2

The Real Numbers

CONTENTS

Chapter 2
Section 2.1: The Real and Extended Real Number System.

2.1.1. (a) If $\frac{1}{0} = x$, then by multiplying by x, we have $1 = x \cdot 0 = 0$. (b) If $\frac{0}{0} = x$, then again, $0 = x \cdot 0 = y \cdot 0$, so $\frac{0}{0} = y$ as well.

2.1.2. Since $x \cdot \frac{1}{x} = 1 \neq 0$, and $x \neq 0$, it must be the case that $\frac{1}{x} \neq 0$. If $x > 0$ but $\frac{1}{x} < 0$, then $1 = x \cdot \frac{1}{x} < 0$, in contradiction to Proposition 2.6, part (4).

2.1.3. Suppose without loss of generality that $x < y$. Then

$$x = \frac{x+x}{2} < \frac{x+y}{2} < \frac{y+y}{2} = y.$$

2.1.4. (a) In the proof of part (3) of Proposition 2.8, replace $<$ and $>$ with $\leq$ and $\geq$. (b) In the proof of part (4) of Proposition 2.8, replace $<$ and $>$ with $\leq$ and $\geq$.

2.1.6. If $x, y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$. If $x, y < 0$, then again $xy > 0$, and so $|xy| = xy$. Since $x, y < 0$, $|x| = -x$, and $|y| = -y$, so $|x||y| = (-x)(-y) = xy$, and again the result holds. Suppose exactly one of x or y is negative, and the other is nonnegative; without loss of generality, suppose $x < 0$ and $y \geq 0$. Then $xy \leq 0$, so $|xy| = -xy$ (or 0), while $|x| = -x$ and $|y| = y$ (or 0), so that $|x||y| = (-x)y = -xy$ (or 0).

2.1.7. If $x \geq 0$, then $|x| = x$ and the result is true. If $x < 0$ then $|x| = -x$ and $x < 0 < -x$, and so $x < |x|$ in that case.

2.1.8. We must have a and b to have the same sign.

2.1.9. (a) Following the hint, replacing $b = c - a$ we have

$$|a+b| \leq |a|+|b| \Leftrightarrow |a+(c-a)| = |c| \leq |a|+|c-a|, \text{ so } |c|-|a| \leq |c-a|.$$

(b) Following the hint, replacing $\tilde{b} = a - c$ we have

$$|c+\tilde{b}| \leq |c|+|\tilde{b}| \Leftrightarrow |c+(a-c)| = |a| \leq |c|+|a-c|, \text{ so } |a|-|c| \leq |a-c| = |c-a|.$$

(c) This follows directly from the previous parts and Assertion (3) of Proposition 2.8.

2.1.10. Writing $0 = 1 - 1$, we would have $\infty \cdot 0 = \infty(1-1) = \infty - \infty$, the latter cannot be defined.

2.1.11. If $x = \frac{\infty}{\infty}$, then

$$x = \frac{\infty}{\infty} = \frac{\infty + \infty}{\infty} = \frac{\infty}{\infty} + \frac{\infty}{\infty} = x + x = 2x.$$

So $x = 0$. But if $x = 0 = \frac{\infty}{\infty}$, it must be the case that $0 \cdot \infty = \infty$, and the former cannot be defined.

2.1.12. Let $x, y \in \mathbb{R}^\sharp$, we construct a neighborhood U of x and a neighborhood V of y so that $U \cap V = \emptyset$. If x and y are both real, then the proof of Theorem 2.14. Suppose then that one of x or y is infinite, suppose without loss of generality that it is x. If y is finite, then choose our neighborhood of y to have radius 1, so that $V = (y-1, y+1)$. If $x = \infty$, then choose $U = (y+2, \infty)$. If $x = -\infty$, choose $U = (-\infty, y-2)$. If x and y are both infinite, say $x = -\infty$ and $y = \infty$, then choose $U = (-\infty, 0$, and $V = (0, \infty)$.

Section 2.2: The Supremum and Infimum.

2.2.1. (a) $\sup(S) = 4, \inf(S) = -7$. (b) $\sup(S) = \infty, \inf(S) = 0$. (c) $\sup(S) = \infty, \inf(S) = -\infty$. (d) $\sup(S) = \infty, \inf(S) = -\infty$.

2.2.2. Suppose S is bounded, and that M is an upper bound for S, and m is a lower bound for S. Let $N = \max\{|m|, |M|\}$. Then $N \geq |M|$, and $N \geq |m|$, so that $-N \leq M \leq N$ and $-N \leq m \leq N$. So, for every $s \in S$, we have

$$-N \leq m \leq s \leq M \leq N, \text{ so } 0 \leq |s| \leq N.$$

Conversely, suppose N is an upper bound for $|S|$. Then, for every $|s| \in |S|$, we have $|s| \leq N$. But this means that $-N \leq s \leq N$, and so N and $-N$ are upper and lower bounds for S, respectively.

2.2.3. b is an upper bound for S since $S \subseteq [a, b]$ and b is an upper bound for $[a, b]$. Thus $\sup(S) \leq b$. Since S is nonempty, there exists an $x \in S$, so since $S \subseteq [a, b]$, $a \leq x \leq b$. So, $\sup(S) \geq x \geq a$. So, $\sup(S) \in [a, b]$.

2.2.4. (a) Let $\alpha = \sup(S)$ for convenience. Notice that for every $s \in S$, $s \leq \alpha$, and so since $a > 0$, $as \leq a\alpha$. So, $a\alpha$ is an upper bound for aS. Now suppose $b < a\alpha$. Then $\frac{b}{a} < \alpha$, and so $\frac{b}{a}$ is not an upper bound for S. Therefore there exists an $s \in S$ with $\frac{b}{a} < s$, so $b < as$. So, b is not an upper bound for aS. (b) Let $\beta = \inf(S)$ for convenience. Notice for every $s \in S$, we have $s \geq \beta$. Since $a > 0$, we have $as \geq a\beta$. So, $a\beta$ is a lower bound for S. Now suppose $b > a\beta$.

Then $\frac{b}{a} > \beta$, and so $\frac{b}{a}$ is not a lower bound for S. Therefore there exists an $s \in S$ so that $\frac{b}{a} > s$, so $b > as$. So, b is not a lower bound for S.

2.2.5. Let $\alpha = \sup(S)$, and let $s \in S$. Then $s \leq \alpha$ and so since $a < 0$ we have $as \geq a\alpha$. So, $a\alpha$ is a lower bound for aS. Now let $b > a\alpha$. Then $\frac{b}{a} < \alpha$, and so there exists an $s \in S$ with $\frac{b}{a} < s$, so that $b > as$, so that b is not a lower bound for aS. To prove the next assertion, let $a = -1$.

2.2.6. Suppose, without loss of generality, that $\alpha = \sup(A) = \max\{\sup(A), \sup(B)\}$. Let $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \leq \alpha$. If $x \in B$, then $x \leq \sup(B) \leq \alpha$. So α is an upper bound for $A \cup B$. Now suppose $b < \alpha$. Since $\alpha = \sup(A)$, and $b < \alpha$, there exists an element $a \in A$ with $b < a$. Therefore, b is not an upper bound for $A \cup B$ since $a \in A \subseteq A \cup B$, and $b < a$.

2.2.7. Suppose, without loss of generality, that $\beta = \inf(A) = \min\{\inf(A), \inf(B)\}$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \geq \inf(A) = \beta$. If $x \in B$, then $x \geq \inf(B) \geq \inf(A) = \beta$. So, β is a lower bound for $A \cup B$. Now let $b > \beta$. Since $\beta = \inf(A)$, there exists an $a \in A$ so that $b > a$. So, b is not a lower bound for $A \cup B$, since $a \in A \subseteq A \cup B$, and $b > a$.

2.2.8. (a) Since A and B are each bounded, there exists upper bounds M_A and M_B for A and B, respectively. In addition, there exists lower bounds m_A and m_B for A and B, respectively. For each $a \in A$ and $b \in B$, we have

$$m_A + m_B \leq a + b \leq M_A + M_B.$$

Therefore, $A + B$ is bounded. (b) Let $\alpha = \sup(A)$ and $\beta = \sup(B)$, both real numbers. Let $a \in A$, and $b \in B$. Then $a + b \leq \alpha + \beta$, so $\alpha + \beta$ is an upper bound for $A + B$. Since $\alpha + \beta$ is an upper bound for $A + B$, $\alpha + \beta$ is either the $\sup(A + B)$, or, $\sup(A + B) < \alpha + \beta$. If the latter were true, there would exist $a + b \in A + B$ with $\alpha + \beta < a + b$. But for this to be true, one of a or b must be greater than α or β, respectively. Either possibility is a contradiction, so $\alpha + \beta = \sup(A + B)$.

2.2.9. Let $A = \{1, -1\}$, and $B = \{-2, 1\}$. Then $\sup(A) = 1, \sup(B) = 1$, but $AB = \{-2, -1, 1, 2\}$, so $\sup(AB) = 2 \neq 1 \cdot 1 = 1$.

2.2.10. (a) Let $\epsilon > 0$. Notice that $\sup(S) - \epsilon$ is not an upper bound for S since it is strictly less than $\sup(S)$, the smallest upper bound. So, there exists an $s \in S$ with $\sup(S) - \epsilon < s$. That $s \leq \sup(S)$ follows from the fact that $\sup(S)$ is an upper bound for S. (b) Let $\epsilon > 0$. Notice that $\inf(S) + \epsilon$ is not a lower bound for S since it is strictly greater than $\inf(S)$, the largest upper bound. So, there exists an

$s \in S$ with $s < \inf(S) + \epsilon$. That $\inf(S) \leq s$ follows from the fact that $\inf(S)$ is a lower bound for S.

2.2.11. We determine whether or not $\inf(U)$ is greater or less than $\sup(S)$. Suppose $\inf(U) < \sup(S)$. Then $\inf(U)$ is not an upper bound for S, and so there exists an $s \in S$ with $\inf(U) < s$. Consider the number $\frac{1}{2}(\inf(U)+s)$: this is larger than $\inf(U)$, and so there exists a $u \in U$ with $u < \frac{1}{2}(\inf(U)+s) < s$. This contradicts the fact that every element in U is an upper bound for s. The argument that $\inf(U)$ is not greater than $\sup(S)$ is similar.

2.2.12. Let $x \in A$. Then $f(x) + g(x) \leq \sup(f(A)) + \sup(g(A))$. Therefore, $\sup(f(A)) + \sup(g(A))$ is an upper bound for $(f+g)(A)$, and the result follows, since $\sup((f+g)(A))$ is the smallest upper bound of $(f+g)(A)$. An example that illustrates inequality is $A = [-1, 1]$, $f(x) = x$, and $g(x) = -x$. We have $(f+g)(x) = 0$ for all x, but $\sup(f(A)) = \sup(g(A)) = 1$.

2.2.13. We use Mathematical Induction on the cardinality of S. Suppose $S = \{x_1\}$ contains one element, then $\sup(S) = x_1 \in S$. Now suppose $S = \{x_1, \ldots, x_n, x_{n+1}\}$, and any finite set containing n or fewer elements has a supremum within the set. Consider $S - \{x_{n+1}\}$. This set has n elements, and therefore $\sup(S - \{x_{n+1}\}) = x_i \in S$ for some $i = 1, \ldots, n$. Now $S = (S - \{x_{n+1}\}) \cup \{x_{n+1}\}$, and so by exercise 6, $\sup(S)$ is either x_i or x_{n+1}, whichever is larger.

Section 2.3: The Completeness Axiom.

2.3.1. (a) The inf is 0. This is clearly a lower bound, and if $0 < \epsilon$, the Archimedean Property asserts the existence of $N \in \mathbb{N}$ for which $\frac{1}{N} < \epsilon$, so ϵ is not a lower bound for this set. (b) The sup is 1. This is clearly an upper bound, and if $b < 1$, then $0 < 1 - b$, and so $0 < \sqrt{1-b}$, so again by the Archimedean Property, there exists an $N \in \mathbb{N}$ for which $\frac{1}{N} < \sqrt{1-b}$, so that $b < 1 - \frac{1}{N^2}$. (c) The sup is 1. This is clearly an upper bound, and if $b < 1$ there exists a rational number between them, since $\mathbb{Q}$ is dense in $\mathbb{R}$.

2.3.2. The inf exists in $\mathbb{R}^\sharp$. Suppose that $\inf(S) = \gamma$, and $\inf(S) = \delta$, with $\gamma < \delta$. Both are greatest lower bounds. Since γ is such, and $\delta > \gamma$, δ must not be a lower bound, which is a contradiction to it being an infimum of this set.

2.3.3. Suppose that $\mathbb{N}$ were bounded above by M. Then consider applying the Archimedean Property to $\epsilon = \frac{1}{M}$. There exists an $N \in \mathbb{N}$ for which $\frac{1}{N} < \frac{1}{M}$, or, $M < N$, contradicting M being an upper bound for $\mathbb{N}$.

2.3.4. Let $\epsilon > 0$ be given. Since S is not bounded above, there exists an $s \in S$ for which $\frac{1}{\epsilon} < s$. Or rather, $\frac{1}{s} < \epsilon$.

2.3.5. Let $s \in \mathbb{S}$ be the largest element in S, and consider $x = s + 1$ and $y = s + 2$. Every element in s is strictly smaller than x (which is the smaller of x and y), and so there is no element of S between x and y. Thus, S is not dense in $\mathbb{R}$.

2.3.6. An example is $\mathbb{N}$, which is countably dense, but there is no natural number between $\frac{1}{4}$ and $\frac{1}{2}$.

2.3.7. Let α be given. For any $n \in \mathbb{N}$, $\alpha - \frac{1}{n} < \alpha$, so by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists an $r_n \in \mathbb{Q}$ so that $\alpha - \frac{1}{n} < r_n < \alpha$. Since $\frac{1}{n}$ can be arbitrarily close to 0 by the Archimedean Property, $\alpha - \frac{1}{n}$ can be made arbitrarily close to α, and so r_n can be found arbitrarily close to α.

2.3.8. Let $a < b$ be given, and suppose there were only finitely many rational numbers between a and b, and that r is the largest of these. Then $r < b$ and so there exists a rational number q so that $r < q < b$. This q is also between a and b and contradicts the assumption that r was the largest of the rational numbers between a and b.

2.3.9. We first claim that if q is rational, then $\sqrt{2}q$ is not rational. If $\sqrt{2}q = r \in \mathbb{Q}$, then $\sqrt{2} = \frac{r}{q}$. But $\frac{r}{q}$ is rational since both r and q are, but $\sqrt{2}$ is not rational. Now let $a < b$ be given. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a $q \in \mathbb{Q}$ for which $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$. So, $\sqrt{2}q$ is an irrational number, and $a < \sqrt{2}q < b$.