CHAPTER 2

# Solving Linear Systems (Exer. 64–132)

## 1. The Linear System

**64.** Find the coefficient matrix $\mathbf{A}$, the variable vector $\mathbf{x}$, and the right-hand side vector $\mathbf{b}$ for this system:

$$\begin{aligned} 2x_1 - 3x_2 \qquad + x_4 &= 12 \\ 2x_2 - 7x_3 - x_4 &= -5 \\ -4x_1 + x_2 - 5x_3 - 2x_4 &= 0 \end{aligned}$$

**Solution:**

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & 2 & -7 & -1 \\ -4 & 1 & -5 & -2 \end{bmatrix}, \qquad \mathbf{x} = (x_1, x_2, x_3, x_4) \qquad \mathbf{b} = (12, -5, 0)$$

$\square$

**66.** If $\mathbf{A}$ and $\mathbf{b}$ are the matrix and vector given by

$$\mathbf{A} = \begin{bmatrix} -8 & -5 & 9 \\ 6 & 3 & -1 \\ 9 & -5 & 7 \\ -8 & -3 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -3 \\ 8 \\ 2 \\ -5 \end{pmatrix}$$

What linear system (written out in full) corresponds to the vector equation $\mathbf{Ax} = \mathbf{b}$?

**Solution:**

$$\begin{aligned} -8x - 5y + 9z &= -3 \\ 6x + 3y - \phantom{9}z &= 8 \\ 9x - 5y + 7z &= 2 \\ -8x - 3y - 8z &= -5 \end{aligned}$$

$\square$

**68.** Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Which of the vectors $\mathbf{x}$ below solve $\mathbf{Ax} = \mathbf{b}$? (Show your work.)

$$\text{a)}\ \ \mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \quad \text{b)}\ \ \mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad \text{c)}\ \ \mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{d)}\ \ \mathbf{x} = \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix} \quad \text{e)}\ \ \mathbf{x} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

**Solution:** Simply compute the matrix/vector product $\mathbf{Ax}$ for each given $\mathbf{x}$, and check: Does it equal $\mathbf{b} = (2, 1)$ or not? By direct calculation, we get these results:

$$\text{a)}\ \ \mathbf{Ax} = (2, 1) \qquad \text{b)}\ \ \mathbf{Ax} = (-10, -4) \qquad \text{c)}\ \ \mathbf{Ax} = (2, 1)$$

$$\text{d)}\ \ \mathbf{Ax} = (2, 1) \qquad \text{e)}\ \ \mathbf{Ax} = (4, 2)$$

So the vectors in (a), (c), and (d) solve the equation. The others don't. □

**70.** a) Express the "column problem"

$$u\,\mathbf{a}_1 + v\,\mathbf{a}_2 + w\,\mathbf{a}_3 = \mathbf{b}$$

as a linear system in the variables $u, v, w$, assuming $\mathbf{a}_1 = (1, -2, 3)$, $\mathbf{a}_2 = (-2, 1, -2)$, and $\mathbf{a}_3 = (3, -2, 1)$.

**Solution:**

$$\begin{aligned} u - 2v + 3w &= b_1 \\ -2u + \phantom{2}v - 2w &= b_2 \\ 3u - 2v + \phantom{3}w &= b_3 \end{aligned}$$

□

b) Express the linear system below in "column" form $x\,\mathbf{a}_1 + y\,\mathbf{a}_2 = \mathbf{b}$:

$$\begin{array}{rcr} 3x - 2y & = & 0 \\ 2x - 3y & = & 1 \\ 2x + 3y & = & 0 \\ 3x + 2y & = & -1 \end{array}$$

**Solution:**

$$x\begin{pmatrix}3\\2\\2\\3\end{pmatrix}+y\begin{pmatrix}-2\\-3\\3\\2\end{pmatrix}=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

□

**72.** Consider this problem:

*What vectors* $\mathbf{x} \in \mathbf{R}^4$ *dot to zero with all three vectors* $(1,1,2,3)$, $(1,2,3,1)$, *and* $(2,1,1,3)$ *simultaneously?*

What linear system expresses this problem? You needn't solve it.

**Solution:** If we write $\mathbf{x} = (x_1, x_2, \ldots, x_4)$, the vanishing of its dot product with each given vector produces one linear equation, as follows

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 &= 0\\ x_1 + 2x_2 + 3x_3 + x_4 &= 0\\ 2x_1 + x_2 + x_3 + 3x_4 &= 0 \end{aligned}$$

□

## 2. The Augmented Matrix and RRE Form

**74.** What are the augmented and coefficient matrices for these systems?

a) $\begin{aligned} 2x+3y &= 7\\ 3x-2y &= 4 \end{aligned}$ b) $\begin{aligned} y &= 12\\ x &= -1 \end{aligned}$ c) $\begin{aligned} 7x-2y &= 5\\ 6x-3y &= 0 \end{aligned}$

**Solution:**

a) $\mathbf{A} = \begin{bmatrix} 2 & 3\\ 3 & -2 \end{bmatrix}$, $[\mathbf{A}\,|\,\mathbf{b}] = \begin{bmatrix} 2 & 3 & \vdots & 7\\ 3 & -2 & \vdots & 4 \end{bmatrix}$

b) $\mathbf{A} = \begin{bmatrix} 0 & 1\\ 1 & 0 \end{bmatrix}$, $[\mathbf{A}\,|\,\mathbf{b}] = \begin{bmatrix} 0 & 1 & \vdots & 12\\ 1 & 0 & \vdots & -1 \end{bmatrix}$

$$\text{c)} \quad \mathbf{A} = \begin{bmatrix} 7 & -2 \\ 6 & -3 \end{bmatrix}, \quad [\mathbf{A}\,|\,\mathbf{b}] = \begin{bmatrix} 7 & 2 & \vdots & 5 \\ 6 & -3 & \vdots & 0 \end{bmatrix}$$

$\square$

**76.** What systems have the augmented matrices below?

$$\text{a)} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

**Solution:**

$$\text{a)} \begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \qquad \text{b)} \begin{aligned} 3y &= 0 \\ 2x &= 1 \end{aligned} \qquad \text{c)} \begin{aligned} x + y &= -1 \\ x - y &= \phantom{-}1 \end{aligned}$$

$\square$

**78.** Find all solutions to the systems represented by the remaining two augmented matrices in Example 2.15, namely

$$\begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$

**Solution:** The first augmented matrix represents the system

$$\begin{aligned} 0x + \phantom{0}y &= a \\ 0x + 0y &= 0 \end{aligned}$$

The first equation here is clearly satisfied by any vector of the form $(x, a) \in \mathbf{R}^2$, and any such vector solves the second equation too. So $(x, a)$, where $x$ can be any scalar, is the general solution (i.e. all solutions).

The second matrix represents the system

$$\begin{aligned} x + 0y &= a \\ 0x + \phantom{0}y &= b \end{aligned}$$

which clearly says $x = a$, $y = b$. So $(x, y) = (a, b)$ is the only solution for this system. $\square$

**80.** Which matrices below are in reduced row-echelon (RRE) form?

$$\text{a)} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c) $\begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Solution:** Matrices (a), (b), and (f) have RRE form. Matrices (c), (d), and (e) do not. □

**82.** Write down all 16 different 4-by-4 RRE forms. Use $a$ and $b$ to represent arbitrary numbers as in Example 2.15.

**Solution:**

$$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&*&*&*\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&*&*\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&*\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&*&0&*\\0&0&1&*\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&*&*&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix},$$

$$\begin{bmatrix} 0&1&0&*\\0&0&1&*\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&*&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&*\\0&1&0&*\\0&0&1&*\\0&0&0&0 \end{bmatrix},$$

$$\begin{bmatrix} 1&0&*&0\\0&1&*&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&*&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}$$

□

**84.** If an $n \times n$ matrix in RRE form has $n$ pivots, what matrix must it be?

**Solution:** The identity matrix.

No row or column of an RRE matrix can have more than one pivot. So an $n \times n$ matrix with $n$ pivots must have exactly one pivot in each row, and exactly one pivot in each column. Since each pivot is to the right of all pivots above it, each pivot in this situation must be exactly one column to the right of the pivot in the previous row, and the first pivot must be in the first column. Only the identity matrix satisfies these conditions. □

## 3. Homogeneous Systems in RRE Form

**86.** Each matrix below is in RRE form. Label each column as pivot (P) or free (F).

**Solution:**

$$\text{a)}\quad \begin{array}{c} \begin{array}{cccc} P & P & P & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \text{b)}\quad \begin{array}{c} \begin{array}{cccc} F & P & P & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\text{c)}\quad \begin{array}{c} \begin{array}{cccc} F & P & F & P \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \text{d)}\quad \begin{array}{c} \begin{array}{cccc} F & F & P & F \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\text{e)}\quad \begin{array}{c} \begin{array}{ccc} P & P & P \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \text{f)}\quad \begin{array}{c} \begin{array}{ccc} P & F & F \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

□

**88.** Consider this homogeneous system:

$$\begin{aligned} x_1 - \tfrac{1}{2}x_3 - \tfrac{1}{4}x_4 &= 0 \\ x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

Verify that its coefficient matrix has RRE form, and find the solution map whose image gives all its solutions.

**Solution:** The coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Columns 3 and 4 are free, so we get the two homogeneous generators whose linear combinations for this solution map:

$$\mathbf{H}(s_1, s_2) = s_1 \begin{pmatrix} 1/2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} 1/4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

□

**90.** Verify that the coefficient matrix of this system has RRE form, and find its solutions. Then check that each of your generators $\mathbf{h}_i$ solves the system.

$$\begin{aligned} x_1 + 2x_3 - x_5 &= 0 \\ x_2 - x_3 + 2x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

**Solution:** The coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Columns 3 and 5 (only) are free, so we get two homogeneous generators: $\mathbf{h}_1 = (-2, 1, 1, 0, 0)$ and $\mathbf{h}_2 = (1, -2, 0, 1, 1)$. To check that they solve the system, that is, that $\mathbf{A}\mathbf{h}_i = \mathbf{0}$ for each $i = 1, 2$, just dot each generator with each row of the coefficient matrix ("fast matrix/vector multiplication"). The product in each case is zero, so they do solve the homogeneous system. □

**92.** Verify that the coefficient matrix of the system below has RRE form, and find its solutions. Check that each of your generators $\mathbf{h}_i$ solves the system.

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= 0 \\ x_2 + 2x_5 &= 0 \end{aligned}$$

**Solution:** The coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Columns 3, 4, and 5 are free, so we get three homogeneous generators: $\mathbf{h}_1 = (2, 0, 1, 0, 0)$ and $\mathbf{h}_2 = (-3, 0, 0, 1, 0)$, and $\mathbf{h}_3 = (0, -2, 0, 0, 1)$. Dotting each generator with each row of the coefficient matrix ("fast matrix/vector multiplication") gives zero in each case, so they do solve the homogeneous system. □

**94.** Suppose an $n \times m$ matrix $\mathbf{A}$ has RRE form.

a) If $\mathbf{A}$ has $r$ leading 1's, how many generators $\mathbf{h}_i$ will show up in the solution of $\mathbf{Ax} = \mathbf{0}$? Explain.

**Solution:** The coefficient matrix has $m$ columns, and $r$ of them have pivots. So the $m - r$ remaining columns are free, and that yields $m - r$ homogeneous generators $\mathbf{h}_i$. □

b) What is the smallest number of free columns $\mathbf{A}$ can have if $m > n$? How about it $n > m$? Explain.

**Solution:** The rules for RRE form mean that no row or column can have more than one pivot. If $m > n$ and there's a pivot in each of the $n$ rows, that leaves $m - n$ columns without pivots, hence $m - n$ free columns. There might be more (if not every row gets a pivot), but the least *possible* number of free columns is $m - n$.

If $n > m$ we could have a pivot in *every* column and still leave some rows without them. In this case (where every column has a pivot) we get *no* free columns. So when $n > m$ the least *possible* number of free columns is zero. □

**96.** What $3 \times 3$ matrix $\mathbf{A}$ in RRE form yields

a) The single homogeneous generator $\mathbf{h} = (3, 1, 0)$?

**Solution:** If there is only one homogeneous generator, the 1 in the second position signals that $x_2$ is the only free variable. So the RRE matrix has pivots in columns 1 and 3. Since the first coordinate of $\mathbf{h}$ is 3, the matrix must have a $-3$ at the top of the second column. Since it is 3-by-3, it therefore must be:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

□

b) These two homogenous generators: $\mathbf{h}_1 = (1, 0, 0)$, and $\mathbf{h}_2 = (0, -2, 1)$?

**Solution:** The 1's in the first and third positions (each where the other homogeneous vector has a zero) signals that $x_1$ and $x_3$ are free. So the RRE matrix only has a pivot in column 2, and it must be in the first row. Since the second coordinate of $\mathbf{h}_2$ is 02, the matrix must have a $+2$ in row 2, column 3. So we must have:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

□

## 4. Inhomogeneous Systems in RRE Form

**98.** Verify that the augmented matrix for this system is in RRE form. Then find a particular solution for it.

$$\begin{aligned} x_2 + 2x_3 + 3x_5 &= 4 \\ x_4 + 5x_5 &= 6 \end{aligned}$$

**Solution:** The augmented matrix is this:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 5 & 6 \end{bmatrix}$$

There are two pivots, and the four rules for RRE form are easily seen to hold. To extract a particular solution, we set the three free variables to zero: $x_1 = x_3 = x_5 = 0$ and set the pivot variables to the right-hand sides: $x_2 = 4,\ x_4 = 6$. So a particular solution is

$$\mathbf{x}_p = (0, 4, 0, 6, 0)$$

It's easy to check that this does solve both equations. □

**100.** This system has two equations in three unknowns:

$$\begin{aligned} x + 2y &= 1 \\ z &= 2 \end{aligned}$$

Verify that its augmented matrix has RRE form and find all solutions.

**Solution:** The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This does have RRE form, with one free variable, $x_2$. We set this equal to zero and set the pivot variables equal to the right-hand sides to get a particular solution: $\mathbf{x}_p = (1, 0, 2)$. By the usual procedure, the homogeneous generator is $\mathbf{h} = (-2, 1, 0)$. So the general solution is

$$\mathbf{x}(s) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

□

**102.** Verify that the augmented matrix for this system has RRE form and find all solutions:

$$\begin{aligned} x_2 - 2x_3 + x_5 &= 0 \\ x_4 - 2x_5 &= 3 \end{aligned}$$

**Solution:** The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 3 \end{bmatrix}$$

This has RRE form, with 3 free variables, $x_1, x_3, x_5$, and two pivot variables: $x_2$ and $x_4$. We set the latter equal to zero and set the pivot variables equal to the right-hand sides to get a particular solution: $\mathbf{x}_p = (0, 0, 0, 3, 0)$. By the usual procedure, the homogeneous generators are

$$\mathbf{h}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{h}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{h}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

The general solution is then

$$\mathbf{x}(s_1, s_2, s_3) = \mathbf{x}_p + s_1\mathbf{h}_1 + s_2\mathbf{h}_2 + s_3\mathbf{h}_3$$

□

**104.** Suppose $\mathbf{A}$ is a $7 \times 3$ matrix, and $\mathbf{b} \in \mathbf{R}^7$. If $(2, -1, 5)$ solves the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$, and $(8, -2, 9)$ solves the *in*homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{b}$, give three other solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$.

**Solution:** By Theorem 4.1, all inhomogeneous solutions can be gotten from a single one (given here is $\mathbf{x}_p = (8, -2, 9)$ ) by adding homogeneous solutions to it. Here we're given only one homogeneous solution, $(2, -1, 5)$ , but all multiples of a homogeneous solution are again homogeneous solutions. (For if $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{A}(k\mathbf{x}) = k(\mathbf{Ax}) = k\mathbf{0} = \mathbf{0}$ too.) So any vector of the form

$$\begin{pmatrix} 8 \\ -2 \\ 9 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

will give another solution of $\mathbf{Ax} = \mathbf{b}$ . Choose any three different scalars for $s$ and you will have three different solutions of the system. □

**106.** The vectors $(3, -1, 2, 5)$, $(0, 1, -1, 0)$ and $(7, -7, 0, 1) \in \mathbf{R}^4$ all solve $\mathbf{Ax} = \mathbf{b}$ for a certain matrix $\mathbf{A}$ and right-hand side $\mathbf{b}$ . Find three solutions of the *homogeneous* system $\mathbf{Ax} = \mathbf{0}$, and three additional solutions of $\mathbf{Ax} = \mathbf{b}$.

**Solution:** The proof of Theorem 4.1 shows that the difference between any two solutions of the inhomogeneous system solves the *homogeneous* system. So, for instance

$$\begin{aligned}
(3, -1, 2, 5) - (0, 1, -1, 0) &= (3, -2, 3, 5) \\
(0, 1, -1, 0) - (7, -7, 0, 1) &= (-7, 8, -1, -1) \\
(7, -7, 0, 1) - (3, -1, 2, 5) &= (4, -6, -2, -4)
\end{aligned}$$

all solve the *homogeneous* system $\mathbf{Ax} = \mathbf{0}$. Adding any of these to any solution of the inhomogeneous system $\mathbf{Ax} = \mathbf{b}$ has to give us three other solutions of the latter, again by Theorem 4.1. So for instance (there are many possibilities)

$$\begin{aligned}
(3, -1, 2, 5) + (3, -2, 3, 5) &= (6, -3, 5, 10) \\
(3, -1, 2, 5) + (-7, 8, -1, -1) &= (-4, 7, 1, 4) \\
(0, 1, -1, 0) + (4, -6, -2, -4) &= (4, -5, -3, -4)
\end{aligned}$$

□

**108.** Suppose an inhomogeneous system has...

a) ...three equations and four variables. Can it have just one solution? If so, give an example. If not, explain why not.

**Solution:** No. The coefficient matrix has four columns, but only three rows. So it can have at most three pivots (at most one in each row). That means at least one column is free, so we *must* get at least one homogeneous generator, whose multiples all solve the homogeneous equation. □

b) ...four equations and three variables. Can it have exactly one solution? If so, give an example. If not, explain why not.

**Solution:** Yes. For instance

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

The augmented matrix row-reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We get a particular solution ( $\mathbf{x}_p = (1,1,1)$ ) but there is no free column, so the homogeneous system has only the trivial solution, and that means there are no additional solutions of the inhomogeneous system. $\square$

c) ...four equations and three variables. Can it have infinitely many solutions? If so, give an example. If not, explain why not.

**Solution:** Yes. For instance

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_1 + x_2 &= 2 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

This augmented matrix row-reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We get a particular solution ( $\mathbf{x}_p = (1,1,0)$ ) *and* a free column, so there is a homogeneous generator ( $\mathbf{h} = (0,0,1)$ ) which gives us infinitely many solutions. Namely, anything of the form $\mathbf{x}_p + s\mathbf{h}$, where $s$ can be any scalar. $\square$

## 5. The Gauss-Jordan Algorithm

**110.** Reduce to RRE form using Gauss–Jordan:

**Solution:**

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

□

**112.** Reduce to RRE form using Gauss–Jordan:

**Solution:**

$$\begin{bmatrix} 3 & 4 & -1 \\ 9 & 12 & -3 \\ 3 & 0 & 1 \\ 3 & -8 & 5 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

**114.** Find the complete solution of each system.

$$\text{a)} \quad \begin{aligned} x+y+z &= 1 \\ x-y+z &= -1 \\ x+y-z &= -1 \end{aligned} \qquad \text{b)} \quad \begin{aligned} x+y+z &= 1 \\ x-y+z &= -1 \\ x+z &= 1 \end{aligned}$$

**Solution:** For (a), the augmented matrix row-reduces as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Since row-reduction puts pivots in all three columns of the coefficient matrix, there are no homogeneous generators, and hence $\mathbf{x}_p = (-1, 1, 1)$ is the complete and only solution.

For (b), the augmented matrix row-reduces like this:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot at the end of the last row tells us the system has *no* solution. □

**116.** Solve the homogeneous system:

$$\begin{array}{rcl} 8x_1 + 4x_2 + 4x_3 + 2x_4 & = & 0 \\ x_2 + 2x_3 + \phantom{2}x_4 & = & 0 \\ 8x_1 + 6x_2 + 8x_3 + 4x_4 & = & 0 \end{array}$$

**Solution:** The coefficient matrix reduces as follows:

$$\begin{bmatrix} 8 & 4 & 4 & 2 \\ 0 & 1 & 2 & 1 \\ 8 & 6 & 8 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last two columns are free, so we get two homogeneous generators:

$$\mathbf{h}_1 = \begin{pmatrix} \frac{1}{2} \\ -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} \frac{1}{4} \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

The system is solved by (and only by) any linear combination of $\mathbf{h}_1$ and $\mathbf{h}_2$. □

**118.** If we let $\mathbf{R}^{m\times n}$ denote the set of all $n \times m$ matrices, *each elementary row operation can be seen as a map from* $\mathbf{R}^{m\times n}$ *to itself.*

a) Let $S_{ij} \colon \mathbf{R}^{m\times n} \to \mathbf{R}^{m\times n}$ be the mapping that swaps rows $i$ and $j$ of any $\mathbf{A} \in \mathbf{R}^{m\times n}$. Is this map one-to-one? Is it onto? What is its inverse?

**Solution:** The map $S_{ij}$ is its own inverse—if you swap rows $i$ and $j$, and then swap them again, you get the original matrix. Like any invertible map, $S_{ij}$ is both one-to-one and onto. □

b) Let $M_i(k) \colon \mathbf{R}^{m\times n} \to \mathbf{R}^{m\times n}$ be the mapping that multiplies row $i$ of any $\mathbf{A} \in \mathbf{R}^{m\times n}$ by the scalar $k$. Is this map one-to-one? Is it onto? Your answers should depend on whether $k = 0$ or not. When $k \neq 0$, what is the inverse of $M_i(k)$?

**Solution:** When $k \neq 0$, the map $M_i(k)$ is inverted by $M_i(k^{-1})$, since multiplying by $k^{-1}$ "undoes" multiplication by $k$. Thus, $M_i(k)$ is both one-to-one and onto when $k \neq 0$.

If, on the other hand, $k = 0$, the map is neither one-to-one nor onto. Its image contains only matrices whose $i$th row is all zeros, so it's not onto. It's not one-to-one because any two matrices that differ *only* in row $i$ will have the same image. $\square$

c) Let $A_{i,j}(k)\colon \mathbf{R}^{m\times n} \to \mathbf{R}^{m\times n}$ be the mapping that adds $k$ times row $i$ of $\mathbf{A}$ to row $j$. Is this map one-to-one? Is it onto? What is its inverse?

**Solution:** The mapping is inverted by $A_{i,j}(-k)$, since subtracting $k$ times row $i$ from row $j$ undoes adding $k$ times row $i$ to row $j$. So like any invertible mapping, $A_{i,j}(k)$ is both one-to-one and onto. $\square$

**120.** Let $\mathbf{R}^{m\times n}$ denote the set of all $n \times m$ matrices as in Exercise 5. The rule $\mathbf{A} \mapsto \text{RRE}(\mathbf{A})$ defines a mapping $GJ$ ("Gauss–Jordan") from $\mathbf{R}^{m\times n}$ to itself.

a) Is $GJ$ onto? If not, what is its image?

**Solution:** It is not onto, since its image contains only the $n \times m$ matrices that have RRE form. $\square$

b) If $GJ$ one-to-one? If not, how many pre-images does a matrix in its image typically have?

**Solution:** It is not one-to-one. For example, take any matrix with a non-zero entry in the $(1,1)$ position. Multiplying this row by any non-zero scalar will give a different matrix, but the new matrix will reduce to the same RRE form as the original matrix, since the first step of the Gauss-Jordan algorithm will divide row 1 by the (new) $(1,1)$ entry, always producing the same result. For this sort of reason, every RRE form matrix (except the zero matrix) will have infinitely many pre-images. The zero matrix has only one—itself. $\square$

**122.** (Cubic interpolation) Find a *cubic* polynomial $c(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ whose graph goes through the points $(-3, 60)$, $(-1, 0)$, $(1, 12)$, and $(3, 24)$.

**Solution:** Saying the graph goes through $(x, y)$ is the same as saying that $c(x) = y$. So here, using $x = -3, -1, 1, 1$ respectively, we have $c(-3) = 60$, $c(-1) = 0$, $c(1) = 12$ and $c(3) = 24$. Using the cubic formula for $c(x)$, this gives us four *linear* equations for the four coefficients $a_0, a_1, a_2, a_3$, namely

$$
\begin{aligned}
a_0 - 3a_1 + 9a_2 - 27a_3 &= 60 \\
a_0 - \phantom{3}a_1 + \phantom{9}a_2 - \phantom{27}a_3 &= 0 \\
a_0 + \phantom{3}a_1 + \phantom{9}a_2 + \phantom{27}a_3 &= 12 \\
a_0 + 3a_1 + 9a_2 + 27a_3 &= 24
\end{aligned}
$$

The augmented matrix for this system row-reduces as follows:

$$
\begin{bmatrix} 1 & -3 & 9 & -27 & 60 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 12 \\ 1 & 3 & 9 & 27 & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 0 & 15/2 \\ 0 & 0 & 1 & 0 & 9/2 \\ 0 & 0 & 0 & 1 & -3/2 \end{bmatrix}
$$

The cubic polynomial we want thus as $a_0 = 3/2$, $a_1 = 15/2$, etc:

$$
c(x) = -\frac{3}{2}x^3 + \frac{9}{2}x^2 + \frac{15}{2}x + \frac{3}{2}
$$

□

## 6. Two Mapping Answers

**124.** Give a short proof that when a "domain = range" linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is onto, it must also be one-to-one. (Imitate the reasoning used in the first paragraph of the proof of Corollary 6.12.)

**Solution:** If $T$ is onto, and $\mathbf{A}$ is the matrix that represents it, then $\text{RRE}(\mathbf{A})$ must have a pivot in every *row*. That gives us $n$ pivots, and since $\mathbf{A}$ is $n \times n$ (because domain and range are both $\mathbf{R}^n$) we have a pivot in every column too. That makes $T$ one-to-one by Proposition 6.3. □

**126.** Give an example (like the ones in Exercise 125 above) of

a) A linear mapping $F : \mathbf{R}^3 \to \mathbf{R}^5$ that is one-to-one. Can you also make it onto? Why or why not?

**Solution:** A one-to-one example is easy. For instance, $F(x, y, z) = (x, y, z, 0, 0)$, which is represented by

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

You can't make it onto, however, because the representing matrix will be 5-by-3, with more rows than columns. □

b) A linear mapping $F : \mathbf{R}^5 \to \mathbf{R}^3$ that is onto. Can you also make it one-to-one? Why or why not?

**Solution:** An onto example is easy. E.g., let, $F(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3)$, which is represented by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

You can't make it one-to-one, however, because the representing matrix will be 3-by-5 with more columns than rows. □

**128.** Show that when a linear map $F : \mathbf{R}^2 \to \mathbf{R}^2$ is represented by a matrix of the form

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$

with $ab \neq 1$, it has to be an isomorphism.

**Solution:** We subtract $b$ times row 1 from row 2 to reduce this matrix to

$$\begin{bmatrix} 1 & a \\ 0 & 1-ab \end{bmatrix}$$

Column 1 now has a pivot, and if $ab \neq 1$, we can divide row 2 by $1-ab$ to get a pivot in column 2 also. Both rows and columns then have pivots, so $F$ is one-to-one and onto—an isomorphism. □

**130.** True or False: *An* $n \times n$ *matrix* $\mathbf{A}$ *represents an isomorphism from* $\mathbf{R}^n$ *to* $\mathbf{R}^n$ *if and only if* $\text{RRE}(\mathbf{A}) = \mathbf{I}_n$ *(the* $n \times n$ *identity matrix).* Explain your conclusion.

**Solution:** True. An isomorphism is both one-to-one and onto, so by Propositions 6.3 and 6.7, $\text{RRE}(\mathbf{A})$ must have a pivot in every one of its $n$ rows and columns. The only $n \times n$ matrix in RRE form with this property is the identity. Conversely, if $\text{RRE}(\mathbf{A}) = \mathbf{I}$, then it has a pivot in every row and column, hence represents a mapping that is both one-to-one and onto. Such a mapping is an isomorphism by definition. □

**132.** Given any three scalars $a, b, c$, form the matrix

$$\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

Show that the linear transformation it represents is never one-to-one, nor onto, no matter what $a, b$, and $c$ are.

**Solution:** First, suppose $c$ is non-zero. Then the matrix row-reduces to

$$\begin{bmatrix} 1 & 0 & -a/c \\ 0 & 1 & -b/c \\ 0 & 0 & 0 \end{bmatrix}$$

If, on the other hand, $c = 0$, but $b \neq 0$, the matrix row-reduces to

$$\begin{bmatrix} 1 & -a/b & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $c = b = 0$ we get a row of zeros even quicker.

So in all possible cases, the matrix reduces to an RRE form with only two pivots. The transformation is therefore neither onto nor one-to-one, by Propositions 6.3 and 6.7. □