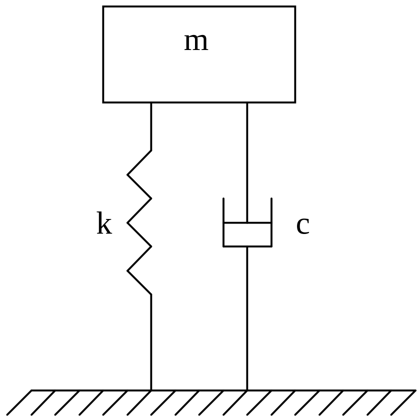
2. Spring Mass Damper System – Unforced Response



Example

Solve for five cycles, the response of an unforced system given by the equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1}$$

For ξ = 0.1; m = 1 kg; k = 100 N/m; x (0) = 2 cms; $\dot{x}(0) = 0$;

Solution

The above equation is a second order constant-coefficient differential equation. To solve this equation we have to reduce it into two first order differential equations. This step is taken because MATLAB uses a Runge-Kutta method to solve differential equations, which is valid only for first order equations.

Let

$$\dot{x} = v \tag{2}$$

From the above expression we see that

$$m\dot{v} + cv + kx = 0$$

so the equation (1) reduces to

$$\dot{v} = [(\frac{-c}{m})v - (\frac{k}{m})x] \tag{3}$$

We can see that the second order differential equation (1) has been reduced to two first order differential equations (2) and (3).

For our convenience, put

x = y (1);

$\dot{x} = v = y(2);$

Equations (2) and (3) reduce to

$\dot{y}(1)$ = y (2); (4)

$\dot{y}(2)$ = [(-c/m)*y (2) ï (k/m)*y (1)]; (5)

To calculate the value of ócô, compare equation (1) with the following generalized equation.

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^{\;2} = 0$$

Equating the coefficients of the similar terms we have

$$\frac{c}{m} = 2\xi\omega_n \tag{6}$$

$$\omega_n^{\;2} = \frac{k}{m} \tag{7}$$

Using the values of ómô and ókô, calculate the different values of ócô corresponding to each value of ξ. Once the values of ócô are known, equations (4) and (5) can be solved using MATLAB.

The problem should be solved for five cycles. In order to find the time interval, we first need to determine the damped period of the system.

Natural frequency ω_n = $\sqrt{(k/m)}$ = 10 rad/sec.

For ξ = 0.1

Damped natural frequency ω_d = $\omega_n \sqrt{1-\xi^2}$ = 9.95 rad/sec.

Damped time period $T_d = 2\pi/\omega_d$ = 0.63 sec.

Therefore for five time cycles the interval should be 5 times the damped time period, i.e., 3.16 sec.

MATLAB Code

In order to apply the ODE45 or any other numerical integration procedure, a separate function file must be generated to define equations (4) and (5). Actually, the right hand side of equations (4) and (5) are what is stored in the file. The equations are written in the form of a vector.

The MATLAB code is given below.

```
function yp = unforced1(t,y)
yp = [y(2);(-((c/m)*y(2))-((k/m)*y(1)))];                    (8)
```

Open a new M-file and write down the above two lines. The first line of the function file must start with the word ñfunctionò and the file must be saved corresponding to the function call; i.e., in this case, the file is saved as unforced1.m. The derivatives are stored in the form of a vector.

This example problem has been solved for $\xi = 0.1$. We need to find the value of óc/mô and ók/mô so that the values can be substituted in equation (8). Substituting the values of ξ and ω_n in equations (6) and (7) the values of óc/mô and ók/mô can be found out. After finding the values, substitute them into equation (8).

Now we need to write a code, which calls the above function and solves the differential equation and plots the required result. First open another M-file and type the following code.

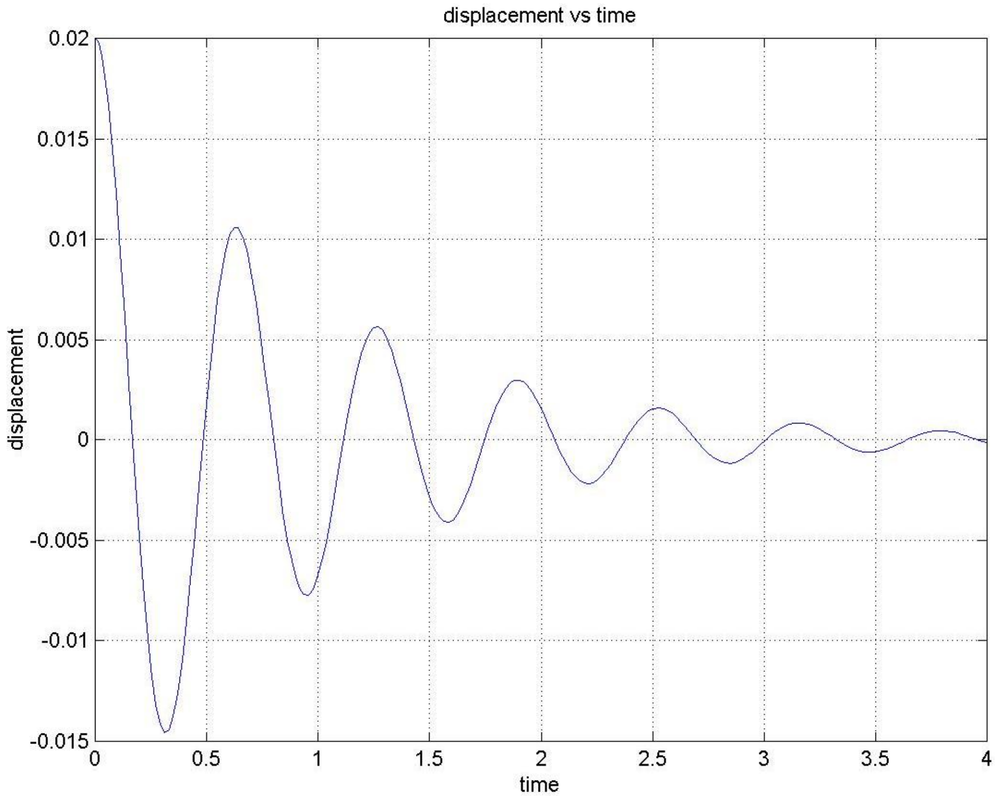
```
tspan=[0 4];
y0=[0.02;0];
[t,y]=ode45('unforced1',tspan,y0);
plot(t,y(:,1));
hold;
```

The ode45 command in the main body calls the function unforced1, which defines the systems first order derivatives. The response is then plotted using the plot command. óTspanô represents the time interval and óy0ô represents the initial conditions for y(1) and y(2) which in turn represent the displacement óxô and the first derivative of óxô. In this example, the initial conditions are taken as 0.02 m for óxô and 0 m/sec for the first derivative of óxô.

In order to solve for different values of ξ, calculate the values of óc/mô for each value of ξ. Substitute each value of ξ in the function file, which has the derivatives, save the file and then run the main program to view the result.

In the above code óy(:,1) represents the displacement óxô. To plot the velocity, change the variable in the plot command line to óy(:,2)ô.

The plot is attached below



Assignment

Solve for six cycles the response of an unforced system given by

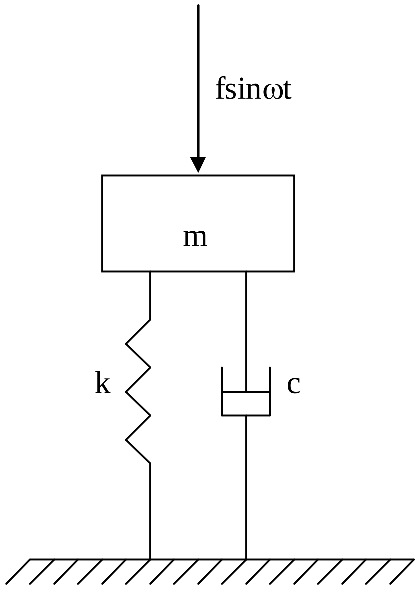
$$m\ddot{x} + c\dot{x} + kx = 0$$

For ξ = {0, 0.1 0.25, 0.5, 0.75, 1.0}.

Take m = 5 kg; k = 1000 N/m; x(0) = 5 cms; $\dot{x}(0) = 0$;

Develop a plot for the solutions corresponding to the seven ξ values and comment on the plots obtained.

Spring Mass Damper System – Forced Response



Example

Plot the response of a forced system given by the equation

$$m\ddot{x} + c\dot{x} + kx = f\sin\omega t \qquad (1)$$

For $\xi = 0.1$; m = 1 kg; k = 100 N/m; f = 100 N; $\omega = 2\omega_n$; x(0) = 2 cms; $\dot{x}(0) = 0$.

Solution

The above equation is similar to the unforced system except that it has a forcing function. To solve this equation we have to reduce it into two first order differential equations. Again, this step is taken because MATLAB uses a Runge-Kutta method to solve differential equations, which is valid only for first order equations.

Let

$$\dot{x} = v \qquad (2)$$

so the above equation reduces to

$$\dot{v} = [(\frac{f}{m})\sin\omega\, t - (\frac{c}{m})v - (\frac{k}{m})x] \qquad (3)$$

We can see that the second order differential equation has been reduced to two first order differential equations.

For our convenience, put

x = y (1);

$\dot{x} = v = y(2)$;

Equations (2) and (3) then reduce to

$\dot{y}(1)$ = y (2);

$\dot{y}(2)$ = [(f/m)*sin(ω * t) + (-c/m)*y (2) ï (k/m)*y (1)];

Again, to calculate the value of ócô, compare equation (1) with the following generalized equation

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^{\,2} = f \sin \omega\, t$$

Equating the coefficients of the similar terms we have

$$\frac{c}{m} = 2\xi\omega_n$$

$$\omega_n^{\,2} = \frac{k}{m}$$

Using the values of ómô and ókô, calculate the different values of ócô corresponding to each value of ξ.

To find the time interval the simulation should run, we first need to find the damped time period.

Natural frequency ω_n = $\sqrt{(k/m)}$ = 10 rad/sec.
For ξ = 0.1;
Damped natural frequency ω_d = ω_n $\sqrt{1-\xi^2}$ = 9.95 rad/sec.
Damped time period T_d = $2\pi/\omega_d$ = 0.63 sec.

Therefore, for five time cycles the interval should be 5 times the damped time period, i.e., 3.16 sec. Since the plots should indicate both the transient and the steady state response, the time interval will be increased.

MATLAB Code

The MATLAB code is similar to that written for the unforced response system, except that there is an extra term in the derivative vector, which represents the force applied to the system.

The MATLAB code is given below.

```
function yp = forced(t,y)
yp = [y(2);(((f/m)*sin(ωn*t))-((c/m)*y(2))-((k/m)*y(1)))];
```

Again the problem is to be solved for ξ = 0.1. So, calculate the value of óc/mô, ók/mô and óf/mô by following the procedure mentioned in the earlier example and then substitute these values into the above expression. Save the file as óforced.mô.

The following code represents the main code, which calls the function and solves the differential equations and plots the required result.

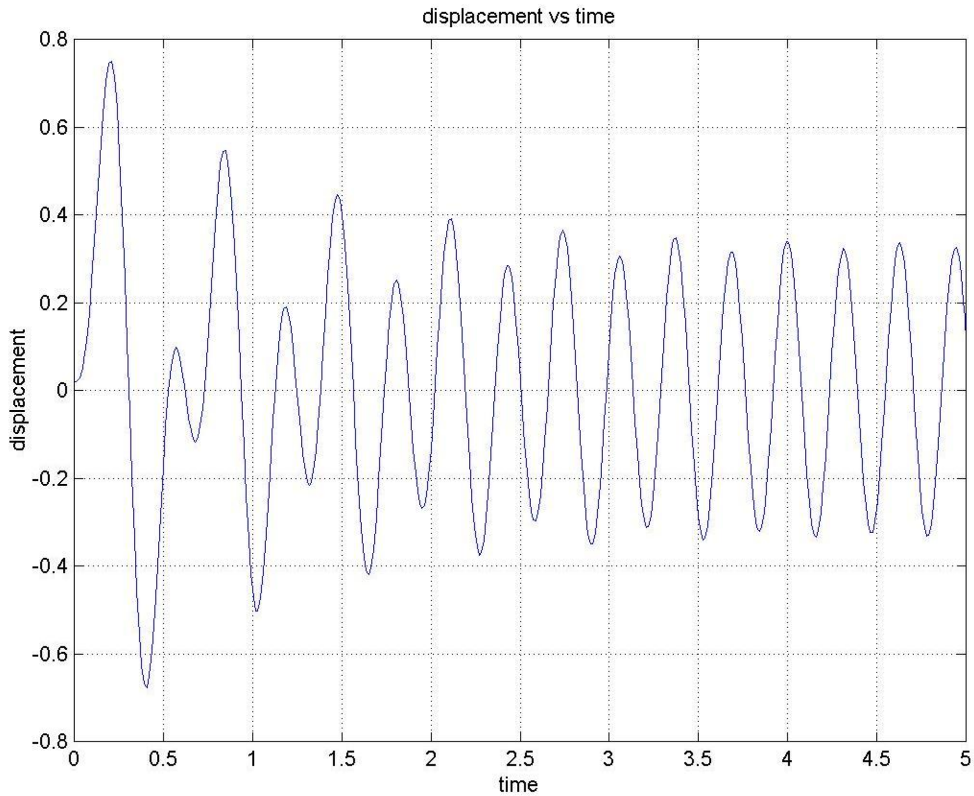
```
tspan=[0 5];
y0=[0.02;0];
[t,y]=ode45('forced',tspan,y0);
plot(t,y(:,1));
hold;
```

Again, ótspanô represents the time interval and óy0ô represents the initial conditions for y(1) and y(2) which in turn represent the displacement óxô and the first derivative of óxô. In this example the initial conditions are taken as 0.02 m for óxô and 0 cm/sec for the first derivative of óxô.

To solve for different values of ξ, calculate the values of óc/mô for each value of ξ. Substitute each value of ξ in the function file, which has the derivatives, save the file and then run the main program to view the result.

In the above code óy(:,1) represents the displacement óxô. To plot the velocity, change the variable in the plot command line to óy(:,2)ô.

The plot is attached below.


displacement vs time
displacement
time
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Assignment

Plot the response of a forced system given by the equation

$$m\ddot{x} + c\dot{x} + kx = f \sin \omega t$$

For ξ = {0, 0.1 0.25, 0.5, 0.75, 1.0}.

Take m = 5 kg; k = 1000 N/m; f = 50 N; $\omega = 4\omega_n$, x(0) = 5 cms; $\dot{x}(0) = 0$.

Develop a plot for the solutions corresponding to the seven ξ values and comment on the plots obtained.