Chapter 2

Connected Graphs and Distance

Section 2.1. Connected Graphs

1. Let $G = K_{2,3}$ and let u and v be the two vertices of G with $\deg u = \deg v = 3$.

2. Consider the path $P_3 = (v, u, w)$ of order 3.

3. (a) The statement is true. **Proof.** Suppose that $A_1 = A_2$. Then A_1 and A_2 are both $n \times n$ matrices for some positive integer n. Hence the orders of G_1 and G_2 are n. Let $u_1, u_2, \ldots, u_n$ be a labeling of the vertices of G_1 that gives the adjacency matrix A_1 and let $v_1, v_2, \ldots, v_n$ be a labeling of the vertices of G_2 that gives the adjacency matrix A_2. Define $f : V(G_1) \to V(G_2)$ by $f(u_i) = v_i$ for $1 \le i \le n$. Since $A_1 = A_2$, two vertices u_i and u_j are adjacent in G_1 if and only if v_i and v_j are adjacent in G_2. Hence $G_1 \cong G_2$. ■

   (b) The statement is false. Let $G_2 = (u_1, u_2, u_3)$ be a path of order 3 and $G_3 = (v_1, v_3, v_2)$ be a path of order 3. Then $G_2 \cong G_3$ but $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and so $A_2 \ne A_3$.

4. (a) The matrix A is the adjacency matrix of the graph K_2 with $V(K_2) = \{v_1, v_2\}$. Since there is one $v_1 - v_1$ walk of length 4, one $v_2 - v_2$ walk of length 4, and no $v_1 - v_2$ walks or $v_2 - v_1$ walks of length 4, the matrix A^4 is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (b) Since $\deg v_1 = \deg v_2 = 1$ and there exist no $v_1 - v_2$ paths or $v_2 - v_1$ paths of length 2, it follows that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since A^2 is the identity matrix $\mathbf{I}$, it follows that $A^4 = A^2 \cdot A^2 = \mathbf{I} \cdot \mathbf{I} = \mathbf{I}$.

5. The adjacency matrix of G_1 is $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. To compute A^2, observe that $\deg v_1 = \deg v_4 = 2$ and $\deg v_2 = \deg v_3 = 3$. For $i \neq j$, the (i,j)-entry of A^2 is the number of different $v_i - v_j$ paths of length 2. Thus $A^2 = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$. To compute A^3, observe that v_1 and v_4 belong to one triangle and v_2 and v_3 belong to two triangles each. For $i \neq j$, the (i,j)-entry of A^3 is the number of different $v_i - v_j$ walks of length 3. Thus $A^3 = \begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 4 & 5 & 5 \\ 5 & 5 & 4 & 5 \\ 2 & 5 & 5 & 2 \end{bmatrix}$.

6. The adjacency matrix of the graph G_2 is $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. The (i,j)-entry of A^k is the number of different $v_i - v_j$ walks of length k in G. Thus

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix},\ A^3 = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \text{ and}$$

$$A^4 = \begin{bmatrix} 2 & 0 & 3 & 0 & 1 \\ 0 & 5 & 0 & 4 & 0 \\ 3 & 0 & 6 & 0 & 3 \\ 0 & 4 & 0 & 5 & 0 \\ 1 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

7. $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_5\}$.

8. Since there is one $v_i - v_i$ walk of length 0 and no $v_i - v_j$ walks of length 0 for $i \neq j$, the (i,j)-entry of A^0 gives the number of $v_i - v_j$ walks of length 0. Therefore, Theorem 2.2 also holds for $k = 0$.

9. Let $P_{k+1} = (u_1, u_2, \ldots, u_{k+1})$ be a path of order $k+1$ and let G be the graph obtained from P_{k+1} by adding k new vertices $v_1, v_2, \ldots, v_k$ and joining v_i to u_i and u_{i+1} for $1 \leq i \leq k$. Then G has the desired property.

   [Note: Also, consider $G = K_1 \wedge kK_2$.]

10. (a) See Figure 2.1.

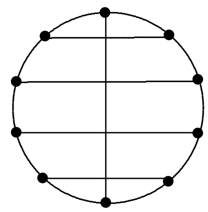

Figure 2.1: The graph in Exercise 10

(b) Let $G = K_{3,3}$. Then $g(G) = 4$ and $c(G) = 6$ but G contains no 5-cycle.

11. (a), (b). Let $P = (u_1, u_2, \ldots, u_\ell)$ be a longest path in G. Thus, u_1 is only adjacent to vertices of P, for otherwise, there is a path longer than P. Let k be the greatest integer, $2 \le k \le \ell$, such that $u_1u_k \in E(G)$. Therefore, $C = (u_1, u_2, \ldots, u_k, u_1)$ is a k-cycle, where $k \ge \delta(G) + 1$.

12. Note that $g(G) = 3$ and $c(G) = 6k$.

13. **Proof.** Let G be a graph. Since (v) is a trivial path for every vertex v of G, the vertex v is connected to itself. Suppose that u is connected to v. Then G contains a $u - v$ path P, say $P = (u = u_1, u_2, \ldots, u_k = v)$. Then $(v = u_k, u_{k-1}, \ldots, u_2, u_1 = u)$ is a $v - u$ path and v is connected to u. Next, suppose that u is connected to v and v is connected to w. Then G contains a $u - v$ path $P = (u = u_1, u_2, \ldots, u_k = v)$ and a $v - w$ path $Q = (v = v_1, v_2, \ldots, u_\ell = w)$. Then

$$(u = u_1, u_2, \ldots, u_k = v = v_1, v_2, \ldots, u_\ell = w)$$

is a $u - w$ walk. By Theorem 2.1, G contains a $u - w$ path. ■

14. **Proof.** Suppose that there is a partition $\{V_1, V_2\}$ of $V(G)$ such that no edge joins a vertex of V_1 and a vertex of V_2. Let $x \in V_1$ and $y \in V_2$. Then x and y are not connected in G and so G is not connected.

For the converse, suppose that for every partition $\{V_1, V_2\}$ of $V(G)$, there exists an edge of G joining a vertex of V_1 and a vertex of V_2. We claim that G is connected, for otherwise, there exist two distinct vertices x and y in G that are not connected. Let U_1 be the set of vertices that are connected to x and let $U_2 = V(G) - U_1$. Then there is no edge joining a vertex of U_1 and a vertex of U_2. ■

15. **Proof.** Assume, to the contrary, that the statement is not true. Then there exists a connected graph G of order n and an integer k with $2 \le k \le n - 1$ such that $\deg u + \deg v \ge k$ for every pair u, v of nonadjacent vertices of G but G contains no path of length k. Let P be a longest path in G. Suppose that P is a $u - v$ path of length ℓ. Then $2 \le \ell < k$. We consider two cases.

Case 1. $uv \in E(G)$. Then G has a cycle C of length $\ell + 1 < n$. Since G is connected, there is a vertex $w \in V(G) - V(C)$ such that w is adjacent to some vertex x on C. Then G has a path of length $\ell + 1$, which is a contradiction.

Case 2. $uv \notin E(G)$. By hypothesis, $\deg u + \deg v \geq k$. Suppose that

$$P = (u = u_0, u_1, \cdots, u_\ell = v).$$

Since P is a longest path in G, every vertex adjacent to u (and to v) belongs to P. If u is adjacent to u_i $(1 \leq i \leq \ell)$, then v is not adjacent to u_{i-1}, for otherwise, G contains a cycle of length $\ell+1$ and we can proceed as in Case 1. Thus $\deg v \leq \ell - \deg u$. However then

$$\deg u + \deg v \leq \ell < k,$$

which is a contradiction. ■

16. **Proof.** Let G be a disconnected graph of order $n \geq 6$ having three components. We show that there exists some vertex $v \in V(G)$ such that $\deg_G v \leq (n-3)/3$.

    Let G_1, G_2 and G_3 be the three components of G with $|V(G_i)| = n_i \geq 1$ for $1 \leq i \leq 3$. Suppose that $n_1 \leq n_2 \leq n_3$. Then $n_1 \leq n/3$. If $v \in V(G_1)$, then

    $$\deg_G v = \deg_{G_1} v \leq n_1 - 1 \leq \frac{n}{3} - 1 = \frac{n-3}{3}.$$

    Since $\delta(G) \leq (n-3)/3$, it follows that $\Delta(\overline{G}) \geq (n-1) - (n-3)/3 = 2n/3$. ■

17. Let $V(G) = \{u = v_1, v_2, \cdots, v_n = v\}$. Since G is connected, it follows that G contains a $v_i - v_{i+1}$ path P_i for $i = 1, 2, \cdots, n-1$. Proceeding along the paths $P_1, P_2, \ldots, P_{n-1}$ in the given order produces a $u - v$ walk containing all vertices of G.

18. A graph G of order n has the property that every induced subgraph of G is connected if and only if $G = K_n$.

    Proof. First, if $G = K_n$, then every induced subgraph of G is complete and therefore connected. For the converse, suppose that G is not complete. Then G contains two nonadjacent vertices u and v. Let $S = \{u, v\}$. Then $G[S] = \overline{K}_2$ is disconnected. ■

19. (a) **Proof.** Let the distinct degrees be a, b and c. Let $\deg x = a$ and $\deg y = b$. Then there exists an $x - y$ path P'. If there is a vertex of degree c on P', then the proof is complete. Otherwise, there are adjacent vertices u and v with $\deg u = a$ and $\deg v = b$. Let $P = (u, v)$ be this subpath of P'. Now, let z be a vertex with $\deg z = c$ and let Q be a shortest path from z to P. This produces a path containing vertices of degrees a, b and c. ■

    (b) No. Consider $K_1 \vee (K_1 + K_2 + K_3)$.

20. **Proof.** If every two vertices of G are of the same parity, then the result follows immediately. If this is not the case, then let $V_1 = \{v \in V(G) : v \text{ is odd}\}$ and $V_2 = \{v \in V(G) : v \text{ is even}\}$. Thus $\{V_1, V_2\}$ is a partition of $V(G)$. Since G is not bipartite, V_1 and V_2 cannot be partite sets for G. So either two vertices of V_1 are adjacent or two vertices of V_2 are adjacent. In either case, G contains two adjacent vertices whose degree sum is even. ■

21. **Proof.** Assume, to the contrary, that G has k (or more) components. Then G has a component G_1 of order at most n/k. Each vertex of G_1 has degree at most $(n/k) - 1 = (n-k)/k$, which is a contradiction. ■

22. **Proof.** Let $M = (k-1)(n-k-1) + \binom{k+1}{2}$. Assume, to the contrary, that there is a graph G of order $n \geq k+1$ and size $m \geq M$, no subgraph of which has minimum degree k. Thus $\delta(F) \leq k-1$ for every subgraph F of G. In particular, $\delta(G) \leq k-1$. Let $v_1 \in V(G)$ such that $\deg_G v_1 \leq k-1$. Consider $G_1 = G - v_1$. The size of G_1 is at least $M-(k-1)$. Since $\delta(G_1) \leq k-1$, there is a vertex $v_2 \in G_1$ such that $\deg_{G_1}(v_2) \leq k-1$. Let $G_2 = G_1 - v_2 = G - \{v_1, v_2\}$. The size of G_2 is at least $M - 2\cdot(k-1)$. In general, let $G_i = G - \{v_1, v_2, \ldots, v_i\}$ such that $\deg_{G_{i-1}}(v_i) \leq k-1$ for $1 \leq i \leq n-k-1$, where $G_0 = G$. Thus the size of of G_i is at least $M - i(k-1)$. In particular, the size of G_{n-k-1} is at least $M - (k-1)(n-k-1) = \binom{k+1}{2}$. Since G_{n-k-1} has order $k+1$, it follows $G_{n-k-1} = K_{k+1}$, which has minimum degree k, producing a contradiction. ■

23. **Proof.** Select some vertex v_1 in G. Since G is connected, there is a vertex v_2 in G that is adjacent to v_1. If $n \geq 3$, then the subgraph induced by $S_2 = \{v_1, v_2\}$ is not a component of G. Thus there is a vertex $v_3 \notin S_2$ that is adjacent to some vertex in S_2. Continuing in this way, we obtain a sequence $v_1, v_2, \ldots, v_n$ where each vertex v_i $(2 \leq i \leq n)$ is adjacent to some vertex in the set $\{v_1, v_2, \ldots, v_{i-1}\}$. ■

24. **Proof.** We show, by induction, for each integer p with $2 \leq p \leq n$ that the subgraph $G[\{v_1, v_2, \ldots, v_p\}]$ is connected. Since $v_1v_2 \in E(G)$, the subgraph $G[\{v_1, v_2\}]$ is connected. Assume for an integer k with $2 \leq k < n$ that $H = G[\{v_1, v_2, \ldots, v_k\}]$ is connected. Suppose that v_{k+1} is adjacent to $v_\ell \in \{v_1, v_2, \ldots, v_k\}$. Since there is a $v_\ell - v_i$ path in H for every i with $1 \leq i \leq k$, there is a $v_{k+1} - v_i$ path in $H' = G[\{v_1, v_2, \ldots, v_{k+1}\}]$ for every i with $1 \leq i \leq k$. Furthermore, since there is a $v_i - v_j$ path in H for every two vertices $v_i, v_j \in V(H)$, there is a $v_i - v_j$ path in H'. Thus, H' is connected. By the Principle of Mathematical Induction, the subgraph $G[\{v_1, v_2, \ldots, v_p\}]$ is connected for every integer p with $2 \leq p \leq n$. In particular, G is connected. ■

Section 2.2. Distance in Graphs

25. **Proof.** Observe that

$$d(u,v) + d(u,w) + d(v,w) = [d(u,v) + d(v,w)] + d(u,w) \geq 2d(u,w)$$

by the triangle inequality. ■

26. **Proof.** If G is a bipartite graph, then G contains no odd cycle by Theorem 1.18 and so G contains no induced odd cycle. For the converse, assume that G contains no induced odd cycle. We show that G does not contain any odd cycle (which implies that G is bipartite by Theorem 1.18). If this is not the case, among all odd cycles of G, let C be one of minimum size. If C is not induced, then there is a chord in C. This, however, produces an odd cycle of smaller size in G, which is impossible. ■

27. Since m is an integer, n is even and so $n = 2k$ for some integer $k \geq 3$. We claim that $G = K_{k-1,k+1}$, which has order n and size $m = \frac{(n-2)(n+2)}{4} = k^2 - 1$.

    If G is a graph of order $n \geq 6$ and size $m = \frac{(n-2)(n+2)}{4}$ containing no odd cycle, then G is a bipartite graph with partite sets U and W where say $|U| \leq |W|$. It is known (by Theorem 1.8) that the size of G is at most k^2. If $|U| = |W| = k$, then $G = K_{k,k} - e$ has size $k^2 - 1$, but then only two vertices of G have degree less than k. Thus, $|U| < |W|$. Then $|U| = k - a$ and $|W| = k + a$ for some positive integer a. If $a \geq 2$, then the size of G is at most $k^2 - a^2 < k^2 - 1$, which does not occur and so $a = 1$. Thus, G is a subgraph of $K_{k-1,k+1}$. Since the size of $K_{k-1,k+1}$ is $k^2 - 1$, it follows that $G = K_{k-1,k+1}$.

28. (a) **Proof.** Assume, to the contrary, that there exists a connected graph G containing two paths P' and P'' of maximum length ℓ that have no vertex in common. Suppose that P' is $u' - v'$ path and P'' is a $u'' - v''$ path. Let

    $$s = \min\{d(w', w'') : w' \in V(P'), w'' \in V(P'')\}.$$

    Thus $s \geq 1$. Let $x \in V(P')$ and $y \in V(P'')$ such that $d(x, y) = s$ and let P be an $x - y$ path of length s. Let Q' be the $u' - x$ subpath or $x - v'$ subpath of P', whichever has length at least $\lceil \ell/2 \rceil$. Similarly, let Q'' be either the $u'' - y$ subpath or the $y - v''$ subpath of P'', whichever has length at least $\lceil \ell/2 \rceil$. Then Q', P, Q'' is a path of length more than ℓ, which is a contradiction. ■

    (b) **Proof.** Suppose that ℓ is odd, say $\ell = 2k + 1$ for some positive integer k. Suppose that P is a $u - v$ path and Q an $x - y$ path such that $V(P) \cap V(Q) = \{w\}$. Then either the $u - w$ subpath or the $w - v$ subpath of P has length at least $k + 1$, say the $w - v$ path P' has length at least $k + 1$. Similarly, we may assume that the $x - w$ subpath Q' of Q has length at least $k + 1$. Then P' and Q' form a path of length at least $2k + 2 > 2k + 1 = \ell$, a contradiction. ■

29. (a) $d_1(v) = \deg v$.

    (b) For $1 \leq k \leq n - 1$, let m_k denote the number of pairs u, v of vertices of G such that $d(u, v) = k$. Then $\sum_{v \in V(G)} d_k(v) = 2m_k$.

    (c) $\sum_{v \in V(G)} \left(\sum_{k=1}^{n-1} d_k(v) \right) = \sum_{v \in V(G)} (n - 1) = n(n - 1) = n^2 - n$.

30. **Proof.** Let u and v be two vertices of $\overline{G}$. We show that u and v are connected and $d_{\overline{G}}(u,v) \leq 2$. If u and v are in different components of G, then uv is an edge in $\overline{G}$ and so $d_{\overline{G}}(u,v) = 1$. Suppose that u and v are in the same component G_1 of G. Let w be a vertex that is in another component G_2 of G. Then (u,w,v) is a path in $\overline{G}$. Thus $d_{\overline{G}}(u,v) \leq 2$. Therefore $\text{diam}(\overline{G}) \leq 2$. ■

31. First, let $a = 1$. For $b = 1$, let $G = P_2$; while for $b = 2$, let $G = P_3$. For $a \geq 2$, let $G = C_{2a}$ if $b = a$; while for $b > a$, let G be obtained from C_{2a} and P_{b-a+1} by identifying an end-vertex of P_{b-a+1} with a vertex of C_{2a}.

32. (a) Assume, to the contrary, that $2,3,3,3$ is the eccentricity sequence of some graph G (necessarily of order 4). Let u be a vertex with eccentricity 3 and suppose that v is a vertex of G such that $e(u) = d(u,v) = 3$. Let $P = (u,w,x,v)$ be a $u-v$ geodesic. Then neither w nor x has eccentricity 3, a contradiction.

    (b) Since a graph with eccentricity sequence a,b,b,b has order 4, it follows that either (i) $a = 1$ and $b = 2$ or (ii) $a = 2$ and $b = 3$. The eccentricity sequence of the graph $K_{1,3}$ is $1,2,2,2$; while $2,3,3,3$ is not the eccentricity sequence of any graph by (a). Thus $a = 1$ and $b = 2$.

33. For $n \geq 4$, let e_1, e_2 and e_3 be the edges of a triangle in K_n. Let $F = K_n - e_1 - e_2$ and $G = K_n - e_1 - e_2 - e_3$. Then $F \not\cong G$ but the eccentricity sequence of both is $1,1,\ldots,1,2,2,2$.

34. (a) K_2. (b) P_3.

35. Let $H = K_{2,k}$ where the partite sets of H are $U = \{u_1,u_2\}$ and $W = \{w_1,w_2,\ldots,w_k\}$ and let G be the graph obtained from H by adding two vertices v_1 and v_2 and the two edges u_1v_1 and u_2v_2. Then $\deg w_i = e(w_i) = 2$ for $i = 1,2,\ldots,k$.

36. **Proof.** Let w be a vertex with $e(w) = k$ and let u be a vertex with $d(w,u) = e(w) = k$. For a central vertex v of G, let P be a $u-v$ path of length $d(u,v)$. Thus $d(u,v) \leq e(v) = \text{rad}(G)$. Since $d(u,w) = k$, it follows that $e(v) < k \leq e(u)$. By Theorem 2.7, there is a vertex x on P such that $e(x) = k$. Because $d(u,x) \leq d(u,v) < k$ and $d(w,u) = k$, it follows that $x \neq w$. ■

37. **Proof.** Let $n = 2 + (r-1)\sum_{i=1}^{s-1}(r-2)^i$ and suppose that G is a connected graph of order n such that $\Delta(G) < r$ and $\text{diam}(G) < s$. Let $x \in V(G)$ such that $e(x) = \text{diam}(G) < s$. Let $A_i(x)$ denote the set of vertices at distance i from x where $0 \leq i \leq s-1$. Then $|A_0(v)| = 1$ and $|A_i(x)| \leq (r-1)(r-2)^{i-1}$ for $1 \leq i \leq s-1$. Since

$$n = |V(G)| = |\bigcup_{i=0}^{s-1} A_i(v)| \leq 1 + (r-1)\sum_{i=1}^{s-1}(r-2)^i,$$

    this is a contradiction. ■

38. No. Suppose that G is a connected graph with $\text{diam}(G) \geq 2$ such that $H = \text{Per}(G)$ is complete. Let u and v be two antipodal vertices of G. Then $1 = d_G(u, v) = \text{diam}(G) \geq 2$, which is a contradiction.

39. For the graph G in Figure 2.2 and the four distinct vertices v_1, v_2, v_3, v_4 in G, v_{i+1} is an eccentric vertex of v_i for $i = 1, 2, 3$. Also, $e(v_1) = 4$, $e(v_2) = 6$ and $e(v_3) = 8$.

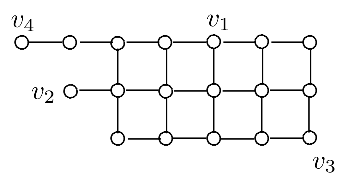

Figure 2.2: The graph in Exercise 39

40. In the graph G of Figure 2.3, each vertex is labeled with its total distance. The median is also shown.

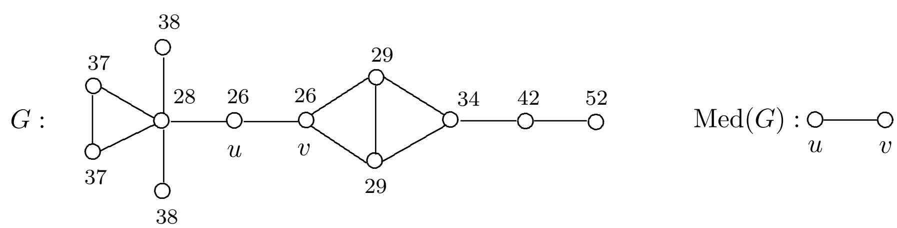

Figure 2.3: The median of a graph

41. Let $2k + 1 \geq 3$ and let G be the graph obtained by identifying the center v of $K_{1,2k+1}$ with the end-vertex v_{2k+1} of the path $P = (v_1, v_2, \ldots, v_{2k+1})$. Then v_{k+1} is the only central vertex of G. Observe that

$$\begin{aligned} td(v) &= \binom{2k+1}{2} + (2k+1) = \binom{2k+2}{2}, \\ td(v_{2k}) &= \binom{2k}{2} + 1 + 2(2k+1) = \binom{2k+2}{2} + 2, \end{aligned}$$

and for an end-vertex x adjacent to v,

$$td(x) = \binom{2k+3}{2} + 2(2k).$$

Since all other vertices of G have a total distance greater than that of v, the vertex v is the only median vertex of G. Thus

$$d(\text{Cen}(G),\ \text{Med}(G)) = k.$$

42. (a) **Proof.** Let u and v be vertices at distance 3 in G. Select any two vertices x and y. We consider three cases. Recall that the closed neighborhood $N[u]$ of u consists of u and all the neighbors of u.

Case 1. $x \in N[u]$ *and* $y \in N[u]$. Then (x, v, y) is a path in $\overline{G}$ and $d(x, y) \leq 2$.

Case 2. $x \in N[u]$ *and* $y \notin N[u]$. Then (x, v, u, y) is a path in $\overline{G}$ and $d(x, y) \leq 3$.

Case 3. $x \notin N[u]$ *and* $y \notin N[u]$. Then (x, u, y) is a path in $\overline{G}$ and $d(x, y) \leq 2$.

Since $d(x, y) \leq 3$ in each case, the diameter is at most 3. ∎

(b) If G has diameter at least 4, then $\overline{G}$ has diameter at most 3; so G cannot be self-complementary. If $\text{diam}(G) = 1$, then G is complete and is not self-complementary. So the diameter must be 2 or 3 for G to be self-complementary.

Figure 2.4(a) shows a self-complementary graph of order $4k$ and diameter 3 and a self-complementary graph of order $4k + 1$ and diameter 2, where the bold lines between two graphs indicate that all possible edges join these two graphs.

(c) If the diameter of G is 2, the diameter of $\overline{G}$ can be any integer k for $2 \leq k \leq n - 1$. For $k = 2$, see Figure 2.4(b). For $3 \leq k \leq n - 1$, let $\overline{G}$ be the graph obtained by identifying a vertex of K_{n-k+1} with an end-vertex of the path P_k. Then $\text{diam}(G) = 2$ and $\text{diam}(\overline{G}) = k$.

43. **Proof.** Certainly, $d(G, H) = 0$ if and only if $G \cong H$. Since each 2-switch from F to F' results in an inverse 2-switch from F' to F, the distance d is symmetric. Let F, G and H be three graphs in $\mathcal{G}_s$ such that $d(F, G) = a$ and $d(G, H) = b$. Then there are a 2-switches that transforms F into G and b 2-switches that transforms G into H. Hence there is a sequence of $a + b$ switches that transforms F into H. Therefore,

$$d(F, H) \leq a + b = d(F, G) + d(G, H)$$

and so the triangle inequality holds. ∎

44. **Proof.** Let G be a connected graph. Since (1) $D(u, v) \geq 0$, (2) $D(u, v) = 0$ if and only if $u = v$ and (3) $D(u, v) = D(v, u)$ for every pair u, v of vertices of G, it remains only to show that detour distance satisfies the triangle inequality.

Let u, v and w be any three vertices of G. Since the inequality $D(u, w) \leq D(u, v) + D(v, w)$ holds if any two of these three vertices are the same vertex, we assume that u, v and w are distinct. Let P be a $u - w$ detour in G of length $k = D(u, w)$. We consider two cases.

Case 1. *v lies on P*. Let P_1 be the $u - v$ subpath of P and let P_2 be the $v - w$ subpath of P. Suppose that the length of P_1 is s and the length of P_2 is t. So

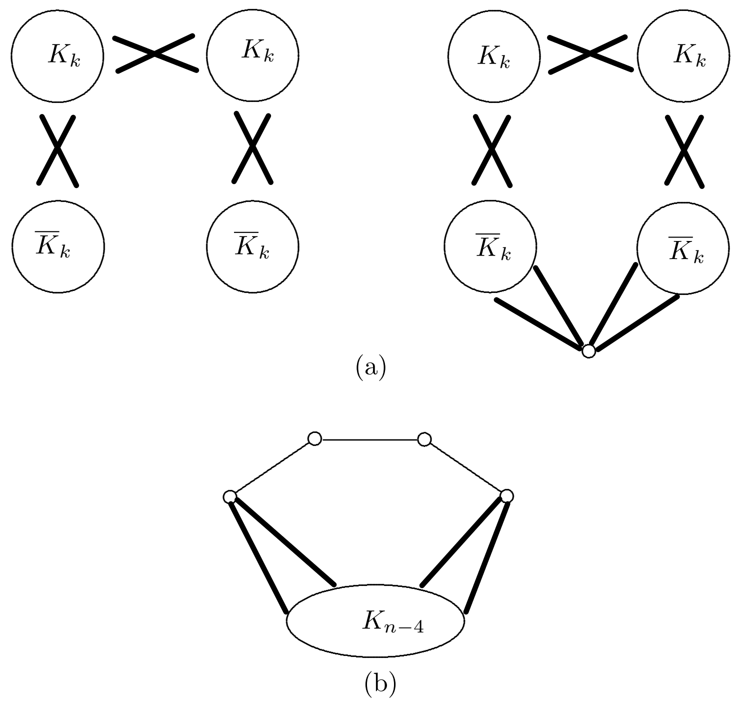

Figure 2.4: Graphs in Exercise 42

$s + t = k$. Therefore,

$$D(u, w) = k = s + t \leq D(u, v) + D(v, w).$$

Case 2. *v does not lie on P.* Since G is connected, there is a shortest path Q from v to a vertex of P. Suppose that Q is a $v - x$ path. Thus x lies on P but no other vertex of Q lies on P. Let r be the length of Q. Thus $r > 0$. Let the $u - x$ subpath P' of P have length a and the $x - w$ subpath P'' of P have length b. Then $a \geq 0$ and $b \geq 0$. Therefore, $D(u, v) \geq a + r$ and $D(v, w) \geq b + r$. So

$$D(u, w) = k = a + b < (a + r) + (b + r) \leq D(u, v) + D(v, w),$$

and so the triangle inequality holds. ■