamma^5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, *so*

$$\bar{u}_3 \gamma^5 u_1 = 2m_p \phi_3^\dagger \left[\frac{\vec{\sigma} \cdot \vec{p}_1}{2m_p} - \frac{\vec{\sigma} \cdot \vec{p}_3}{2m_p} \right] \phi_1 = -\phi_3^\dagger \vec{\sigma} \cdot \vec{q} \phi_1.$$

Therefore,

$$\begin{aligned} V(\vec{r}) &= \int \frac{d^3\vec{q}}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\, \frac{g_\pi^2}{\vec{q}^{\,2} + m_\pi^2}\, \frac{\vec{\sigma}_p \cdot \vec{q}}{2m_p}\, \frac{\vec{\sigma}_n \cdot \vec{q}}{2m_p} \\ &= \frac{-g_\pi^2}{32\pi^3 m_p^2} \left(\vec{\sigma}_p \cdot \vec{\nabla} \right) \left(\vec{\sigma}_n \cdot \vec{\nabla} \right) \int \frac{d^3\vec{q}\, e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^{\,2} + m_\pi^2} \end{aligned}$$

where we have taken $m_p \sim m_n$. *Note that* σ_p *and* σ_n *act on their own spinors. Using contour integration, the integral is just the Yukawa form*

$$\frac{2\pi^2}{r}\, e^{-m_\pi r} = 2\pi^2 m_\pi \frac{e^{-x}}{x}.$$

Also, $\vec{\nabla} = m_\pi \vec{\nabla}_x$ *where* $\vec{\nabla}_x = \frac{d}{d\vec{x}}$, *so that*

$$V(\vec{r}) = \frac{-g_\pi^2 m_\pi^3}{16\pi m_p^2} \left(\vec{\sigma}_p \cdot \vec{\nabla}_x \right) \left(\vec{\sigma}_n \cdot \vec{\nabla}_x \right) \left(\frac{e^{-x}}{x} \right).$$

But

$$\left(\vec{\sigma}_n \cdot \vec{\nabla}_x \right) \frac{e^{-x}}{x} = \sigma_n^i \frac{d}{dx^i} \left(\frac{e^{-x}}{x} \right) = \sigma_n^i \frac{x^i}{x} \frac{d}{dx} \left(\frac{e^{-x}}{x} \right) = -\sigma_n^i x^i \left[\frac{1}{x^2} + \frac{1}{x^3} \right] e^{-x}.$$

Therefore,

$$\begin{aligned} \left(\vec{\sigma}_p \cdot \vec{\nabla}_x \right) \left(\vec{\sigma}_n \cdot \vec{\nabla}_x \right) \frac{e^{-x}}{x} &= \sigma_p^j \frac{d}{dx^j} \left[\left(\vec{\sigma}_n \cdot \vec{\nabla}_x \right) \frac{e^{-x}}{x} \right] \\ &= -\vec{\sigma}_p \cdot \vec{\sigma}_n \left[\frac{1}{x^2} + \frac{1}{x^3} \right] e^{-x} - (\vec{\sigma}_n \cdot \vec{x}) \left(\vec{\sigma}_p \cdot \vec{\nabla}_x \right) \left[\frac{1}{x^2} + \frac{1}{x^3} \right] e^{-x}. \end{aligned}$$

The second term reduces to

$$+ (\vec{\sigma}_p \cdot \vec{x}\,)\, (\sigma_n \cdot \vec{x}\,)\, \frac{1}{x^2} \left[\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3} \right] e^{-x},$$

yielding the result after a slight rearrangement.

2.34 Suppose that a new lepton-flavor violating interaction leads to the effective interaction

$$\mathcal{L}_{eff} = -i\frac{e}{2} F^{\mu\nu} \bar{e} \sigma_{\mu\nu} \left[A + B\gamma^5 \right] \mu + h.c.,$$

where A and B are respectively magnetic and electric dipole transition moments. Calculate the decay rate for $\mu \to e\gamma$, neglecting m_e.

Solution: $\mathcal{L}_{eff}$ *leads to the matrix element*

$$M = -e\bar{u}_e \not{q} \not{\epsilon}^* \left[A + B\gamma^5\right] u_\mu,$$

where $q = p_\mu - p_e$ *is the photon momentum,* ϵ *is the polarization vector, and we have used* $q \cdot \epsilon^* = 0$*. But* $p_\mu \cdot \epsilon^* = 0$ *also, and one can choose a linear polarization basis so that* ϵ *is real. Then*

$$M = +e\, m_\mu \bar{u}_e \not{\epsilon}\left[A - B\gamma^5\right] u_\mu,$$

and

$$|\bar{M}|^2 = \frac{1}{2}\sum_\lambda e^2\, m_\mu^2\; Tr(\not{p}_e \not{\epsilon} \not{p}_\mu \not{\epsilon})\left(|A|^2 + |B|^2\right) = 4e^2 m_\mu^2\, p_e \cdot p_\mu \left(|A|^2 + |B|^2\right).$$

But $p_e \cdot p_\mu = m_\mu^2/2$*, so*

$$\Gamma = \frac{p_f}{8\pi m_\mu^2}|\bar{M}|^2 = \frac{e^2\, m_\mu^3}{8\pi}\left(|A|^2 + |B|^2\right).$$