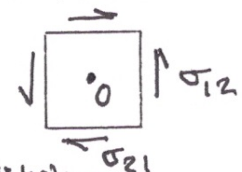
# Chapter 2

2.1

For two dimensional plane stress in the 12 plane, the shear stresses act as shown on the element below

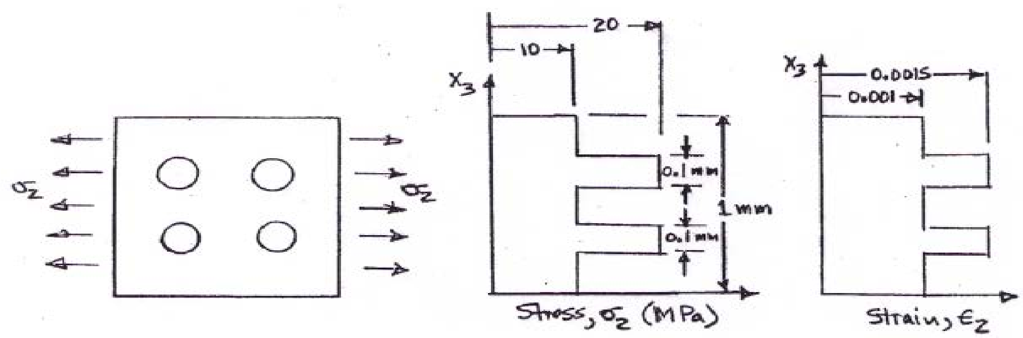



For static equilibrium:

$\sum M_0 = 0$ at the center of the element implies that

$\sigma_{12} = \sigma_{21}$. Similarly, in general $\sigma_{ij} = \sigma_{ji}$ when $i \neq j$

2.2




The average transverse stress is

$$\bar{\sigma}_2 = \frac{\int_A \sigma_2 \, dA}{\int_A dA} = \frac{20\left[4\left(\frac{\pi}{4}\right)(0.1)^2\right] + 10\left[(1)^2 - 4\left(\frac{\pi}{4}\right)(0.1)^2\right]}{(1)^2}$$

$$= 10.314 \text{ MPa}$$

The average transverse strain is

$$\bar{\epsilon}_2 = \frac{\int_A \epsilon_2 \, dA}{\int_A dA} = \frac{(0.0015)4\left(\frac{\pi}{4}\right)(0.1)^2 + 0.001 - 4\left(\frac{\pi}{4}\right)(0.1)^2}{(1)^2}$$

$$= 0.001$$

The effective transverse modulus is

$$E_2 = \frac{\bar{\sigma}_2}{\bar{\epsilon}_2} = \frac{10.314}{0.001} = 10314 \text{ MPa} = 10.314 \text{ GPa}$$

2.3

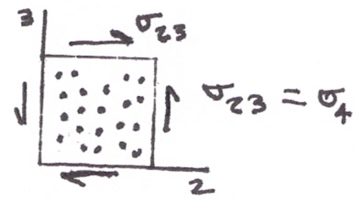
Due to the interchangability of subscripts 2 and 3 for isotropy in the 2-3 plane,

$C_{22} = C_{33}$ and $C_{12} = C_{13}$

$\sigma_{13} = \sigma_{31} = \sigma_5 = \sigma_{12} = \sigma_{21} = \sigma_6$

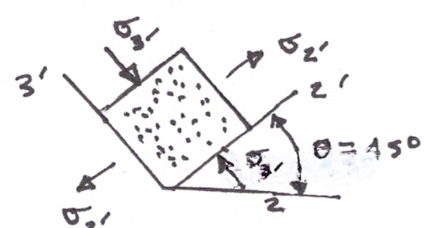
$\gamma_{13} = \gamma_{31} = \epsilon_5 = \gamma_{13} = \gamma_{31} = \epsilon_6$

It can be shown that $C_{44}$ is not independent of $C_{22}$ and $C_{23}$ by considering an element subjected to a pure shear stress $\sigma_{23} = \sigma_4$ as shown below:




From Eqs 2.4 and 2.16, $\sigma_4 = C_{44}\,\epsilon_4$

From mechanics of materials, the corresponding stresses on an element oriented at $\theta = 45°$ to the 2-3 axes are obtained from stress transformations as




$\sigma_2' = 2\sigma_{23}\cos 45° \sin 45° = \sigma_{23} = \sigma_4$

$\sigma_3' = -2\sigma_{23}\cos 45° \sin 45° = -\sigma_{23} = -\sigma_4$

Similarly, the strain transformations are

$\epsilon_2' = 2\,\frac{\gamma_{23}}{2}\cos 45° \sin 45° = \frac{\gamma_{23}}{2} = \epsilon_{23}$

$\epsilon_3' = -2\,\frac{\gamma_{23}}{2}\cos 45° \sin 45° = -\frac{\gamma_{23}}{2} = -\epsilon_{23}$

Using Eqs 2.4 and 2.17 for the 2'3' axes and noting that $C_{22}' = C_{22}$ and $C_{23}' = C_{23}$ due to transverse isotropy,

$\sigma_2' = C_{22}\epsilon_2' + C_{23}\epsilon_3' = \frac{\gamma_{23}}{2}(C_{22} - C_{23})$

But recall that $\sigma_2' = \sigma_{23} = \frac{\gamma_{23}}{2}(C_{22} - C_{23}) = C_{44}\epsilon_4 = C_{44}\gamma_{23}$

There fore,

$$C_{44} = \frac{1}{2}(C_{22} - C_{23})$$

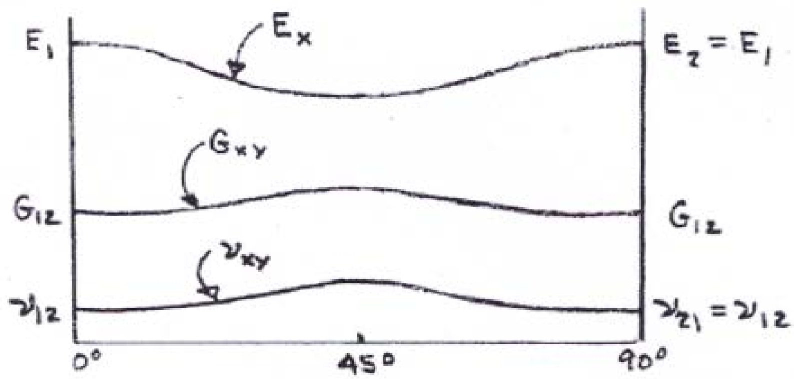
2.4

(a) For the balanced orthotropic, or square symmetric lamina shown in Fig. 2.9 we must have $E_1 = E_2$ and $\nu_{12} = \nu_{21}$, so that the lamina stress-strain relationships will be of the form

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

therefore, this material only has <u>three</u> independent elastic constants.

(b)




2.5

From Eqs. (2.24), we have

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

inverting the compliance matrix, we have

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

where the stiffnesses $Q_{ij}$ are given by

$$[Q] = [S]^{-1} = \frac{[a]^T}{|S|}$$

the cofactor matrix $[a]$ consists of elements

$$a_{ij} = (-)^{i+j} M_{ij}$$

continued

2.5 continued

where the minor element $M_{ij}$ is found by deleting row i and column j from [S] and taking the determinant of the remaining terms. the determinant |S| is given by

$$|S| = \sum_{i=1}^{n} S_{ij} a_{ij} = (S_{11} S_{22} - S_{12}^2) S_{66}$$

the cofactor matrix is

$$[a] = \begin{bmatrix} S_{22} S_{66} & -S_{12} S_{66} & 0 \\ -S_{12} S_{66} & S_{11} S_{66} & 0 \\ 0 & 0 & S_{11} S_{22} - S_{12}^2 \end{bmatrix} = [a]^T$$

Finally, the stiffness matrix is

$$[Q] = [S]^{-1} = \frac{[a]^T}{|S|} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

where $Q_{11} = \dfrac{S_{22}}{S_{11} S_{22} - S_{12}^2}$ ; $Q_{12} = -\dfrac{S_{12}}{S_{11} S_{22} - S_{12}^2}$

$Q_{22} = \dfrac{S_{11}}{S_{11} S_{22} - S_{12}^2}$ ; $Q_{66} = \dfrac{1}{S_{66}}$

By substituting Eqs. (2.25) in the above expressions, we can show that

$$Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}} \; ; \; Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12} \nu_{21}}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}} \; ; \; Q_{66} = G_{12}$$

2.6

From Table 2.2, for AS/3501 carbon/epoxy,

$E_1 = 138\,GPa$, $E_2 = 9\,GPa$, $G_{12} = 6.9\,GPa$, $\nu_{12} = 0.3$

$$\nu_{21} = \nu_{12}\frac{E_2}{E_1} = (0.3)\frac{9}{138} = 0.0196$$

From Eqs. (2.27):

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = \frac{138}{1-0.3(0.0196)} = 138.8\,GPa$$

$$Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{(0.3)(9)}{1-0.3(0.0196)} = 2.716\,GPa = Q_{21}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} = \frac{9}{1-0.3(0.0196)} = 9.05\,GPa$$

$$Q_{66} = G_{12} = 6.9\,GPa$$

so that

$$[Q] = \begin{bmatrix} 138.8 & 2.716 & 0 \\ 2.716 & 9.05 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} GPa$$

From Eqs. (2.25):

$$S_{11} = \frac{1}{E_1} = \frac{1}{138} = 0.00725\,(GPa)^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.3}{138} = -0.00217\,(GPa)^{-1} = S_{21}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{9} = 0.111\,(GPa)^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{6.9} = 0.145\,(GPa)^{-1}$$

so that

$$[S] = \begin{bmatrix} 0.00725 & -0.00217 & 0 \\ -0.00217 & 0.111 & 0 \\ 0 & 0 & 0.145 \end{bmatrix} (GPa)^{-1}$$

and $[S] = [Q]^{-1}$

2.7

Substituting the given conditions $\sigma_2 = \sigma_3 = \sigma$, $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_3 = \epsilon$ into the 3-D stress-strain relationships for a specially orthotropic, transversely isotropic material (i.e., Eqs 2.22 with $E_2 = E_3$, $\nu_{21} = \nu_{31}$, etc.), we find that

$$\epsilon_1 = 0 = \frac{\sigma_1}{E_1} - \frac{\nu_{12}\sigma_2}{E_1} - \frac{\nu_{12}\sigma_3}{E_1} = \frac{\sigma_1}{E_1} - \frac{2\nu_{12}\sigma}{E_1}$$

or $\sigma_1 = 2\nu_{12}\sigma$

Furthermore,

$$\epsilon_2 = \epsilon = -\frac{\nu_{12}}{E_1}\sigma_1 + \frac{\sigma_2}{E_2} - \frac{\nu_{23}\sigma_3}{E_2}$$

So that

$$\epsilon_2 = \epsilon = -\frac{\nu_{12}}{E_1}(2\nu_{12}\sigma) + \frac{\sigma}{E_2} - \frac{\nu_{23}\sigma}{E_2}$$

or

$$\sigma = 2K_{23}\epsilon$$

where $$2K_{23} = \frac{1}{-\frac{2\nu_{12}^2}{E_1} + \frac{1}{E_2} - \frac{\nu_{23}}{E_2}}$$

Rearranging, we get

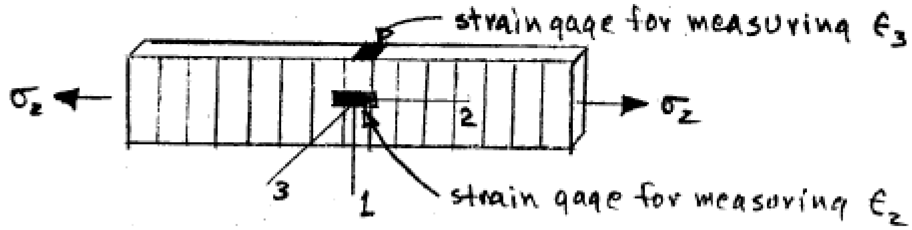
$$\frac{1}{K_{23}} = -\frac{4\nu_{12}^2}{E_1} + \frac{2}{E_2} - \frac{2\nu_{23}}{E_2}$$

$$\frac{1}{K_{23}} = -\frac{4\nu_{12}^2}{E_1} + \frac{4}{E_2} - \frac{2}{E_2}(1+\nu_{23})$$

$$\frac{1}{K_{23}} = -\frac{4\nu_{12}^2}{E_1} + \frac{4}{E_2} + \frac{1}{G_{23}}\text{; since } G_{23} = \frac{E_2}{2(1+\nu_{23})}$$

2.8

Conduct a uniaxial tensile test along the transverse direction by applying a stress condition $\sigma_2 \neq 0$, $\sigma_1 = \sigma_3 = \tau_{12} = \tau_{13} = \tau_{23} = 0$.




From strain gage measurements of strains $\epsilon_2$ and $\epsilon_3$, we can find

$$E_2 = \frac{\sigma_2}{\epsilon_2} \quad \text{and} \quad \nu_{23} = -\frac{\epsilon_3}{\epsilon_2} = \nu_{32}$$

then, for the transversely isotropic material,

$$G_{23} = \frac{E_2}{2(1+\nu_{32})}$$

2.9

the off-axis modulus of elasticity is

$$E_x = \frac{\sigma_x}{\epsilon_x} = \frac{\sigma_x}{\bar{S}_{11}\sigma_x} = \frac{1}{\bar{S}_{11}} \quad \text{where } \sigma_y = \tau_{xy} = 0$$

where $\bar{S}_{11} = S_{11}\cos^4\theta + S_{22}\sin^4\theta + (2S_{12} + S_{66})\sin^2\theta\cos^2\theta$

$$= \frac{1}{E_1}\cos^4\theta + \frac{1}{E_2}\sin^4\theta + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta$$

Thus,

$$E_x = \frac{1}{\frac{1}{E_1}\cos^4\theta + \frac{1}{E_2}\sin^4\theta + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta}$$

2.10

The off-axis shear modulus is

$$G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} = \frac{\tau_{xy}}{\bar{S}_{66}\tau_{xy}} = \frac{1}{\bar{S}_{66}} \quad \text{where } \sigma_x = \sigma_y = 0$$

and where
$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\sin^2\theta\cos^2\theta + S_{66}(\sin^4\theta + \cos^4\theta)$$
$$= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta + \frac{1}{G_{12}}(\sin^4\theta + \cos^4\theta)$$

thus,

$$G_{xy} = \frac{1}{2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta + \frac{1}{G_{12}}(\sin^4\theta + \cos^4\theta)}$$

2.11

Rearranging the result from Problem 2.10, we have

$$\frac{1}{G_{xy}} = A\sin^2\theta\cos^2\theta + B(\sin^4\theta + \cos^4\theta)$$

where $A = 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)$

and $B = \frac{1}{G_{12}}$

(a) For a possible maximum, minimum or inflection point, we must have

$$\frac{d}{d\theta}\left(\frac{1}{G_{xy}}\right) = (2A - 4B)(\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta) = 0$$

The three solutions for the angle $\theta$ are therefore

1. $\sin\theta = 0$, or $\theta = 0^\circ$
2. $\cos\theta = 0$, or $\theta = 90^\circ$
3. $\cos^2\theta - \sin^2\theta = 0$, or $\theta = 45^\circ$

(b) In order to find the bounds on $G_{12}$ for a minimum or maximum of $G_{xy}$, we consider the sign of the derivative $\frac{d^2}{d\theta^2}\left(\frac{1}{G_{xy}}\right)$ at $\theta = 45^\circ$.

continued

$$\frac{d^2}{d\theta^2}\left(\frac{1}{G_{xy}}\right) = \frac{d}{d\theta}\left[(2A-4B)(\sin\theta\cos\theta)(\cos^2\theta-\sin^2\theta)\right]$$

$$= (2A-4B)(\cos^4\theta+\sin^4\theta-6\sin^2\theta\cos^2\theta)$$

and

$$\left.\frac{d^2}{d\theta^2}\left(\frac{1}{G_{xy}}\right)\right|_{@\theta=45^\circ} = (2A-4B)(-1) = 4B-2A$$

For $\left.\frac{d^2}{d\theta^2}\left(\frac{1}{G_{xy}}\right)\right|_{@\theta=45^\circ} > 0$, $4B-2A > 0$, or

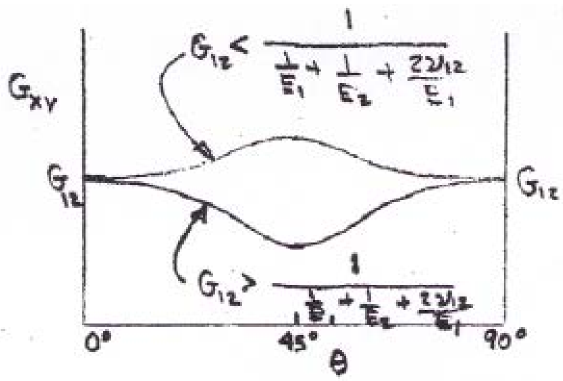
$$G_{12} < \frac{1}{\frac{1}{E_1}+\frac{1}{E_2}+\frac{2\nu_{12}}{E_1}} \quad \text{for a minimum}$$

of $\frac{1}{G_{xy}}$ (or a <u>maximum</u> of $G_{xy}$).

For $\left.\frac{d^2}{d\theta^2}\left(\frac{1}{G_{xy}}\right)\right|_{@\theta=45^\circ} < 0$, $4B-2A < 0$, or

$$G_{12} > \frac{1}{\frac{1}{E_1}+\frac{1}{E_2}+\frac{2\nu_{12}}{E_1}} \quad \text{for a } \underline{\text{minimum}} \text{ of } G_{xy}.$$

(c)




(d) From Table 2.2, for E-glass/epoxy (Scotchply 1002),
$E_1 = 38.6$ GPa, $E_2 = 8.27$ GPa, $G_{12} = 4.14$ GPa, $\nu_{12} = 0.26$

continued

2.11 continued

therefore

$$\frac{1}{\frac{1}{E_1}+\frac{1}{E_2}+\frac{2\nu_{12}}{E_1}} = \frac{1}{\frac{1}{38.6}+\frac{1}{8.27}+\frac{2(0.26)}{38.6}} = 6.24\ GPa$$

and since $6.24\ GPa > (G_{12} = 4.14\ GPa)$, the value of $G_{xy}$ at $\theta = 45^\circ$ must be a maximum for this material, as shown by the upper curve in part (c).

2.12

From longitudinal tension test with $\sigma_1 \neq 0$, $\sigma_2 = \tau_{12} = 0$, we can measure longitudinal strains, $\epsilon_1$, and calculate the longitudinal modulus $E_1 = \sigma_1 / \epsilon_1$. From measured transverse strains, $\epsilon_2$, we can find the major Poisson's ratio $\nu_{12} = -\epsilon_2/\epsilon_1$. Similarly, from transverse tension test with $\sigma_2 \neq 0$, $\sigma_1 = \tau_{12} = 0$, we can find the transverse modulus $E_2 = \sigma_2 / \epsilon_2$. Finally, from an off-axis tension test at a known angle, $\theta$, we can determine the off-axis modulus $E_x = \sigma_x / \epsilon_x$. Then from Eq. (2.40), we can solve for $G_{12}$, since $E_x, E_1, E_2, \nu_{12}$ and $\theta$ are known.

2.13

First, we transform the stresses to the 1,2 system

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta\\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta\\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

$$= \begin{bmatrix}0.75 & 0.25 & 0.866\\ 0.25 & 0.75 & -0.866\\ -0.433 & 0.433 & 0.5\end{bmatrix}\begin{Bmatrix}100\\ -50\\ 50\end{Bmatrix} = \begin{Bmatrix}105.8\\ -55.8\\ -39.95\end{Bmatrix} MPa$$

the strains in the 1,2 system are then

continued

2.13 continued

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

2.14

Eqs. (2.37) are as follows:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

the $\bar{S}_{ij}$ can be expressed in terms of off-axis engineering constants by considering the strains that are generated by simple uniaxial tension and shear experiments.

For example, a uniaxial tension test with $\sigma_x \neq 0$, $\sigma_y = \tau_{xy} = 0$ will generate the strains

$$\epsilon_x = \frac{\sigma_x}{E_x} = \bar{S}_{11}\, \sigma_x$$

$$\epsilon_y = -\nu_{xy}\, \epsilon_x = -\nu_{xy} \frac{\sigma_x}{E_x} = \bar{S}_{12}\, \sigma_x$$

$$\gamma_{xy} = \eta_{x,xy}\, \epsilon_x = \eta_{x,xy} \frac{\sigma_x}{E_x} = \bar{S}_{16}\, \sigma_x$$

continued

2.14 continued

Similarly, a uniaxial tension test with $\sigma_y \neq 0$, $\sigma_x = \tau_{xy} = 0$ will generate the strains

$$\epsilon_y = \frac{\sigma_y}{E_y} = \bar{S}_{22}\,\sigma_y$$

$$\epsilon_x = -\nu_{yx}\,\epsilon_y = -\nu_{xy}\frac{\sigma_y}{E_y} = \bar{S}_{22}\,\sigma_y$$

$$\gamma_{xy} = \eta_{y,xy}\,\epsilon_y = \eta_{y,xy}\frac{\sigma_y}{E_y} = \bar{S}_{26}\,\sigma_y$$

Finally, a pure shear test with $\tau_{xy} \neq 0$, $\sigma_x = \sigma_y = 0$ will generate the strains

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} = \bar{S}_{16}\,\tau_{xy}$$

$$\epsilon_x = \eta_{xy,x}\,\gamma_{xy} = \eta_{xy,x}\frac{\tau_{xy}}{G_{xy}} = \bar{S}_{16}\,\tau_{xy}$$

$$\epsilon_y = \eta_{xy,y}\,\gamma_{xy} = \eta_{xy,y}\frac{\tau_{xy}}{G_{xy}} = \bar{S}_{26}\,\tau_{xy}$$

Using superposition of the strains for a general state of stress $\sigma_x$, $\sigma_y$, $\tau_{xy}$, we find that

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

or, due to the symmetry of the $\bar{S}_{ij}$ matrix,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ -\frac{\nu_{yx}}{E_y} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{xy,x}}{G_{xy}} & \frac{\eta_{xy,y}}{G_{xy}} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

2.15

from Eqs.(2.36):

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta$$

substituting in the following trigonometric identities:

$$\cos^4\theta = \frac{1}{8}(3+4\cos 2\theta + \cos 4\theta)$$

$$\sin^4\theta = \frac{1}{8}(3-4\cos 2\theta + \cos 4\theta)$$

$$\cos^2\theta\sin^2\theta = \frac{1}{8}(1-\cos 4\theta)$$

we find that

$$\begin{aligned}\bar{Q}_{11} &= \frac{1}{8}(3+4\cos 2\theta + \cos 4\theta)Q_{11} \\ &\quad + \frac{1}{8}(3-4\cos 2\theta + \cos 4\theta)Q_{12} \\ &\quad + \frac{1}{4}(1-\cos 4\theta)(Q_{12}+2Q_{66}) \\ &= U_1 + U_2\cos 2\theta + U_3\cos 4\theta\end{aligned}$$

where

$$U_1 = \frac{1}{8}(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66})$$

$$U_2 = \frac{1}{2}(Q_{11}-Q_{22})$$

$$U_3 = \frac{1}{8}(Q_{11}+Q_{22}-2Q_{12}-4Q_{66})$$

Similarly,

$$\begin{aligned}\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta \\ &\quad + Q_{12}(\cos^4\theta+\sin^4\theta) \\ &= U_1 - U_3\cos 4\theta\end{aligned}$$

Where

$$U_4 = \frac{1}{8}(Q_{11}+Q_{22}+6Q_{12}-4Q_{66})$$

2.16

From Eqs. (2.45):

$$U_1 = \frac{1}{8}[3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}]$$
$$= \frac{1}{8}[(3)(138.8) + (3)(9.05) + (2)(2.716) + (4)(6.9)] = 59.57 \text{ GPa}$$
$$U_2 = \frac{1}{2}[Q_{11} - Q_{22}] = \frac{1}{2}[138.8 - 9.05] = 64.87 \text{ GPa}$$
$$U_3 = \frac{1}{8}[Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}]$$
$$= \frac{1}{8}[138.8 + 9.05 - (2)(2.716) - (4)(6.9)] = 14.35 \text{ GPa}$$
$$U_4 = \frac{1}{8}[Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}]$$
$$= \frac{1}{8}[138.8 + 9.05 + (6)(2.716) - (4)(6.9)] = 17.07 \text{ GPa}$$

From Eqs. (2.47):

$$V_1 = \frac{1}{8}[3S_{11} + 3S_{22} + 2S_{12} + S_{66}]$$
$$= \frac{1}{8}[(3)(0.00725) + (3)(0.111) + (2)(-0.00217) + 0.145]$$
$$= 0.0619 \text{ (GPa)}^{-1}$$
$$V_2 = \frac{1}{2}[S_{11} - S_{12}] = \frac{1}{2}[0.00725 - (-0.00217)] = 0.0047 \text{ (GPa)}^{-1}$$
$$V_3 = \frac{1}{8}[S_{11} + S_{22} - 2S_{12} - S_{66}]$$
$$= \frac{1}{8}[0.00725 + 0.111 - (2)(-0.00217) - 0.145]$$
$$= -0.0028 \text{ (GPa)}^{-1}$$
$$V_4 = \frac{1}{8}[S_{11} + S_{22} + 6S_{12} - S_{66}]$$
$$= \frac{1}{8}[0.00725 + 0.111 + (6)(-0.00217) - 0.145]$$
$$= -0.00497 \text{ (GPa)}^{-1}$$

2.17

From Eqs. (2.44), for $\theta = +45^\circ$, we have

$$\bar{Q}_{11} = U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta$$

$$= 59.57 + 64.87 \cos 90^\circ + 14.35 \cos 180^\circ = 45.22 \text{ GPa}$$

$$\bar{Q}_{12} = U_4 - U_3 \cos 4\theta = 17.07 - 14.35 \cos 180^\circ = 31.42 \text{ GPa}$$

$$\bar{Q}_{22} = U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta$$

$$= 59.57 - 64.87 \cos 90^\circ + 14.35 \cos 120^\circ = 45.22 \text{ GPa}$$

$$\bar{Q}_{16} = \frac{U_2}{2} \sin 2\theta + U_3 \sin 4\theta$$

$$= \frac{64.87}{2} \sin 90^\circ + 14.35 \sin 180^\circ = 32.44 \text{ GPa}$$

$$\bar{Q}_{26} = \frac{U_2}{2} \sin 2\theta - U_3 \sin 4\theta$$

$$= \frac{64.87}{2} \sin 90^\circ - 14.35 \sin 180^\circ = 32.44 \text{ GPa}$$

$$\bar{Q}_{66} = \frac{1}{2}(U_1 - U_4) - U_3 \cos 4\theta$$

$$= \frac{1}{2}(59.57 - 17.07) - 14.35 \cos 180^\circ = 35.6 \text{ GPa}$$

the same results can be obtained by using Eqs. (2.36).

For $\theta = -45^\circ$, using either Eqs. (2.44) or Eqs. (2.36), we find that $\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{22}$ and $\bar{Q}_{66}$ are the same as for $\theta = +45^\circ$. the values of $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are just the negatives of the corresponding values at $\theta = +45^\circ$, however. Thus, for $\theta = -45^\circ$, we get

$$\bar{Q}_{11} = 45.22 \text{ GPa}$$

$$\bar{Q}_{12} = 31.42 \text{ GPa}$$

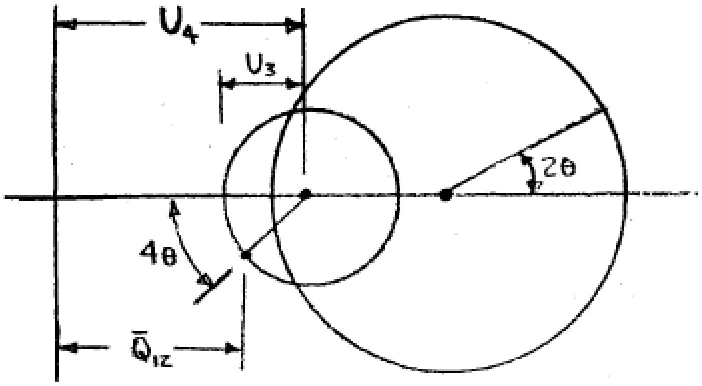
$$\bar{Q}_{22} = 45.22 \text{ GPa}$$

$$\bar{Q}_{16} = -32.44 \text{ GPa}$$

$$\bar{Q}_{26} = -32.44 \text{ GPa}$$

$$\bar{Q}_{66} = 35.6 \text{ GPa}$$

2.18




2.19 $\bar{Q}_{12} = U_4 - U_3 \cos 4\theta$

The averaged stiffnesses are:

$$\tilde{Q}_{11} = \frac{\int_0^\pi \bar{Q}_{11}\, d\theta}{\int_0^\pi d\theta} = \frac{\int_0^\pi (U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta)\, d\theta}{\pi} = U_1$$

Similarly,

$$\tilde{Q}_{22} = \frac{\int_0^\pi \bar{Q}_{22}\, d\theta}{\pi} = \frac{\int_0^\pi (U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta)\, d\theta}{\pi} = U_1$$

$$\tilde{Q}_{12} = \frac{\int_0^\pi \bar{Q}_{12}\, d\theta}{\pi} = \frac{\int_0^\pi (U_4 - U_3 \cos 4\theta)\, d\theta}{\pi} = U_4 = \tilde{Q}_{21}$$

$$\tilde{Q}_{66} = \frac{\int_0^\pi \bar{Q}_{66}\, d\theta}{\pi} = \frac{\int_0^\pi \left[\frac{1}{2}(U_1 - U_4) - U_3 \cos 4\theta\right] d\theta}{\pi} = \frac{U_1 - U_4}{2}$$

$$\tilde{Q}_{16} = \tilde{Q}_{26} = 0$$

The stress-strain relationships are therefore

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & 0 \\ \tilde{Q}_{12} & \tilde{Q}_{22} & 0 \\ 0 & 0 & \tilde{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & \frac{U_1 - U_4}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

continued

2.19 continued

Since the material is planar isotropic, we have

$$\tilde{Q}_{11} = U_1 = \frac{\tilde{E}}{1-\tilde{\nu}^2} = \tilde{Q}_{22}$$

$$\tilde{Q}_{12} = U_4 = \frac{\tilde{\nu}\tilde{E}}{1-\tilde{\nu}^2}$$

$$\tilde{Q}_{66} = \frac{U_1 - U_4}{2} = \tilde{G} = \frac{\tilde{E}}{2(1+\tilde{\nu})}$$

where $\tilde{E}$ = averaged modulus of elasticity

$\tilde{G}$ = averaged shear modulus

$\tilde{\nu}$ = averaged Poisson's ratio

Solving these equations simultaneously, we get

$$\tilde{E} = \frac{(U_1 - U_4)(U_1 + U_4)}{U_1}$$

$$\tilde{G} = \frac{U_1 - U_4}{2}$$

$$\tilde{\nu} = \frac{U_4}{U_1}$$

2.20

$$E_x = \frac{\sigma_x}{\epsilon_x} = \frac{100\text{ MPa}}{0.00647} = 15{,}456\text{ MPa} = 15.456\text{ GPa}$$

$$\nu_{xy} = -\frac{\epsilon_y}{\epsilon_x} = -\frac{(-0.00324)}{0.00647} = 0.5$$

$$\eta_{x,xy} = \frac{\gamma_{xy}}{\epsilon_x}$$

Where the shear strain $\gamma_{xy}$ is found from the measured strains $\epsilon_x$, $\epsilon_y$ and $\epsilon_1$ and the strain transformation equation (Eq. 2.33) evaluated at $\theta = 45°$.

$$\epsilon_1 = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \left(\frac{\gamma_{xy}}{2}\right)(2\cos\theta\sin\theta)$$

or

$$\gamma_{xy} = \frac{\epsilon_1 - \epsilon_x \cos^2\theta - \epsilon_y \sin^2\theta}{\cos\theta \sin\theta}$$

$$\gamma_{xy} = \frac{0.00809 - 0.00647(0.5) - (-0.00324)(0.5)}{0.5}$$

$$\gamma_{xy} = 0.01295$$

so that

$$\eta_{x,xy} = \frac{\gamma_{xy}}{\epsilon_x} = \frac{0.01295}{0.00647} = 2.0$$

2.21

From the first of Eqs (2.40):

$$\frac{1}{E_x} = \frac{c^4}{E_1} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right) s^2 c^2 + \frac{1}{E_2} s^4$$

From Table 2.2 for AS/3501:

$$E_1 = 138\,GPa, \quad E_2 = 9\,GPa, \quad \nu_{12} = 0.3, \quad G_{12} = 6$$

$$\therefore \frac{1}{E_x} = \frac{(0.707)^4}{138} + \left(\frac{1}{6.9} - \frac{2(0.3)}{138}\right)(0.707)^4 + \frac{(0.707)}{9}$$

$$E_x = 15.447\ GPa$$

So that

$$E_x = \frac{\sigma_x}{\epsilon_x} = \frac{15.44\ MPa}{\epsilon_x} = 15.447\ GPa$$

$$\therefore \epsilon_x = \frac{\sigma_x}{E_x} = \frac{15.44\ MPa}{15.447\ GPa} = 0.001$$

2.22 (a)

$$\tau_{xy} = \sigma_1 cs - \sigma_2 cs + \tau_{12}(c^2 - s^2)$$
$$= 1000(0.707)^2 - (-1000)(0.707)^2 + 0$$
$$= 1000\ MPa$$

(b)

$$\epsilon_1 = \frac{\sigma_1}{E_1} - \frac{\nu_{12}}{E_1}\sigma_2$$
$$= \frac{1.0}{131} - 0.22\left(\frac{-1.0}{131}\right)$$
$$= 0.0093$$

$$\epsilon_2 = -\frac{\nu_{12}}{E_1} + \frac{\sigma_2}{E_2}$$
$$= \frac{-0.22}{131} + \frac{(-1)}{10.3}$$
$$= -0.0988$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}} = 0$$

2.23

In terms of invariants, $\bar{Q}_{66} = \frac{U_1 - U_4}{2} - U_3 \cos 4\theta$

For a maximum of $\bar{Q}_{66}$, we must have

$$\frac{d\bar{Q}_{66}}{d\theta} = 4U_3 \sin 4\theta = 0 \text{ and } \frac{d^2\bar{Q}_{66}}{d\theta^2} = 16 U_3 \cos 4\theta <$$

the possible solutions are $\theta = 0, \frac{\pi}{4}$ and $\frac{\pi}{2}$

at $\theta = 0^\circ$, $\frac{d^2\bar{Q}_{66}}{d\theta^2} = 16U_3 \cos(0) = 16U_3 > 0$ (since $U_3$:

at $\theta = \frac{\pi}{4}$, $\frac{d^2 Q_{66}}{d\theta^2} = 16U_3 \cos(\pi) = -16U_3 < 0$

at $\theta = \frac{\pi}{2}$, $\frac{d^2\bar{Q}_{66}}{d\theta^2} = 16U_3 \cos(2\pi) = 16U_3 > 0$

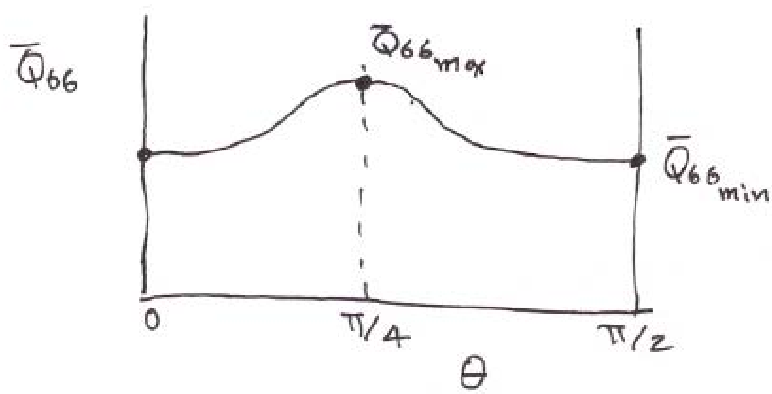
$\therefore$ $\bar{Q}_{66_{max}}$ occurs at $\theta = \frac{\pi}{4}$ and

$$\bar{Q}_{66_{max}} = \frac{U_1 - U_4}{2} - U_3 \cos(\pi) = \frac{U_1 - U_4}{2} + U_3$$

while $\bar{Q}_{66_{min}}$ occurs at $\theta = 0$ and $\frac{\pi}{2}$, where

$$\bar{Q}_{66_{min}} = \frac{U_1 - U_4}{2} - U_3 \cos(0) = \frac{U_1 - U_4}{2} - U_3$$

Graphically, the variation of $\bar{Q}_{66}$ with $\theta$ is as shown below



# Chapter 3

3.1

Consider the rectangular area in Fig. 3.5, the total composite area is $A_c = ab$, while the area of fiber enclosed in the total area is

$$A_f = \pi\left(\frac{s_a}{2}\right)\left(\frac{s_b}{2}\right)$$

the fiber area fraction, which is the same as the fiber volume fraction, is then

$$v_f = \frac{A_f}{A_c} = \frac{\pi s_a s_b}{4ab}$$

the maximum fiber volume fraction is achieved when

$$s_a = a \text{ and } s_b = b$$

or

$$v_{f\,max} = \frac{\pi}{4} = 0.785$$

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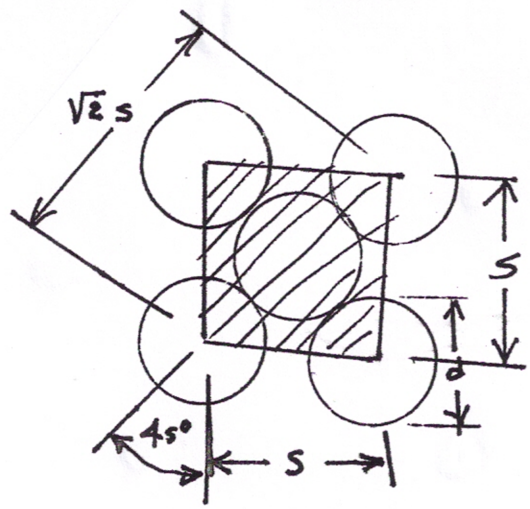
3.2

Fcc array:




For the cross-hatched area above,

$$v_f = \frac{(2)(\pi d^2/4)}{s^2} = \frac{\pi d^2}{2s^2}$$

When the fibers contact, $\sqrt{2}\,s = 2d$ and

$$v_{f\,max} = \frac{\pi d^2}{4d^2} = \frac{\pi}{4} = 0.785$$