CHAPTER 2

Euclidean Geometry

In this chapter we start off with a very brief review of basic properties of angles, lines, and parallels. When presenting such material, one has to make a choice. One can present the basic results of plane geometry from first principles, starting with an axiomatic system, such as Hilbert's Axioms, carefully laying out such concepts as betweenness, incidence, congruence, and continuity. This approach has several virtues. Students see, perhaps for the first time in their mathematical careers, a logical system built entirely from first principles. They also can clearly determine what theorems, definitions, and axioms are fair game to use in their own proofs of results. On the other hand, a thorough and *complete* development of Hilbert's axioms would necessarily take a substantial portion of a semester-long course in geometry, leaving little time for other, equally important topics such as non-euclidean geometry and transformational geometry.

A second approach is to review, in summary form, some of the most important logical problems of classical Euclidean geometry that axiom writers such as Hilbert attempted to fix, and then to move on to more substantial results in plane geometry. This is the approach taken in Chapter 2. It has the advantage of exposing students to the logical issues facing mathematicians over the last several hundred years and, at the same time, covering significant geometric ideas such as the definition of area, cevians, and circle inversion. One disadvantage of this approach is that students may feel unsure of what they can assume and not assume when working on proofs. In each section of Chapter 2 the author tried to carefully describe what results and assumptions were made in that section. For example, in section 2.1, students are instructed to use the notion of *betweenness* in the way one's tuition would dictate, while at the same time pointing out that this is one of those geometric properties that needs an axiomatic base.

If the more rigorous approach to Euclidean Geometry is desired, the complete foundational development can be found in on-line chapters at the author's website: http://www.gac.edu/~hvidsten/geom-text.

SOLUTIONS TO EXERCISES IN CHAPTER 2

2.1 Angles, Lines, and Parallels

This section is perhaps the least satisfying section in the chapter for students, since many theorems are referenced without proof. It may be helpful to remind students that these results were no doubt covered in great detail in their high school geometry course, and that a full development of such results would entail a "filling in" of many days (weeks/months) of foundational work based on Hilbert's axioms.

A significant number of the exercises deal with parallel lines. This is for two reasons. First of all, historically there was a great effort to prove Euclid's fifth Postulate by converting it into a logically equivalent statement that was hoped to be easier to prove. Thus, many of the exercises nicely echo this history. Secondly, parallels and the parallel postulate are at the heart of one of the greatest revolutions in math—the discovery of non-Euclidean geometry. This section foreshadows that development, which is covered in Chapters 7 and 8.

2.1.1 It has already been shown that $\angle FBG \cong \angle DAB$. Also, by the vertical angle theorem (Theorem 2.3) we have $\angle FBG \cong \angle EBA$ and thus, $\angle DAB \cong \angle EBA$.

Now, $\angle DAB$ and $\angle CAB$ are supplementary, thus add to two right angles. Also, $\angle CAB$ and $\angle ABF$ are congruent by the first part of this exercise, as these angles are alternate interior angles. Thus, $\angle DAB$ and $\angle ABF$ add to two right angles.

2.1.2 Let ΔABC be a triangle, and consider the sum of the angles at A and B. Extend the angle at A to create an exterior angle. Then, the sum of this exterior angle and the angle at A is 180 degrees, as they make up a line. However, by the Exterior Angle Theorem we know that the exterior angle is greater than the angle at B. Thus, the sum of the angles at A and B is less than the sum of the angle at A and its exterior angle, which is 180.

2.1.3 a. False, right angles are defined solely in terms of congruent angles.

b. False, an angle is defined as *just* the two rays plus the vertex.

c. True. This is part of the definition.

d. False. The term "line" is undefined.

2.1.4 a. A point M is the midpoint of segment $\overline{AB}$ if M is between A and B and $\overline{AM} \cong \overline{MB}$.

b. The perpendicular bisector of segment $\overline{AB}$ is a line through the midpoint M of $\overline{AB}$ that is perpendicular to $\overleftrightarrow{AB}$.

c. The triangle defined by three non-collinear points A, B, C is the union of the line segments $\overline{AB}$, $\overline{AC}$, $\overline{BC}$.

d. An equilateral triangle is a triangle whose sides are congruent.

2.1.5 Proposition I-23 states that angles can be copied. Let A and B be points on l and n respectively and let m be the line through A and B. If $t = m$ we are done. Otherwise, let D be a point on t that is on the same side of n as l. (Assuming the standard properties of betweenness) Then, $\angle BAD$ is smaller than the angle at A formed by m and n. By Theorem 2.9 we know that the interior angles at B and A sum to two right angles, so $\angle CBA$ and $\angle BAD$ sum to less than two right angles. By Euclid's fifth postulate t and l must meet.

2.1.6 Given the assumptions stated in the exercise, if we copy $\angle CBA$ to A, creating line n, then by Theorem 2.8 n and l will be parallel. Also, the sum of the interior angles at B and A for lines l and n will sum to two right angles. Thus, lines t and n cannot be coincident. Thus, by Playfair line t cannot be parallel to l.

To show that t and l intersect on the same side of m as D and C, we assume that they intersect on the other side, at some point E. Let F be a point on n that is on the other side of m from C, and let G a point on t on this same side. Then, $\angle BAF$ is less than $\angle BAG$ in measure, and since $\angle BAF \cong \angle CBA$ by Theorem 2.9, we have that the exterior angle $\angle CBA$ to ΔBAE is smaller than an opposite interior angle ($\angle BAG$), which contradicts the Exterior Angle Theorem.

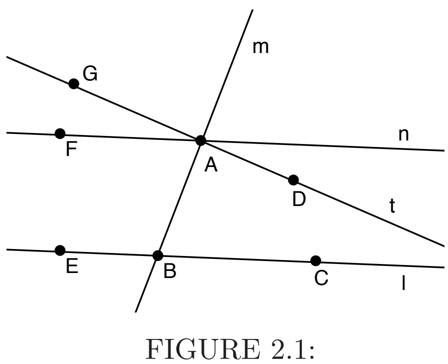

FIGURE 2.1:

2.1.7 First, assume Playfair's Postulate, and let lines l and m be parallel, with line t perpendicular to l at point A. If t does not intersect m then, t and l are both parallel to m, which contradicts Playfair. Thus, t intersects m and by Theorem 2.9 t is perpendicular at this intersection.

Now, assume that whenever a line is perpendicular to one of two parallel lines, it must be perpendicular to the other. Let l be a line and P a point not on l. Suppose that m and n are both parallel to l at P. Let t be a perpendicular from P to l. Then, t is perpendicular to m and n at P. By Theorem 2.4 it must be that m and n are coincident.

2.1.8 To create a parallel to $\overleftrightarrow{BC}$ at A we could just copy $\angle CBA$ to A and use Theorem 2.8. The three angles defined by the triangle at A sum to two right angles, and by Theorem 2.9 we have that the sum of these angles equals the sum of the angles in the triangle.

2.1.9 Assume Playfair and let lines m and n be parallel to line l. If $m \neq n$ and m and n intersect at P, then we would have two different lines parallel to l through P, contradicting Playfair. Thus, either m and n are parallel, or are the same line.

Conversely, assume that two lines parallel to the same line are equal or themselves parallel. Let l be a line and suppose m and n are parallel to l at a point P not on l. Then, n and m must be equal, as they intersect at P.

2.1.10 Assume Playfair and let line t intersect one of the parallel lines m and n, say it intersects m at P. If m did not intersect n, then t and m would be two different lines both parallel to n at P, which contradicts Playfair.

Conversely, assume that if a line intersects one of two parallel lines, it must intersect the other. Let l be a line and suppose m and n are parallel to l at a point P not on l, with $m \neq n$. Then, m intersects n, which is parallel to l. By assumption, m must intersect l, and thus cannot be parallel to l.

2.2 Congruent Triangles and Pasch's Axiom

This section introduces many results concerning triangles and also discusses several axiomatic issues that arose from Euclid's treatment of triangles.

This may be a good point to review Euclid's "proof" of SAS congruence. An interesting discussion point would be to have students voice their opinion as to whether the proof was valid or not.

Also, the history of axiom systems would be a good supplemental activity at this point. Hilbert's axioms did not arise overnight. He took the best of those who came before him, including Pasch, and molded these separate strands into a complete system.

2.2.1 Yes, it could pass through points A and B of ΔABC. It does not contradict Pasch's axiom, as the axiom stipulates that the line cannot pass through A, B, or C.

2.2.2 Construct the diagonal $\overline{AC}$ of the rectangle $ABCD$. Then, a line passing through a side of the rectangle will be a line passing through a side of one of the two triangles defined by the diagonal and the original sides of the rectangle. By Pasch's axiom, this line will either pass through one of the other sides of the triangle, which include the rectangle sides and the diagonal. If it passes through a side of the rectangle, we are done. If it passes through the diagonal, then using Pasch's axiom a second time, we get that it must pass through one of the other two sides of the other triangle, and thus through a side of the rectangle.

The same argument can be used repeatedly to show that a line passing through a side (but not a vertex) of an arbitrary n-gon (and not just a regular n-gon) will intersect a side. Just pick a vertex and construct interior triangles by taking all diagonals from this vertex.

2.2.3 No. Here is a counter-example.

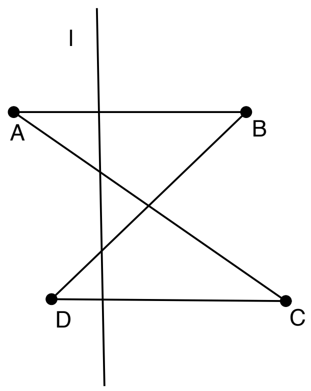

FIGURE 2.2:

2.2.4 If $A = B$, or $B = C$, or $A = C$ the result follows immediately. Otherwise, we can assume all three points are distinct.

If they all lie on a line, then, one is between the other two. In every case, we get that $\overline{AC}$ cannot intersect l.

Assume the points are non-collinear. and that A and C are on opposite sides of l. Then, l intersects $\overline{AC}$ and does not pass through A or C. By Pasch's axiom, it must intersect $\overline{AB}$ or $\overline{BC}$ at a point other than A, B, or C. Say it intersects $\overline{AB}$. This contradicts the assumption that A and B are on the same side. Likewise, if it intersects $\overline{BC}$ we get a contradiction. Thus, A and C are on the same side of l.

2.2.5 If $A = C$ we are done. If A, B, and C are collinear, then B cannot be between A and C, for then we would have two points of intersection for two lines. If A is between B and C, then l cannot intersect $\overline{AC}$. Likewise, C cannot be between A and B.

If the points are not collinear, suppose A and C are on opposite sides. Then l would intersect all three sides of ΔABC, contradicting Pasch's axiom.

2.2.6 A point P is in the interior of a ΔABC if P is in the interior of $\angle ABC$ and in the interior of $\angle BCA$ and in the interior of $\angle CAB$.

2.2.7 Let $\angle ABC \cong \angle ACB$ in ΔABC. Let $\overrightarrow{AD}$ be the angle bisector of $\angle BAC$ meeting side $\overline{BC}$ at D. Then, by AAS, ΔDBA and ΔDCA are congruent and $\overline{AB} \cong \overline{AC}$.

2.2.8 Referring to Fig. 2.1, we can use the SSS triangle congruence theorem on ΔADE and ΔABE to show that $\angle EAB \cong \angle BAE$.

2.2.9 Suppose that two sides of a triangle are not congruent. Then, the angles opposite those sides cannot be congruent, as if they were, then by the previous exercise, the triangle would be isosceles.

Suppose in ΔABC that $\overline{AC}$ is greater than $\overline{AB}$. On $\overline{AC}$ we can find a point D between A and C such that $\overline{AD} \cong \overline{AB}$. Then, $\angle ADB$ is an exterior angle to ΔBDC and is thus greater than $\angle DCB$. But, ΔABD is isosceles and so $\angle ADB \cong \angle ABD$, and $\angle ABD$ is greater than $\angle DCB = \angle ACB$.

2.2.10 In the figure accompanying Theorem 2.11, suppose that $\overline{BC}$ was greater than $\overline{YZ}$. Then, we could find a point D between B and C such that $\overline{BD} \cong \overline{YZ}$, and by SAS ΔABD would be congruent to ΔXYZ. This implies that $\angle BAD \cong \angle YXZ$. But, we are given that $\angle BAC \cong \angle YXZ$, and so $\angle BAD \cong \angle BAC$. This implies that D lies on $\overleftrightarrow{AC}$, and that A, B, C are collinear, which is impossible.

2.2.11 Let ΔABC and ΔXYZ be two right triangles with right angles at A and X, and suppose $\overline{BC} \cong \overline{YZ}$ and $\overline{AC} \cong \overline{XZ}$. Suppose $\overline{AB}$ is greater than $\overline{XY}$. Then, we can find a point D between A and B such that $\overline{AD} \cong \overline{XY}$. By SAS $\Delta ADC \cong \Delta XYZ$. Now, $\angle BDC$ is exterior to ΔADC and thus must be greater than 90 degrees. But, ΔCDB is isosceles, and thus $\angle DBC$ must also be greater than 90 degrees. This is impossible, as then ΔCDB would have angle sum greater than 180 degrees.

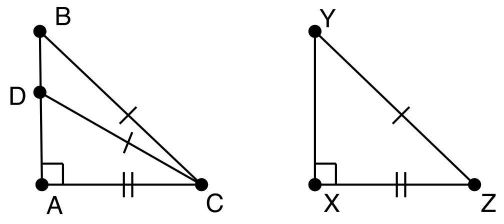

FIGURE 2.3:

2.2.12 Show that ΔADB and ΔBCA are congruent, and then show that ΔADC and ΔBDC are congruent.

2.2.13 We use AAS to show that $\Delta BFH \cong \Delta AFG$ and $\Delta CEI \cong \Delta AEG$. Thus $\overline{BH} \cong \overline{AG} \cong \overline{CI}$ and $BHIC$ is Saccheri. Also, by adding congruent angles in the left case we get that the sum of the angles in the triangle is the same as the sum of the summit angles. In the right case, we need to re-arrange congruent angles.

2.2.14 Assume Playfair and let $ABCD$ be a Saccheri Quadrilateral with base $\overline{AB}$. By Theorem 2.8 we know that $\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{BC}$. By Theorem 2.9 the summit angles must add to 180 degrees. This, each angle must be 90 degrees.

Conversely, assume that the summit angles of a Saccheri quadrilateral are always right angles. Let ΔABC be a triangle. By the previous exercise, we know that we can construct a Saccheri quadrilateral based on the triangle whose summit angles add to the angle sum of the triangle. Thus, the sum of the angles in a triangle is always 180 degrees. By Exercise 2.1.8, this implies Playfair's axiom is true.

2.2.15 Given quadrilaterals $ABCD$ and $WXYZ$ we say the two quadrilaterals are congruent if there is some way to match vertices so that corresponding sides are congruent and corresponding angles are congruent.

SASAS Theorem: If $\overline{AB} \cong \overline{WX}$, $\angle ABC \cong \angle WXY$, $\overline{BC} \cong \overline{XY}$, $\angle BCD \cong \angle XYZ$, and $\overline{CD} \cong \overline{YZ}$, then quadrilateral $ABCD$ is congruent to quadrilateral $WXYZ$.

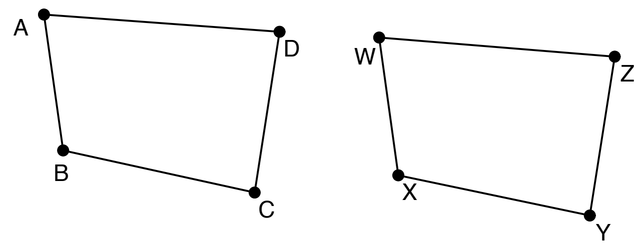

FIGURE 2.4:

Proof: ΔABC and ΔWXY are congruent by SAS. This implies that ΔACD and ΔWYZ are congruent. This shows that sides are correspondingly congruent, and two sets of angles are congruent ($\angle ABC \cong \angle WXY$ and $\angle CDA \cong \angle YZW$). Since $\angle BAC \cong \angle XWY$ and $\angle CAD \cong \angle YWZ$, then by angle addition $\angle BAD \cong \angle XWZ$. Similarly, $\angle BCD \cong \angle XYZ$. □

2.3 Project 3 - Special Points of a Triangle

Students should be encouraged to explore and experiment in this lab project. Ask them if there are any other sets of intersecting lines that one could construct. Or, are there interesting properties of constructed intersecting lines in other polygons?

Some of the students in the class will be future secondary math teachers. This project is one that could be easily transferred to the high school setting. Students could have as an extra credit exercise the task of preparing a similar project for a high school class.

2.3.1 ΔDGB and ΔDGA are congruent by SAS, as are ΔEGB and ΔEGC. Thus, $\overline{AG} \cong \overline{BG} \cong \overline{CG}$. By SSS $\Delta AFG \cong \Delta CFG$ and since the angles at F must add to 180 degrees, the angles at F must be congruent right angles.

2.3.2 We proved in the previous exercise that, if G is the circumcenter, then $\overline{AG} \cong \overline{BG} \cong \overline{CG}$. Thus, the circle centered at G with radius $\overline{AG}$ must pass through the other two vertices.

2.3.3 The angle pairs in question are all pairs of an exterior angle and an interior angle on the same side for a line falling on two parallel lines. These are congruent by Theorem 2.9.

Since $\angle DAB$, $\angle BAC$, and $\angle CAE$ sum to 180 degrees, and $\angle BDA$, $\angle BAD$, and $\angle ABD$ sum to 180 then, using the congruences shown in the diagram, we get that $\angle DBA \cong \angle BAC$. Likewise, $\angle BAD \cong \angle ABC$. By ASA we get that $\Delta ABC \cong \Delta BAD$. Similarly, $\Delta ABC \cong \Delta CEA$ and $\Delta ABC \cong \Delta FCB$.

2.3.4 An altitude of ΔABC will meet a side at right angles. Thus, the altitude meets the side of the associated triangle at right angles, as this side is parallel to the side of ΔABC. Also, by the triangle congruence result of the previous exercise, the altitude bisects the side of the associated triangle.

2.3.5 Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ define an angle and let $\overrightarrow{AD}$ be the bisector. Drop perpendiculars from D to $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and assume these intersect at B and C. Then, by AAS, ΔABD and ΔACD are congruent, and $\overline{BD} \cong \overline{CD}$.

Conversely, suppose D is interior to $\angle BAC$ with $\overline{BD}$ perpendicular to $\overrightarrow{AB}$ and $\overline{CD}$ perpendicular to $\overrightarrow{AC}$. Also, suppose that $\overline{BD} \cong \overline{CD}$. Then, by the Pythagorean Theorem $AB^2 + BD^2 = AD^2$ and $AC^2 + CD^2 = AD^2$. Thus, $\overline{AB} \cong \overline{AC}$ and by SSS $\Delta ABD \cong \Delta ACD$. This implies that $\angle BAD \cong \angle CAD$.

2.4.1 Mini-Project:Area in Euclidean Geometry

This section includes the first "mini-project" for the course. These projects are designed to be done in the classroom, in groups of three or four students. Each group should elect a Recorder. The Recorder's sole job is to outline the group's solutions to exercises. The summary should not be a formal write-up of the project, but should give enough a brief synopsis of the group's reasoning process.

The main goal for the mini-projects is to have students discuss geometric ideas with one another. Through the group process, students clarify their own understanding of concepts, and help each other better grasp abstract ways of thinking. There is no better way to conceptualize an idea than to have to explain it to another person.

In this mini-project, students are asked to grapple with the notion of "area". You may want to precede the project by a general discussion of how to best define area. Students will quickly find that it is not such an obvious notion as they once thought. For example, what does it mean for two figures to have the same area?

2.4.1 Construct a diagonal and use the fact that alternate interior angles of a line falling on parallel lines are congruent to generate an ASA congruence for the two sub-triangles created in the parallelogram.

2.4.2 We know that $\overline{AD} \cong \overline{EF}$. Thus, $\overline{AE} \cong \overline{DF}$, as the length of either of these differ by the length of $\overline{DE}$. Also, $\angle AEB \cong \angle DFC$, by Theorem 2.9. By SAS we get that $\Delta AEB \cong \Delta DFC$. Thus, parallelogram $ABCD$ can be split into ΔAEB and $EBCD$ and parallelogram $EBCF$ can be split into ΔDFC and $EBCD$. Clearly, these are two pairs of congruent polygons.

2.4.3 We have that $\overline{AE} \cong \overline{DF}$. Theorem 2.9 says that $\angle AEB \cong \angle DFC$. So, by SAS $\Delta AEB \cong \Delta DFC$. Let G be the point where $\overline{CD}$ intersects $\overline{BE}$. (Such a point exists by Pasch's axiom applied to ΔAEB) Now, parallelogram $ABCD$ can be split into ΔAEB plus ΔBGC minus ΔDGE. Also, parallelogram $EBCF$ can be split into ΔDFC plus ΔBGC minus ΔDGE.

2.4.4 Use Theorem 2.8 and Exercise 2.4.1.

2.4.5 By Theorem 2.9 we know that $\angle BAE$ and $\angle FBA$ are right angles, and thus $ABFE$ is a rectangle. By Theorem 2.9 we have that $\angle DAB \cong \angle CBG$, where G is a point on $\overrightarrow{AB}$ to the right of B. Subtracting the right angles, we get $\angle DAE \cong \angle CBF$. By SAS, $\Delta DAE \cong \Delta CBF$. Then rectangle $AEFB$ can be split into $AECB$ and ΔCBF and parallelogram $DABC$ can be split into $AECB$ and ΔDAE and the figures are equivalent.

Hidden Assumptions? One hidden assumption is the notion that areas are additive. That is, if we have two figures that are not overlapping, then the area of the union is the sum of the separate areas.

2.4.2 Cevians and Area

2.4.6 Since a median is a cevian to a midpoint, then the fractions in the ratio product of Theorem 2.24 are all equal to 1.

2.4.7 Let the triangle and medians be labeled as in Theorem 2.24. The area of ΔAYB will be equal to AYh, where h is the length of a perpendicular dropped from B to $\overleftrightarrow{AC}$. The area of ΔCYB will be equal to CYh, Since $\overline{AY} \cong \overline{CY}$, these areas will be the same and ΔABC will balance along $\overleftrightarrow{BY}$. A similar argument shows that ΔABC balances along each median, and thus the centroid is a balance point for the triangle.

2.4.8 Refer to the figure below. By the previous exercise we know that $1+2+3 = 4+5+6$ (in terms of areas). Also, since 1 and 2 share the same base and height we have $3 = 4$. Similarly, $1 = 2$ and $5 = 6$. Thus, $1 = 6$.

Similarly, $2+3+4 = 1+5+6$ will yield $4 = 5$, and $3+4+5 = 1+5+5$ yields $2 = 3$. Thus, all 6 have the same area.

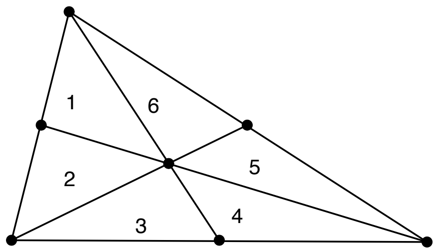

FIGURE 2.5:

2.4.9 Consider median $\overline{BD}$ in ΔABC, with E the centroid. Let $a = BE$ and $b = DE$. Then, the area of ΔDBC is $\frac{(a+b)h}{2}$ where h is the height of the triangle. This is 3 times the area of ΔDEC by the previous exercise. Thus, $\frac{(a+b)h}{2} = 3\frac{bh}{2}$, or $a = 2b$.

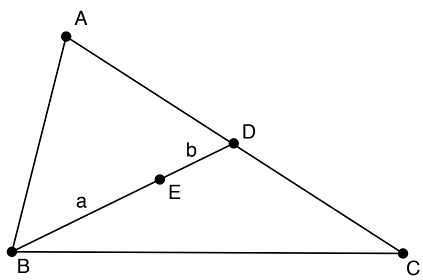

FIGURE 2.6:

2.5 Similar Triangles

As stated in the text, similarity is one of the most useful tools in the geometer's toolkit. Several of the exercises could be jumping off points for further discussion—the definition of the trigonometric functions, the Pythagorean Theorem (this could be a place to review some of the myriad of ways that this theorem has been proved), and Pascal's Theorem and its use in Hilbert's development of arithmetic.

2.5.1 Since $\overleftrightarrow{DE}$ cuts two sides of triangle at the midpoints, then by Theorem 2.27, this line must be parallel to the third side $\overline{BC}$. Thus $\angle ADE \cong \angle ABC$ and $\angle AED \cong \angle ACB$. Since the angle at A is congruent to itself, we have by AAA that ΔABC and ΔADE are similar, with proportionality constant of $\frac{1}{2}$.

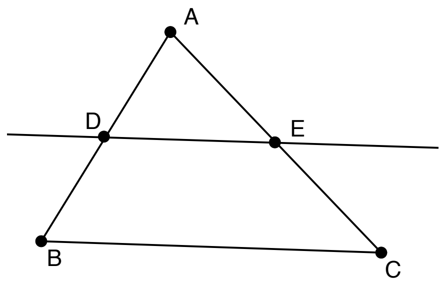

FIGURE 2.7:

2.5.2 Since $\overline{AG} \cong \overline{DE}$ then $\frac{AB}{AG} = \frac{AB}{DE}$ and so $\frac{AB}{AG} = \frac{AC}{DF}$. Since $\overline{AH} \cong \overline{DF}$ we get $\frac{AB}{AG} = \frac{AC}{AH}$. By Theorem 2.27, $\overleftrightarrow{GH}$ and $\overleftrightarrow{BC}$ are parallel.

2.5.3 Let ΔABC and ΔDEF have the desired SSS similarity property. That is sides $\overline{AB}$ and $\overline{DE}$, sides $\overline{AC}$ and $\overline{DF}$, and sides $\overline{BC}$ and $\overline{EF}$ are proportional. We can assume that $\overline{AB}$ is at least as large as $\overline{DE}$. Let G be a point on $\overline{AB}$ such that $\overline{AG} \cong \overline{DE}$. Let $\overleftrightarrow{GH}$ be the parallel to $\overleftrightarrow{BC}$ through G. Then, $\overleftrightarrow{GH}$ must intersect $\overleftrightarrow{AC}$, as otherwise $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ would be parallel. By the properties of parallels, $\angle AGH \cong \angle ABC$ and $\angle AHG \cong \angle ACB$. Thus, ΔAGH and ΔABC are similar.

Therefore, $\frac{AB}{AG} = \frac{AC}{AH}$. Equivalently, $\frac{AB}{DE} = \frac{AC}{AH}$. We are given that $\frac{AB}{DE} = \frac{AC}{DF}$. Thus, $\overline{AH} \cong \overline{DF}$.

Also, $\frac{AB}{AG} = \frac{BC}{GH}$ and $\frac{AB}{AG} = \frac{AB}{DE} = \frac{BC}{EF}$. Thus, $GH \cong EF$.

By SSS ΔAGH and ΔDEF are congruent, and thus ΔABC and ΔDEF are similar.

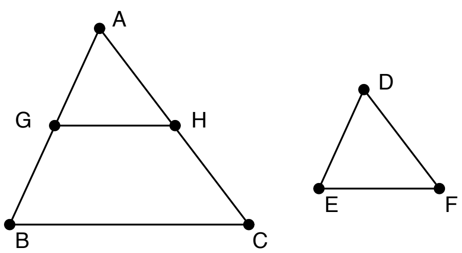

FIGURE 2.8:

2.5.4 Since $\angle ACD$ and $\angle DCB$ have measures summing to 90 degrees, and since $\angle DAC$ and $\angle ACD$ sum to 90, then $\angle DCB \cong \angle CAD (\cong \angle CAB)$. Likewise, $\angle CBD (\cong \angle ABC) \cong \angle ACD$. By AAA ΔDCB and ΔCAB are similar, as are ΔACD and ΔABC. Thus, $\frac{y}{a} = \frac{a}{c}$ and $\frac{x}{b} = \frac{b}{c}$. Or, $y = \frac{a^2}{c}$ and $x = \frac{b^2}{c}$. Thus, $c = x + y = \frac{a^2+b^2}{c}$. The result follows immediately.

2.5.5 Any right triangle constructed so that one angle is congruent to $\angle A$ must have congruent third angles, and thus the constructed triangle must be similar to ΔABC. Since *sin* and *cos* are defined in terms of ratios of sides, then proportional sides will have the same ratio, and thus it does not matter what triangle one uses for the definition.

2.5.6 Drop a perpendicular from C to $\overleftrightarrow{AB}$ intersecting at D. There are two cases. If D is not between A and B, then it is to to the left of A or to the right of B. We can assume it is to the left of B. Then, the angle at A must be obtuse, as $\angle BAC$ is exterior to right triangle ΔACD. If D is between A and B then we can assume the angles at A and B are acute, again by an exterior angle argument.

In the first case, $sin(\angle A) = \frac{CD}{b}$ and $sin(\angle B) = \frac{CD}{a}$. Then, $\frac{sin(\angle A)}{sin(\angle B)} = \frac{a}{b}$.

In the second case, $sin(\angle A) = sin(\angle DAC)$. An exactly analogous argument to the first case finishes the proof.

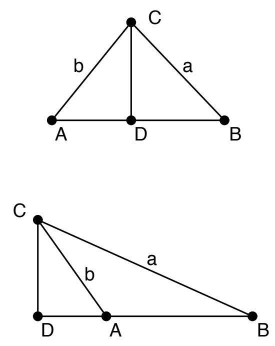

FIGURE 2.9:

2.5.7 If the parallel to $\overleftrightarrow{AC}$ does not intersect $\overleftrightarrow{RP}$, then it would be parallel to this line, and since it is already parallel to $\overleftrightarrow{AC}$, then by Exercise 2.1.9 $\overleftrightarrow{RP}$ and $\overleftrightarrow{AC}$ would be parallel, which is impossible.

By the properties of parallels, $\angle RAP \cong \angle RBS$ and $\angle RPA \cong \angle RSB$. Thus, by AAA ΔRBS and ΔRAP are similar. ΔPCQ and

ΔSBQ are similar by AAA using an analogous argument for two of the angles and the vertical angles at Q.

Thus, $\frac{CP}{BS} = \frac{CQ}{BQ} = \frac{PQ}{QS}$, and $\frac{AP}{BS} = \frac{AR}{BR} = \frac{PR}{SR}$. So, $\frac{CP}{AP}\frac{BQ}{QC} = \frac{CP}{AP}\frac{BS}{CP} = \frac{BS}{AP}$ And, $\frac{CP}{AP}\frac{BQ}{QC}\frac{AR}{RB} = \frac{BS}{AP}\frac{AR}{RB} = \frac{BS}{AP}\frac{AP}{BS} = 1$.

2.5.8 By Theorem 2.25 applied to ΔADE and ΔABC we get $\frac{AD}{AB} = \frac{AE}{AC}$. Again, using Theorem 2.25 on ΔAFE and ΔABG we get $\frac{AF}{AB} = \frac{AE}{AG}$. Thus, $\frac{AD}{AF} = \frac{AG}{AC}$.

2.5.1 Mini-Project: Finding Heights

This mini-project is a good example of an activity future high school geometry teachers could incorporate into their courses. It is a very practical application of the notion of similarity. The mathematics in the first example for finding height is extremely easy, but the interesting part is the data collection. Students need to determine how to get the most accurate measurements using the materials they have on hand.

The second method of finding height is again a simple calculation using two similar triangles, but the students may not see this at first. The interesting part of the project is having them see the connection between the mirror reflection and the calculation they made in part I.

Again, have the students work in small groups with a Recorder, but make sure the Recorder position gets shifted around from project to project.

2.6 Circle Geometry

This section introduces students to the basic geometry of the circle. The properties of inscribed angles and tangents are the most important properties to focus on in this section.

2.6.1 Case 2: A is on the diameter through $\overline{OP}$. Let $\alpha = m\angle PBO$ and $\beta = m\angle POB$. Then, $\beta = 180 - 2\alpha$. Also, $m\angle AOB = 180 - \beta = 2\alpha$.

Case 3: A and B are on the same side of $\overleftrightarrow{PO}$. We can assume that $m\angle OPB > m\angle OPA$. Let $m\angle OPB = \alpha$ and $m\angle OPA = \beta$. Then, we can argue in a similar fashion to the proof of the Theorem using $\alpha - \beta$ instead of $\alpha + \beta$.

2.6.2 Let quadrilateral $ABCD$ be inscribed in the circle, with center O. Then, by Theorem 2.31 $a = \angle OAB = \frac{1}{2}\angle EOB$, where E is the point of intersection of $\overrightarrow{AO}$ with the circle. Also, $b = \angle OAD = \frac{1}{2}\angle EOD$.

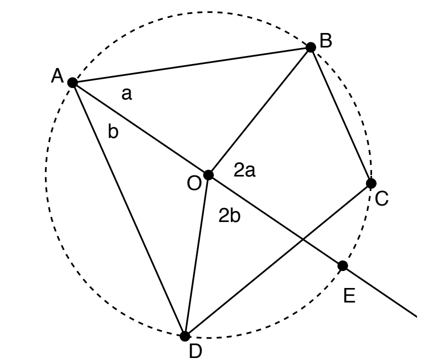

FIGURE 2.10:

Likewise, we would have this relationship for angles generated by $\overrightarrow{CO}$.

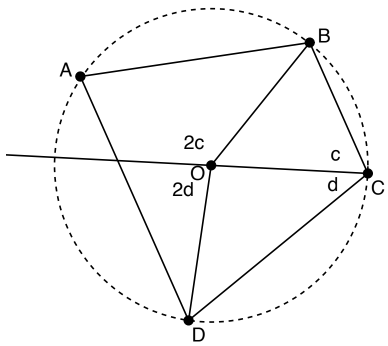

FIGURE 2.11:

Clearly, the sum of the angles at O $(2a + 2b + 2c + 2d)$ is 360, and so the sum of the angles at A and C is 180.

2.6.3 Consider $\angle AQP$. This must be a right angle by Corollary 2.33. Similarly, $\angle BQP$ must be a right angle. Thus, A, Q, and B are collinear.

2.6.4 By Theorem 2.31 we know that $m\angle AOP = 2m\angle ABP$ and $m\angle POC = 2m\angle PBC$. But, $\overrightarrow{BP}$ bisects $\angle ABC$ and so $\angle AOP \cong \angle POC$. Let Q be the point of intersection of $\overline{OP}$ and $\overline{AC}$. Then, $\Delta OQA \cong \Delta OQC$ by SAS. The result follows.

2.6.5 Let $\overline{AB}$ be the chord, O the center, and M the midpoint of $\overline{AB}$. Then $\Delta AOM \cong \Delta BOM$ by SSS and the result follows.

2.6.6 $\angle BAD \cong \angle BCD$ by Corollary 2.32. Likewise, $\angle CBA \cong \angle CDA$. Thus, ΔABP and ΔCDP are similar. The result follows immediately.

2.6.7 Consider a triangle on the diagonal of the rectangle. This has a right angle, and thus we can construct the circle on this angle. Since the other triangle in the rectangle also has a right angle on the same side (the diameter of the circle) then it is also inscribed in the same circle.

2.6.8 First, $m\angle BDA + m\angle CAD = 180 - m\angle DPA = m\angle CPD$. Then, by Theorem 2.31, we have $m\angle BDA + m\angle CAD = \frac{1}{2}(m\angle BOA + m\angle COD)$. Since $m\angle CPD = m\angle BPA$ (Vertical angles), the result follows.

2.6.9 If point P is inside the circle c, then Theorem 2.41 applies. But, this theorem says that $m\angle BPA = \frac{1}{2}(m\angle BOA + m\angle COD)$, where C and D are the other points of intersections of $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ with the circle. If P is inside c, then C and D are different points. The assumption of Theorem 2.42 says that $m\angle BPA = \frac{1}{2}m\angle BOA$. But, $m\angle BPA = \frac{1}{2}(m\angle BOA + m\angle COD)$ would then imply that $m\angle COD = 0$, which is impossible as C and D are not collinear with O.

2.6.10 By Theorem 2.36 $m\angle PTA = 90 - m\angle ATO$. Since triangle OAT is isosceles ($\overline{OA}$ and $\overline{OT}$ are radii of c), then $\angle ATO \cong \angle OAT$. Since $m\angle TOA = 180 - (m\angle ATO + m\angle OAT = 180 - 2m\angle ATO$, then $m\angle ATO = \frac{1}{2}(180 - m\angle TOA = 90 - \frac{1}{2}m\angle TOA)$.

So, $m\angle PTA = 90 - m\angle ATO = 90 - (90 - \frac{1}{2}m\angle TOA) = \frac{1}{2}m\angle TOA$.

2.6.11 The angle made by $\overline{BT}$ and l must be a right angle by Theorem 2.36. Likewise, the angle made by $\overline{AT}$ and l is a right angle. Thus, A, T, and B are collinear.

2.6.12 Suppose they intersected at another point P. Then, ΔTBP and ΔTAP are both isosceles triangles. But, this would imply, by the previous exercise, that there is a triangle with two angles greater than a right angle, which is impossible.

2.6.13 Suppose one of the circles had points A and B on opposite sides of the tangent line l. Then $\overline{AB}$ would intersect l at some point P which is interior to the circle. But, then l would pass through an interior point of the circle and by continuity must intersect the circle in two points which is impossible. Thus, either all points of one circle are on opposite sides of l from the other circle or are on the same side.

2.6.14 Let P and Q be points on the tangent, as shown. Then, $\angle BDT \cong \angle BTP$, as both are inscribed angles on the same arc. Likewise, $\angle ACT \cong \angle ATQ$. Since, $\angle BTP \cong \angle ATQ$ (vertical angles), then $\angle BDT \cong \angle ACT$ and the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are parallel.

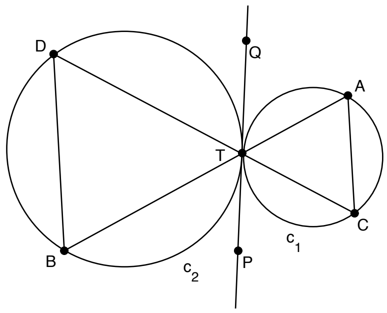

FIGURE 2.12:

2.6.15 By Theorem 2.36, we have that $\angle OAP$ is a right angle, as is $\angle OBP$. Since the hypotenuse ($\overline{OP}$) and leg ($\overline{OA}$) of right triangle ΔOAP are congruent to the hypotenuse ($\overline{OP}$) and leg ($\overline{OB}$) of right triangle ΔOBP, then by Exercise 2.2.10 the two triangles are congruent. Thus $\angle OPA \cong \angle OPB$.

2.6.16 Suppose that the bisector did not pass through the center. Then, construct a segment from the center to the outside point. By the previous theorem, the line continued from this segment must bisect the angle made by the tangents. But, the bisector is unique, and thus the original bisector must pass through the center.

2.6.17 Let A and B be the centers of the two circles. Construct the two perpendiculars at A and B to $\overleftrightarrow{AB}$ and let C and D be the intersections with the circles on one side of $\overleftrightarrow{AB}$.

If $\overleftrightarrow{CD}$ does not intersect $\overleftrightarrow{AB}$, then these lines are parallel, and the angles made by $\overleftrightarrow{CD}$ and the radii of the circles will be right angles. Thus, this line will be a common tangent.

Otherwise, $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{AB}$ at some point P. Let $\overleftrightarrow{PE}$ be a tangent to the circle with center A. Then, since ΔPAC and ΔPBD are similar, we have $\frac{AP}{BP} = \frac{AC}{BD}$. Let $\overleftrightarrow{BF}$ be parallel to $\overleftrightarrow{AE}$ with F the intersection of the parallel with the circle centered at B. Then, $\frac{AC}{BD} = \frac{AE}{BF}$. So, $\frac{AP}{BP} = \frac{AE}{BF}$. By SAS similarity, ΔPAE and ΔPBF are similar, and so F is on $\overleftrightarrow{PE}$ and $\angle PFB$ is a right angle. Thus, $\overleftrightarrow{PE}$ is a tangent to the circle centered at B.

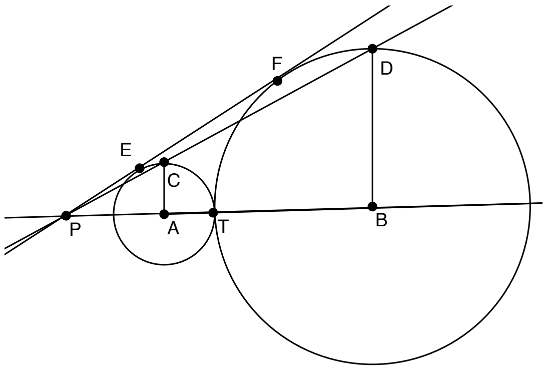

FIGURE 2.13:

2.7 Project 4 - Circle Inversion and Orthogonality

This section is crucial for the later development of the Poincaré model of non-Euclidean (hyperbolic) geometry. It is also has some of the most elegant mathematical results found in the course.

2.7.1 By Theorem 2.31, $\angle Q_2P_1P_2 \cong \angle Q_2Q_1P_2$. Thus, $\angle PP_1Q_2 \cong \angle PQ_1P_2$. Since triangles ΔPP_1Q_2 and ΔPQ_1P_2 share the angle at P, then they are similar. Thus, $\frac{PP_1}{PQ_1} = \frac{PQ_2}{PP_2}$, or $(PP_1)(PP_2) = (PQ_1)(PQ_2)$.

2.7.2 Choose a line from P passing through the center. Then, $PP_1PP_2 = (PO - OP_1)(PO + OP_1) = PO^2 - r^2$.

2.7.3 By similar triangles $\frac{OP}{OT} = \frac{OT}{OP'}$. Since $OT = r$ the result follows.

2.7.4 As P approaches O, the distance OP goes to zero, so the distance OP' must get larger without bound, for the product to remain equal to r^2. Thus, the orthogonal circle radius grows larger without bound as well.