Chapter 2: Theorems of Euclidean Plane Geometry

2.1 **The Exterior Angle Theorem**

1. Suppose that a triangle does not have two acute interior angles. Then at least two of its interior angles are right or obtuse. By the Supplement Postulate, the exterior angle adjacent to the right or obtuse interior angle is acute. Hence its measure is less than or equal to that of the other right or obtuse interior angle. Since this contradicts the Exterior Angle Theorem, a triangle has at least two acute interior angles.

2.2 **Triangle Congruence Theorems**

1. Let A, B, and C be distinct points. We proceed by cases:

   <u>Case 1</u>: A, B, and C are collinear. Then either $A - B - C$, $A - C - B$, or $C - A - B$.

   <u>Subcase 1a</u>: $A - B - C$. Then $AB + BC = AC$.

   <u>Subcase 1b</u>: $A - C - B$. Then $AC + BC = AB$ so that $AB + BC > AB > AC$.

   <u>Subcase 1c</u>: $C - A - B$. Then $AC + AB = BC$ so that $AB + BC > BC > AC$.

   <u>Case 2</u>: A, B, and C are non-collinear. By the Ruler Postulate, there exists a point D such that $A - B - D$ and $BD = BC$. Then $\mu(\angle ACD) > \mu(\angle BCD)$ by the Angle Addition Postulate, and $\mu(\angle BCD) = \mu(\angle BDC) = \mu(\angle ADC)$ by the Isosceles Triangle Theorem. Thus $\mu(\angle ACD) > \mu(\angle ADC)$ so that $AD > AC$ by the Scalene Inequality. Now $AD = AB + BD = AB + BC$. Therefore $AB + BC > AC$ by substitution.

   In all cases we have proved that $AB + BC \geq AC$, and the equality holds if and only if $A - B - C$ (Subcase 1a).

2. Let $\triangle ABC$ and $\triangle DEF$ be triangles such that $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\overline{AC} \cong \overline{DF}$. We must show that $\triangle ABC \cong \triangle DEF$.

   By the Ruler Postulate, there exists a point B' on $\overrightarrow{CB}$ such that $\overline{B'C} \cong \overline{EF}$. Then $\triangle AB'C \cong \triangle DEF$ by SAS so that $\angle AB'C \cong \angle DEF$ by CPCTC. Thus $\angle ABC \cong \angle AB'C$. Now to show that $B' = B$, suppose that $B' \neq B$. Then by definition of ray $\overrightarrow{CB}$, either $C - B' - B$ or $C - B - B'$. If $C - B' - B$, then $\mu(\angle AB'C) > \mu(\angle ABC)$ by the Exterior Angle Theorem; similarly, if $C - B - B'$, then $\mu(\angle ABC) > \mu(\angle AB'C)$. In both cases, we have a contradiction since $\angle ABC \cong \angle AB'C$. Therefore $B' = B$ and $\triangle ABC \cong \triangle DEF$.

3. Let $\triangle ABC$ and $\triangle DEF$ be triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ACB \cong \angle DFE$. We must show that $\angle BAC \cong \angle EDF$ or $\mu(\angle BAC) + \mu(\angle EDF) = 180°$.

   By the Ruler Postulate, there exists a point A' on $\overrightarrow{CA}$ such that $\overline{A'C} \cong \overline{DF}$. Then $\triangle A'BC \cong \triangle DEF$ by SAS so that $\angle BA'C \cong \angle EDF$ and $\overline{A'B} \cong \overline{DE}$ by CPCTC. If $A' = A$, then $\angle BAC \cong \angle EDF$; if $A' \neq A$, then either $A - A' - C$ or $A' - A - C$. Without loss of generality, assume $A - A' - C$. Since $\overline{AB} \cong \overline{DE}$ and $\overline{A'B} \cong \overline{DE}$, we have $\overline{A'B} \cong \overline{AB}$ so that $\angle BA'A \cong \angle BAC$ by the Isosceles Triangle Theorem. Since $\angle BA'A$ and $\angle BA'C$ form a linear pair, $\mu(\angle BA'A) + \mu(\angle BA'C) = 180°$ by the Supplement Postulate. Therefore $\mu(\angle BAC) + \mu(\angle EDF) = 180°$ by substitution.

4. Let $\triangle ABC$ and $\triangle DEF$ be triangles such that $\angle ACB$ and $\angle DFE$ are right angles, $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$. We must show that $\triangle ABC \cong \triangle DEF$.

   Since $\angle ACB$ and $\angle DFE$ are right angles, $\angle ACB \cong \angle DFE$. Thus $\angle BAC$ and $\angle EDF$ are congruent or supplementary by Exercise 2.2.3. Since $\angle BAC$ and $\angle EDF$ are both acute by Exercise 2.1.1, they are not supplementary, so $\angle BAC \cong \angle EDF$. Therefore $\triangle ABC \cong \triangle DEF$ by AAS.

5. Let A and B be distinct points. We must show that a point P lies on the perpendicular bisector of $\overline{AB}$ if and only if $PA = PB$. Let M be the midpoint of $\overline{AB}$. If $P = M$, then the statement is trivial, so assume that $P \neq M$.

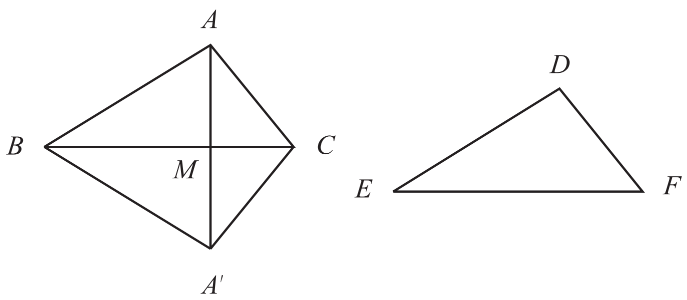
   Suppose P lies on the perpendicular bisector of $\overline{AB}$. Then line $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$. Note that $AM = BM$, $\mu(\angle PMA) = \mu(\angle PMB) = 90°$, and $PM = PM$. Thus $\triangle PAM \cong \triangle PBM$ by SAS so that $PA = PB$ by CPCTC.

   Suppose $PA = PB$. Note that $AM = BM$, and $\angle PAM \cong \angle PBM$ by the Isosceles Triangle Theorem. Thus $\triangle PAM \cong \triangle PBM$ by SAS so that $\angle PMA \cong \angle PMB$ by CPCTC. Since $\mu(\angle PMA) + \mu(\angle PMB) = 180°$ by the Supplement Postulate, $\mu(\angle PMA) = \mu(\angle PMB) = 90°$. Therefore line $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$. In particular, P lies on the perpendicular bisector of $\overline{AB}$.

6. Let $\triangle ABC$ and $\triangle DEF$ be triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$. We must show that $\triangle ABC \cong \triangle DEF$.

   By the Angle Construction Postulate and the Ruler Postulate, there exists a point A' on the opposite side of $\overleftrightarrow{BC}$ from A, such that $\angle A'BC \cong \angle DEF$ and $A'B = DE$. Then $\triangle A'BC \cong \triangle DEF$ by SAS so that $A'C =$

DF by CPCTC. It suffices to show that $\angle ABC \cong \angle A'BC$, since this will imply $\angle ABC \cong \angle DEF$ so that $\triangle ABC \cong \triangle DEF$ by SAS.



Since $AB = DE$ and $A'B = DE$, we have $AB = A'B$ so that B lies on the perpendicular bisector of $\overline{AA'}$ by Theorem 48. A similar argument shows that C lies on the perpendicular bisector of $\overline{AA'}$. Thus line $\overleftrightarrow{BC}$ is the perpendicular bisector of $\overline{AA'}$, so $\overleftrightarrow{BC}$ cuts $\overline{AA'}$ at the midpoint M of $\overline{AA'}$. If $M = B$, then $\angle ABC \cong \angle A'BC$ since they are both right angles. If $M \neq B$, $\triangle ABM \cong \triangle A'BM$ by SAS since $AM = A'M$, $\mu(\angle AMB) = \mu(\angle A'MB) = 90°$, and $BM = BM$. Hence $\angle ABM \cong \angle A'BM$ by CPCTC.

Case 1: $M \in \overrightarrow{BC}$. Then $\angle ABC = \angle ABM$ and $\angle A'BC = \angle A'BM$ so that $\angle ABC \cong \angle A'BC$ by substitution.

Case 2: $M - B - C$. Then $\mu(\angle ABC) + \mu(\angle ABM) = 180°$ and $\mu(\angle A'BC) + \mu(\angle A'BM) = 180°$ by the Supplement Postulate. Thus $\mu(\angle ABC) = 180° - \mu(\angle ABM) = 180° - \mu(\angle A'BM) = \mu(\angle A'BC)$ so that $\angle ABC \cong \angle A'BC$.

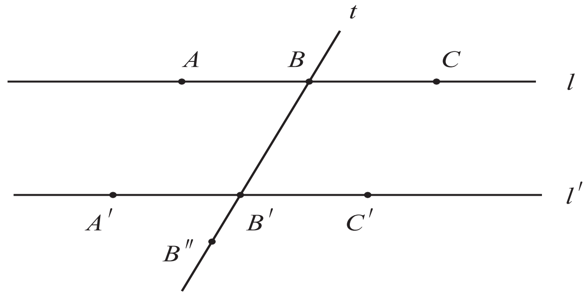
7. Let A, B, and C be non-collinear points, and let P be a point in the interior of $\angle BAC$. Let D and E be the feet of the perpendiculars from P to $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively. We must show that P lies on the angle bisector of $\angle BAC$ if and only if $PD = PE$.

   Suppose P lies on the angle bisector of $\angle BAC$. Then $\mu(\angle PAD) = \mu(\angle PAE)$ by definition. Since we also have $\mu(\angle PDA) = \mu(\angle PEA) = 90°$ and $PA = PA$, $\triangle PAD \cong \triangle PAE$ by AAS so that $PD = PE$ by CPCTC.

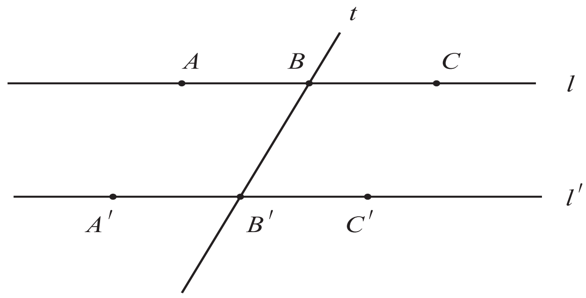
   Suppose $PD = PE$. Since $\mu(\angle PDA) = \mu(\angle PEA) = 90°$ and $PA = PA$, we have $\triangle PAD \cong \triangle PAE$ by the Hypotenuse-Leg Theorem. Thus $\mu(\angle PAD) = \mu(\angle PAE)$ by CPCTC so that $\overrightarrow{AP}$ is the angle bisector of $\angle BAC$ by definition. In particular, P lies on the angle bisector of $\angle BAC$.

2.3 The Alternate Interior Angles Theorem and the Angle Sum Theorem

1. Let l and l' be distinct lines cut by a transversal t at point B on l and point B' on l'. Choose points A, A', C, C', and B'' as in Definition 52. By the Vertical Angles Theorem, $\angle A'B'B'' \cong \angle C'B'B$. Thus by the Alternate Interior Angles Theorem, $l \parallel l'$ if and only if $\angle ABB' \cong \angle C'B'B$ if and only if $\angle A'B'B'' \cong \angle ABB'$. Congruence of the other pairs of corresponding angles follows by a similar argument.



2. Let l and l' be distinct lines cut by a transversal t at point B on l and point B' on l'. Choose points A, A', C, and C' as in Definition 52. By the Supplement Postulate, $\angle A'B'B$ and $\angle C'B'B$ are supplementary. Thus by the Alternate Interior Angles Theorem, $l \parallel l'$ if and only if $\angle ABB' \cong \angle C'B'B$ if and only if $\angle A'B'B$ and $\angle ABB'$ are supplementary. Congruence of the other pairs of non-alternate interior angles follows by a similar argument.



3. Let $\triangle ABC$ be a triangle and D a point on $\overleftrightarrow{BC}$ such that $B - C - D$. We must prove that $\mu(\angle ACD) = \mu(\angle BAC) + \mu(\angle ABC)$.

   By the Angle Sum Theorem, $\mu(\angle BAC) + \mu(\angle ABC) + \mu(\angle ACB) = 180°$. By the Supplement Postulate, $\mu(\angle ACD) + \mu(\angle ACB) = 180°$. Thus $\mu(\angle ACD) = 180° - \mu(\angle ACB) = \mu(\angle BAC) + \mu(\angle ABC)$.

4. Let $\square ABCD$ be a parallelogram. Then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ by definition.

   (a) By the Alternate Interior Angles Theorem, $\angle CAB \cong \angle ACD$ and $\angle ACB \cong \angle CAD$. Since $AC = CA$, $\triangle ABC \cong \triangle CDA$ by ASA. The proof for $\triangle ABD \cong \triangle CDB$ is similar.

   (b) Since $\triangle ABC \cong \triangle CDA$, $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$ by CPCTC.

   (c) Since $\triangle ABD \cong \triangle CDB$, $\angle DAB \cong \triangle BCD$ by CPCTC. Since $\triangle ABC \cong \triangle CDA$, $\angle ABC \cong \triangle CDA$ by CPCTC.

   (d) Let M be the point where $\overline{AC}$ and $\overline{BD}$ intersect each other. By part (b), $\overline{AB} \cong \overline{CD}$. By the Alternate Interior Angles Theorem, $\angle MAB \cong MCD$ and $\angle MBA \cong MDC$. Thus $\triangle ABM \cong \triangle CDM$ by ASA. Therefore $AM = CM$ and $BM = DM$ by CPCTC, i.e., the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.

5. Let l, m, and n be lines. Suppose that $l \parallel m$ and $m \parallel n$, but $l \nparallel n$. By negating the definition of parallels, $l \neq n$ and $l \cap n \neq \varnothing$. Since $l \cap n \neq \varnothing$, there exists a point $P \in l \cap n$. If $P \in m$, then $l = m$ and $m = n$ so that $l = n$; if $P \notin m$, then $l = n$ by the Euclidean Parallel Postulate. In both cases, we have $l = n$, which contradicts our previous assumption that $l \nparallel n$.

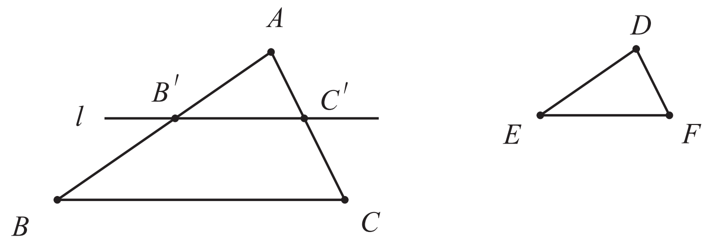
2.4 **Similar Triangles**

1. Given two triangles, assume first that they are similar. Then two pairs of corresponding angles are congruent by definition. For the converse, assume that two pairs of corresponding angles are congruent. Since the angle sum of every triangle is 180°, the third pair of corresponding angles are also congruent. Thus the two triangles are similar by definition.

2. Let $\triangle ABC$ and $\triangle DEF$ be triangles such that $\angle CAB \cong \angle FDE$ and $\frac{AB}{AC} = \frac{DE}{DF}$. We must show that $\triangle ABC \sim \triangle DEF$.

   Case 1: $AB = DE$. Then $AC = DF$ by algebra. Thus $\triangle ABC \cong \triangle DEF$ by SAS so that $\triangle ABC \sim \triangle DEF$.

   Case 2: $AB \neq DE$. Without loss of generality, assume that $AB > DE$. By the Ruler Postulate, there exists a point B' on $\overline{AB}$ such that $AB' = DE$. By the Euclidean Parallel Postulate, there exists a line l through B' and parallel to $\overleftrightarrow{BC}$. By Pasch's Axiom, l cuts $\overline{AC}$ at some point C'. By the Corresponding Angles Theorem, $\angle ABC \cong \angle AB'C'$ and $\angle ACB \cong \angle AC'B'$. Thus $\triangle ABC \sim \triangle AB'C'$ by AA.



By the Similar Triangles Theorem, $\frac{AB}{AB'} = \frac{AC}{AC'}$ so that $\frac{AB}{AC} = \frac{AB'}{AC'}$ by algebra. Since $\frac{AB}{AC} = \frac{DE}{DF}$, we have $\frac{AB'}{AC'} = \frac{DE}{DF}$. Since $AB' = DE$, we have $AC' = DF$. Thus $\triangle AB'C' \cong \triangle DEF$ by SAS. Since $\triangle ABC \sim \triangle AB'C'$, we conclude that $\triangle ABC \sim \triangle DEF$.

Chapter 3: Introduction to Transformations, Isometries, and Similarities

3.1 **Transformations**

1. α is bijective (both injective and surjective).

   β is neither injective nor surjective.

   γ is not injective but is surjective.

   δ is bijective (both injective and surjective).

   ϵ is neither injective nor surjective.

   η is bijective (both injective and surjective).

   ρ is injective but not surjective.

   σ is bijective (both injective and surjective).

   τ is bijective (both injective and surjective).

2. Let α and β be bijective transformations. Given points P and Q, assume $(\alpha \circ \beta)(P) = (\alpha \circ \beta)(Q)$. Then $\alpha(\beta(P)) = \alpha(\beta(Q))$. Since α is injective, $\beta(P) = \beta(Q)$. Since β is injective, $P = Q$. Therefore $\alpha \circ \beta$ is injective. Given a point R, there is a point Q such that $\beta(Q) = R$ since β is surjective, and there is a point P such that $\alpha(P) = Q$ since α is surjective. Therefore $(\alpha \circ \beta)(P) = \alpha(\beta(P)) = \alpha(Q) = R$ and $\alpha \circ \beta$ is surjective.

3. Let α be a transformation. Given a point P, note that $(\alpha \circ \iota)(P) = \alpha(\iota(P)) = \alpha(P)$ so that $\alpha \circ \iota = \iota$. Similarly, $(\iota \circ \alpha)(P) = \iota(\alpha(P)) = \alpha(P)$ so that $\iota \circ \alpha = \iota$. Therefore $\alpha \circ \iota = \iota \circ \alpha = \iota$.