Chapter 2

Portfolio Optimization Solutions

Exercise 2.1
Consider a Portfolio Optimization problem with data $n = 3$, $\mu = (1.1, 1.15, 1.2)'$, and $\Sigma = \mathrm{diag}(10^{-4}, 10^{-3}, 10^{-2})$. Find h_0, h_1, $x(t)$, α_0, α_1, β_0, β_1 and β_2, (as in Example 2.1) and sketch the efficient frontier.

Solution:
First of all, we have

$$\Sigma^{-1}l = \begin{pmatrix} 10^4 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10^4 \\ 10^3 \\ 10^2 \end{pmatrix}$$

and $l'\Sigma^{-1}l = 10^4 + 10^3 + 10^2 = 11100$. Thus,

$$h_0 = \frac{\Sigma^{-1}l}{l'\Sigma^{-1}l} = \frac{1}{111} \begin{pmatrix} 100 \\ 10 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.9 \\ 0.09 \\ 0.009 \end{pmatrix}.$$

Next, we have

$$\Sigma^{-1}\mu = \begin{pmatrix} 10^4 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^2 \end{pmatrix} \begin{pmatrix} 1.1 \\ 1.15 \\ 1.2 \end{pmatrix} = \begin{pmatrix} 11000 \\ 1150 \\ 120 \end{pmatrix}$$

and $l'\Sigma^{-1}\mu = 11000 + 1150 + 120 = 12270$. Then

$$\begin{aligned} h_1 &= \Sigma^{-1}\mu - l'\Sigma^{-1}\mu\left[\frac{\Sigma^{-1}l}{l'\Sigma^{-1}l}\right] \\ &= \begin{pmatrix} 11000 \\ 1150 \\ 120 \end{pmatrix} - \frac{12270}{111}\begin{pmatrix} 100 \\ 10 \\ 1 \end{pmatrix} \\ &= \frac{1}{37}\begin{pmatrix} -2000 \\ 1650 \\ 350 \end{pmatrix} \approx \begin{pmatrix} -54.054 \\ 44.594 \\ 9.459 \end{pmatrix}. \end{aligned}$$

Therefore, the efficient portfolios are given by

$$x(t) = h_0 + th_1 = \frac{1}{111}\begin{pmatrix} 100 \\ 10 \\ 1 \end{pmatrix} + \frac{t}{37}\begin{pmatrix} -2000 \\ 1650 \\ 350 \end{pmatrix}$$

and have expected return $\mu_p = \alpha_0 + t\alpha_1$ and variance $\sigma_p^2 = \beta_0 + t\beta_1 + t^2\beta_2$, where

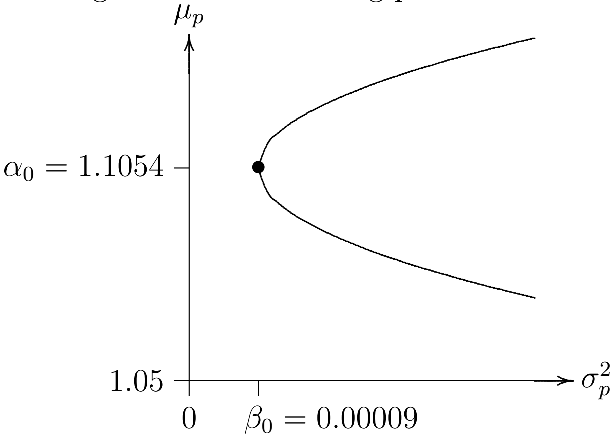
$$\begin{aligned} \alpha_0 &= \mu' h_0 = \frac{409}{370} \approx 1.1054, \\ \alpha_1 &= \mu' h_1 = \frac{235}{74} \approx 3.1756, \\ \beta_0 &= h_0'\Sigma h_0 = \frac{1}{11100} \approx 0.00009, \\ \beta_1 &= h_0'\Sigma h_1 = \frac{1}{11100}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}' h_1 = 0, \\ \beta_2 &= h_1'\Sigma h_1 \\ &= \frac{1}{37}\begin{pmatrix} -1/5 \\ 33/20 \\ 7/2 \end{pmatrix}' h_1 = \frac{1}{1369}(400 + 2722.5 + 1225) = \frac{235}{74} \approx 3.1756. \end{aligned}$$

So we have $\alpha_1 = \beta_2$ and $\beta_1 = 0$ as expected.

The equation of the efficient frontier is

$$\sigma_p^2 - 0.00009 = (\mu_p - 1.1054)^2/3.1757,$$

which gives us the following picture.



$\lozenge$

Exercise 2.2
Consider a Portfolio Optimization problem with data $n = 2$, $\mu = (1.1, 1.2)'$ and $\Sigma = \text{diag}(10^{-2}, 10^{-1})$. Augment this problem with a risk free asset having expected return $r = 1.05$. Determine the efficient portfolios and the two pieces of the efficient frontier.

Solution:
First we calculate the efficient risky assets as

$$x(t) = t\Sigma^{-1}(\mu - rl) = t\begin{pmatrix} 100 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 0.05 \\ 0.15 \end{pmatrix} = t\begin{pmatrix} 5 \\ 1.5 \end{pmatrix}$$

and the efficient risk free asset as

$$x_{n+1}(t) = 1 - tl'\Sigma^{-1}(\mu - rl) = 1 - 6.5t.$$

Letting $t_m = 1/(l'\Sigma^{-1}(\mu - rl)) = 1/6.5$, we get the *market portfolio* (the efficient portfolio which has zero holdings in the risk free asset), which we calculate as

$$x_m = t_m\Sigma^{-1}(\mu - rl) = \begin{pmatrix} 10/13 \\ 3/13 \end{pmatrix}.$$

The market portfolio has expected return

$$\mu_p = \mu' x_m = \frac{73}{65} \approx 1.1231$$

and variance

$$\sigma_p^2 = x_m' \Sigma x_m = \frac{19}{1690} \approx 0.0112,$$

which gives us

$$\sigma_p = \sqrt{\frac{19}{1690}} \approx 0.1060.$$

For $t > t_m$, the efficient portfolios are given by $x(t) = h_0 + th_1$, where

$$\begin{aligned} h_0 &= \frac{\Sigma^{-1} l}{l' \Sigma^{-1} l} = \begin{pmatrix} 10/11 \\ 1/11 \end{pmatrix} \\ h_1 &= \Sigma^{-1}\mu - l'\Sigma^{-1}\mu \left[\frac{\Sigma^{-1} l}{l'\Sigma^{-1} l} \right] = \begin{pmatrix} -10/11 \\ 10/11 \end{pmatrix}. \end{aligned}$$

Therefore, the efficient portfolios are given by

$$x(t) = \begin{cases} t \begin{pmatrix} 5 \\ 1.5 \end{pmatrix}, & \text{for } 0 \le t \le t_m, \\ \begin{pmatrix} 10/11 \\ 1/11 \end{pmatrix} + t \begin{pmatrix} -10/11 \\ 10/11 \end{pmatrix}, & \text{for } t > t_m. \end{cases}$$

The efficient frontier is composed of the linear piece

$$\mu_p - r = \sigma_p [(\mu - rl)' \Sigma^{-1} (\mu - rl)]^{\frac{1}{2}}$$

and the hyperbolic piece

$$\sigma_p^2 - (\mu_p - \alpha_0)^2 / \alpha_1 = \beta_0.$$

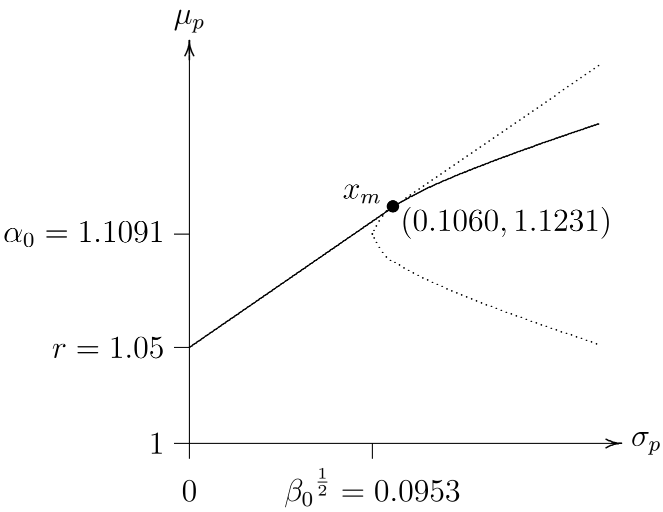
In our case, we have $[(\mu - rl)'\Sigma^{-1}(\mu - rl)]^{\frac{1}{2}} = \sqrt{\frac{19}{40}} \approx 0.6892$, $\alpha_0 = \frac{61}{55} \approx 1.1091$, $\alpha_1 = \frac{1}{11}$, and $\beta_0 = \frac{1}{110} \approx 0.0091$. Therefore, the two pieces of the efficient frontier are

$$\mu_p - 1.05 = (0.6892)\sigma_p$$

and

$$\sigma_p^2 - 11\,(\mu_p - 1.1091)^2 = 0.0091,$$

which gives us the following picture.



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Exercise 2.3
Show $\beta_2 = 0$ if and only if μ is a multiple of l.

Solution:
Note that $\beta_2 = h_1' \Sigma h_1$, where $h_1 = \Sigma^{-1}\mu - \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right)\Sigma^{-1}l$.

($\Rightarrow$) Suppose $\beta_2 = h_1'\Sigma h_1 = 0$. Since Σ is positive definite, we must have $h_1 = 0$. Therefore, $\Sigma^{-1}\mu = \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right)\Sigma^{-1}l$, which implies that

$$\mu = \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right) l,$$

so we see that μ is a multiple of l.

($\Leftarrow$) Suppose $\mu = cl$. Then

$$\begin{aligned} h_1 &= c\Sigma^{-1}l - c\left(\frac{l'\Sigma^{-1}l}{l'\Sigma^{-1}l}\right)\Sigma^{-1}l \\ &= c\Sigma^{-1}l - c\Sigma^{-1}l \\ &= 0, \end{aligned}$$

which implies that $\beta_2 = h_1'\Sigma h_1 = 0$.

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Exercise 2.4
Verify (2.17) ($\alpha_1 = \beta_2$).

Solution:
Note that $\beta_2 = h_1'\Sigma h_1$ and $\alpha_1 = \mu' h_1$, where $h_1 = \Sigma^{-1}\mu - \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right)\Sigma^{-1}l$.
Then,

$$
\begin{aligned}
\beta_2 &= h_1'\Sigma h_1 \\
&= h_1'\left(\mu - \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right) l\right) \\
&= h_1'\mu - \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right) h_1' l \\
&= \alpha_1 - \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right) l' h_1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\alpha_1 - \beta_2 &= \frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\left(l'\Sigma^{-1}\mu - \left(\frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\right) l'\Sigma^{-1}l\right) \\
&= \frac{l'\Sigma^{-1}\mu}{l'\Sigma^{-1}l}\left(l'\Sigma^{-1}\mu - l'\Sigma^{-1}\mu\right) \\
&= 0,
\end{aligned}
$$

so $\beta_2 = \alpha_1$.

$\Diamond$

Exercise 2.5
No solution given.

$\Diamond$

Exercise 2.6

Suppose $\Sigma = \text{diag}(\sigma_i)$. Show the efficient portfolios have components

$$x_i = [\theta_1 + t(\mu_i - \theta_2)] / \sigma_i, \quad i = 1, \ldots, n,$$

where

$$\theta_1 = 1/(\sigma_1^{-1} + \cdots + \sigma_n^{-1}) \text{ and } \theta_2 = \theta_1(\mu_1/\sigma_1 + \cdots + \mu_n/\sigma_n).$$

Also, show the multiplier for the budget constraint is $u = -\theta_1 + t\theta_2$.

Solution:
First note that $\Sigma^{-1} = \text{diag}(\sigma_i^{-1})$. Then

$$\theta_1 = \frac{1}{l'\Sigma^{-1}l} \text{ and } \theta_2 = \frac{\mu'\Sigma^{-1}l}{l'\Sigma^{-1}l},$$

so

$$\begin{aligned} x_i &= (h_0 + th_1)_i \\ &= \theta_1(\Sigma^{-1}l)_i + t(\Sigma^{-1}\mu - \theta_2\Sigma^{-1}l)_i \\ &= \theta_1/\sigma_i + t(\mu_i/\sigma_i - \theta_2/\sigma_i) \\ &= [\theta_1 + t(\mu_i - \theta_2)] / \sigma_i. \end{aligned}$$

Moreover,

$$u = \frac{-1}{l'\Sigma^{-1}l} + t\frac{\mu'\Sigma^{-1}l}{l'\Sigma^{-1}l} = -\theta_1 + t\theta_2.$$

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