Chapter 2

Sequences

2.1 Convergence of Sequences

2. If not then there is an $\epsilon > 0$ such that the interval $(t - \epsilon, t]$ contains no element of S. But then $t - \epsilon/2$ is an upper bound for S, and that contradicts the hypothesis that t is the least upper bound of S.

3. Let $\epsilon > 0$. Choose $N_1 > 0$ such that, if $j > N_1$, then $|a_j - \alpha| < \epsilon$. Likewise, choose $N_2 > 0$ such that, if $j > N_2$, then $|c_j - \alpha| < \epsilon$. Let $N = \max\{N_1, N_2\}$. If $j > N$, then

$$b_j - \alpha \leq c_j - \alpha \leq |c_j - \alpha| < \epsilon\,.$$

Likewise,

$$b_j - \alpha > -\epsilon\,.$$

Thus

$$|b_j - \alpha| < \epsilon\,.$$

That proves the result.

5. The answer is no. We can even construct a sequence with arbitrarily long repetitive strings and with subsequences that converges to *any* real number α. Indeed, order $\mathbb{Q}$ into a sequence $\{q_n\}$. Consider the following sequence

$$\{q_1, q_2, q_2, q_1, q_1, q_1, q_2, q_2, q_2, q_2, q_3, q_3, q_3, q_3, q_3, q_1, q_1, q_1, q_1, q_1, q_1, \cdots\}.$$

In this way we have repeated each rational number infinitely many times, and with arbitrarily long strings. From the above sequence we can find subsequences that converge to any real number.

7. Notice that

$$\int_0^1 \frac{dt}{1+t^2} = \mathrm{Tan}^{-1}(t)\Big|_0^1 = \frac{\pi}{4}\,.$$

Now approximate the integral by its Riemann sums.

8. Notice that $j^2/(j^2+1) \to 1$. Hence the given sequence has terms tending to $+1$ and also terms tending to -1. So the sequence cannot converge.

9. The sequence is majorized by $e^j/e^{2j} = e^{-j} \to 0$. So the sequence converges to 0.

11. The sequence

$$3\,,\ 3.1\,,\ 3.14\,,\ 3.141\,,\ 3.1415\,,\ 3.14.159\ \ldots$$

consists of rational numbers that converge to π.

The sequence

$$a_j = 2 + \sqrt{2}/j$$

consists of irrational numbers that converge to 2.

13. Let $\beta > \alpha$ be irrational numbers. Let $\epsilon = \beta - \alpha$. Let $q > 0$ be a rational number that is smaller than ϵ. Consider the sequence $\{jq\}_{j=-\infty}^{\infty}$. Then some element jq must lie between α and β.

15. Consider the sequence

$$1\,,\ 3\,,\ 2\,,\ 4\,,\ 6\,,\ 5\,,\ 7\,,\ 9\,,\ 8\,,\ldots\,.$$

Then it is clear that the sequence tends to infinity, but it does not do so monotonically.

2.2 Subsequences

1. Clearly any increasing sequence $\{a_j\}$ that is bounded above is bounded. By Bolzano-Weierstrass it has a convergent subsequence $\{a_{j_k}\}$. But the same argument shows that any subsequence has a convergent subsequence with the same limit α. By Exercise 3 of the last section, the full sequence converges to α. In fact α is simply the least upper bound of the sequence.

2. Let $\{q_j\}$ be an enumeration of the rational numbers. Now consider the sequence A

$$q_1, q_1, q_2, q_1, q_2, q_3, q_1, q_2, q_3, q_4, \ldots .$$

Notice that every q_j occurs infinitely many times in this new sequence.

If α is any real number, then there is a sequence $\{q_{j_k}\}$ that converges to α. But the sequence $\{q_{j_k}\}$ can certainly be realized as a subsequence of A.

4. The sequence is bounded, so the Bolzano-Weierstrass theorem guarantees a convergent subsequence.

5. The sequence $\{1, 2, 3, \ldots\}$ is bounded below but does not have a convergent subsequence.

The sequence $\{-1, -2, -3, \ldots\}$ is bounded above but does not have a convergent subsequence.

6. For any positive integer set

$$\phi(n) = n - k\pi,$$

where k is the (unique) integer such that

$$k\pi < n < (k+1)\pi.$$

By the pigeonhole principle, the set of all $\phi(n)$ will contain arbitrarily small elements. So $\{\phi(n)\}$ is dense in $[0, \pi]$.

By calculus we know that $\cos x$ is one-to-one on $[0, \pi]$ with values in $[-1, 1]$. Let $\cos^{-1}$ be the inverse. For $\alpha \in [-1, 1]$ we have $\cos^{-1} \alpha \in [0, \pi]$. Thus there exists a sequence

$$\phi(j_k) \longrightarrow \cos^{-1} \alpha \text{ as } k \to \infty.$$

By the continuity of the cosine function,

$$\cos(j_k) = \cos(\phi(j_k)) \longrightarrow \alpha \quad \text{as} \quad k \to \infty .$$

8. Certainly Proposition 2.13 shows that the b_j converge to some limit β (see also Exercise 1 above). But the limit of the b_j is also the lim inf of the original sequence a_j. It follows then that there is a subsequence of the a_j that converges to β.

9. The sequence

$$3\,,\ 3\,,\ 3.1\,,\ 3\,,\ 3.1\,,\ 3.14\,,\ 3\,,\ 3.1\,,\ 3.14\,,\ 3.141\,,\ 3\,,\ 3.1\,,\ 3.14\,,\ 3.141\,,\ 3.1315\,, \dots$$

has infinitely many different subsequences that converge to π.

11. If it is not true that every subsequence has a convergent subsequence, then some subsequence lacks a convergent subsequence. But then the full sequence cannot converge.

The converse direction is similar.

2.3 Limsup and Liminf

1. Consider the sequence

$$0, 1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots .$$

Then the supremum of this set of numbers is 3, while the limsup is 0. A similar example applies to the inf and liminf.

3. Let $\alpha \equiv \limsup a_j$ and $\beta \equiv \liminf a_j$. Let $A_j = \sup\{a_j, a_{j+1}, a_{j+2}, \dots\}$ and $B_j = \inf\{a_j, a_{j+1}, a_{j+2}, \dots\}$. Then

$$\begin{aligned} \sup\{1/a_j, 1/a_{j+1}, 1/a_{j+2}, \cdots\} &= 1/\inf\{a_j, a_{j+1}, a_{j+2}, \cdots\} \\ &= 1/B_j . \end{aligned}$$

Thus $\limsup 1/a_j = 1/\beta$.

Analogously one shows that $\liminf_{j\to\infty} 1/a_j = 1/\alpha$.

4. We write
$$(|\sin j|)^{\sin j} = e^{\sin j \log|\sin j|}.$$
Now we look at the function
$$f(x) = x \log|x|$$
when $x \in [-1, 1]$. We have
$$f'(x) = \log|x| + 1.$$
Thus f has a maximun when $x = -e^{-1}$ and a minimum when $x = e^{-1}$. Moreover
$$f(-e^{-1}) = e^{-1}$$
and
$$f(e^{-1}) = -e^{-1}.$$
We know that $\{\sin j\}$ is dense in $[-1, 1]$. Thus there exist sequences $\{j_k\}$ and $\{j_\ell\}$ such that $\sin j_k \to 1/e$ and $\sin j_\ell \to -1/e$. Then
$$\limsup_{j\to\infty} |\sin j|^{\sin j} = e^{1/e}$$
and
$$\liminf_{j\to\infty} |\sin j|^{\sin j} = e^{-1/e}.$$

5. Let $\alpha = \liminf a_j = \limsup a_j$. Seeking a contradiction suppose that $\{a_j\}$ does not converge. Then there exist $\epsilon > 0$ and a subsequence $\{a_{j_k}\}$ such that for all k
$$|a_{j_k} - \alpha| > \epsilon.$$
Let $\beta = \limsup a_{j_k} (\neq \alpha)$ and $a_{j_{k_\ell}}$ be a subsequence such that $\lim_{\ell\to\infty} a_{j_{k_\ell}} = \beta$. But $\{a_{j_{k_\ell}}\}$ is a subsequence of the original sequence. By Corollary 2.33,
$$\liminf a_j \le \lim_{\ell\to\infty} a_{j_{k_\ell}} \le \limsup a_j$$
and by the Pinching Principle
$$\lim_{\ell\to\infty} a_{j_{k_l}} = \alpha.$$
This contradiction shows that $\{a_j\}$ converges to α.

6. Let

$$\alpha_j = b - 1/2^j \qquad \text{if } j \text{ is even}$$

and

$$\alpha_j = a + 1/2^j \qquad \text{if } j \text{is odd}\,.$$

Then it is clear that the limsup is b and the liminf is a.

7. Let $a_j = (-1)^j$ and $b_j = (-1)^{j+1}$. Then $\limsup a_j = 1$, $\limsup b_j = 1$, yet $\limsup(a_j + b_j) = 0$. There are similar examples for the liminf.

8. The complex numbers cannot be ordered (see Exercise 8 in Section 1.2). So the concepts of limsup and liminf make no sense.

9. When dealing with $\limsup(a_j \cdot b_j)$ we have to be careful of the signs. If a_j and b_j are all non-negative numbers, then

$$\begin{aligned} \limsup(a_j \cdot b_j) &= \lim_{k\to\infty} (a_{j_k} \cdot b_{j_k}) \\ &= \lim_{k\to\infty} a_{j_k} \cdot \lim_{k\to\infty} b_{j_k} \\ &\leq \alpha \cdot \beta. \end{aligned}$$

Notice that in the inequality we have used that fact that all the quantities involved are non-negative ($x_1 < y_1$ and $x_2 < y_2$ implies $x_1 \cdot x_2 \leq y_1 \cdot y_2$ only if x_1, x_2, y_1, y_2 are non-negative). Using this comment, it is easy to construct sequences $\{a_j\}$ and $\{b_j\}$ of negative numbers for which

$$\limsup(a_j \cdot b_j) > \limsup a_j \cdot \limsup b_j.$$

10. Let $a_j \geq 0$. Then clearly the liminf of $-a_j$ is the negative of the limsup of a_j. Similar statements are true for other sign values of the a_j.

11. Since the values $\cos j$ are dense in the interval $[-1, 1]$, it follows that the limsup of $\cos j$ is $+1$ and the liminf of the sequence is -1. A similar assertion holds for the limsup and liminf of $\sin j$.

2.4 Some Special Sequences

1. Let $r = p/q = m/n$ be two representations of the rational number r. Recall that for any real α, the number α^r is defined as the real number β for which

$$\alpha^m = \beta^n.$$

Let β' satisfy

$$\alpha^p = \beta'^q \, .$$

We want to show that $\beta = \beta'$. we have

$$\begin{aligned} \beta^{n \cdot q} &= \alpha^{m \cdot q} \\ &= \alpha^{p \cdot n} \\ &= \beta'^{q \cdot n}. \end{aligned}$$

By the uniqueness of the $(n \cdot q)^{th}$ root of a real number it follows that

$$\beta = \beta',$$

proving the desired equality. The second equality follows in the same way. Let

$$\alpha = \gamma^n.$$

Then

$$\alpha^m = \gamma^{n \cdot m}.$$

Therefore, if we take the n^{th} root on both sides of the above inequality, we obtain

$$\gamma^m = (\alpha^m)^{1/n}.$$

Recall that γ is the n^{th} root of α. Then we find that

$$(\alpha^{1/n})^m = (\alpha^m)^{1/n}.$$

Using similar arguments, one can show that for all real numbers α and β and $q \in \mathbb{Q}$

$$(\alpha \cdot \beta)^q = \alpha^q \cdot \beta^q.$$

Finally, let α, β, and γ be positive real numbers. Then

$$\begin{aligned} (\alpha \cdot \beta)^\gamma &= \sup\{(\alpha \cdot \beta)^q : q \in \mathbb{Q}, q \leq \gamma\} \\ &= \sup\{\alpha^q \beta^q : q \in \mathbb{Q}, q \leq \gamma\} \\ &= \sup\{\alpha^q : q \in \mathbb{Q}, q \leq \gamma\} \cdot \sup\{\beta^q : q \in \mathbb{Q},\ q \leq \gamma\} \\ &= \alpha^\gamma \cdot \beta^\gamma. \end{aligned}$$

3. It suffices to notice that, for any fixed x,

$$\begin{aligned}
\lim_{j\to\infty}\left(1+\frac{x}{j}\right)^{j} &= \lim_{j\to\infty}\left\{\left(1+\frac{x}{j}\right)^{j/x}\right\}^{x} \\
&= \left\{\lim_{j/x\to\infty}\left\{1+\frac{x}{j}\right\}^{j/x}\right\}^{x} \\
&= e^{x}.
\end{aligned}$$

4. We see that

$$(a^b)^c = \sup\{(a^b)^\gamma : \gamma < c \text{ and } \gamma \text{ rational}\}$$

$$= \sup\{(\sup\{a^\beta : \beta < b \text{ and } \beta \text{ rational}\})^\gamma : \gamma < c \text{ and } \gamma \text{ rational}\}$$

$$= \sup\{a^{\beta\gamma} : \beta < b \text{ and } \beta \text{ rational}, \gamma < c \text{ and } \gamma \text{ rational}\}\,.$$

But this last is a^{bc}.

The proof of the other identity is similar.

5. Write

$$\begin{aligned}
\frac{j^j}{(2j)!} &= \frac{j\cdots j}{1\cdots j\cdot j+1\cdots 2j} \\
&\leq \frac{1}{1\cdots j} \\
&= \frac{1}{j!}.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{j\to\infty}\frac{j^j}{(2j)!} &\leq \lim_{j\to\infty}\frac{1}{j!} \\
&= 0.
\end{aligned}$$

7. We write $F(x) = a_0 + a_1x + a_2x^2 + \cdots$. Here the a_j's are the terms of the Fibonacci sequence and the letter x denotes an unspecified variable. What is curious here is that we do not care about what x is. We intend to manipulate the function F in such a fashion that we will be able to solve for the coefficients a_j. Just think of $F(x)$ as a polynomial with a *lot* of coefficients.

Notice that

$$xF(x) = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots$$

and

$$x^2F(x) = a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \cdots .$$

Thus, grouping like powers of x, we see that

$$\begin{aligned} &F(x) - xF(x) - x^2F(x) \\ &\quad = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 \\ &\qquad +(a_3 - a_2 - a_1)x^3 + (a_4 - a_3 - a_2)x^4 + \cdots . \end{aligned}$$

But the basic property that defines the Fibonacci sequence is that $a_2 - a_1 - a_0 = 0$, $a_3 - a_2 - a_1 = 0$, etc. Thus our equation simplifies drastically to

$$F(x) - xF(x) - x^2F(x) = a_0 + (a_1 - a_0)x.$$

We also know that $a_0 = a_1 = 1$. Thus the equation becomes

$$(1 - x - x^2)F(x) = 1$$

or

$$F(x) = \frac{1}{1 - x - x^2}. \tag{$*$}$$

It is convenient to factor the denominator as follows:

$$F(x) = \frac{1}{\left[1 - \frac{-2}{1-\sqrt{5}}x\right] \cdot \left[1 - \frac{-2}{1+\sqrt{5}}x\right]}$$

(just simplify the right hand side to see that it equals $(*)$.

A little more algebraic manipulation yields that

$$F(x) = \frac{5+\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1-\sqrt{5}}x}\right] + \frac{5-\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1+\sqrt{5}}x}\right].$$

Now we want to apply the formula for the sum of a geometric series to each of the fractions in brackets ([ ]). For the first fraction, we think of $-\frac{2}{1-\sqrt{5}}x$ as λ. Thus the first expression in brackets equals

$$\sum_{j=0}^{\infty} \left(-\frac{2}{1-\sqrt{5}}x\right)^j .$$

Likewise the second sum equals

$$\sum_{j=0}^{\infty} \left(-\frac{2}{1+\sqrt{5}}x\right)^j .$$

All told, we find that

$$F(x) = \frac{5+\sqrt{5}}{10} \sum_{j=0}^{\infty} \left(-\frac{2}{1-\sqrt{5}}x\right)^j + \frac{5-\sqrt{5}}{10} \sum_{j=0}^{\infty} \left(-\frac{2}{1+\sqrt{5}}x\right)^j .$$

Grouping terms with like powers of x, we finally conclude that

$$F(x) = \sum_{j=0}^{\infty} \left[\frac{5+\sqrt{5}}{10} \left(-\frac{2}{1-\sqrt{5}}x\right)^j + \frac{5-\sqrt{5}}{10} \left(-\frac{2}{1+\sqrt{5}}x\right)^j \right] x^j .$$

But we began our solution of this problem with the formula

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The two different formulas for $F(x)$ must agree. In particular, the coefficients of the different powers of x must match up. We conclude that

$$a_j = \frac{5+\sqrt{5}}{10} \left(-\frac{2}{1-\sqrt{5}}\right)^j + \frac{5-\sqrt{5}}{10} \left(-\frac{2}{1+\sqrt{5}}\right)^j .$$

We rewrite

$$\frac{5+\sqrt{5}}{10} = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2} \qquad\qquad \frac{5-\sqrt{5}}{10} = -\frac{1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2}$$

and

$$-\frac{2}{1-\sqrt{5}} = \frac{1+\sqrt{5}}{2} \qquad\qquad -\frac{2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2} .$$

Making these four substitutions into our formula for a_j, and doing a few algebraic simplifications, yields

$$a_j = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^j - \left(\frac{1-\sqrt{5}}{2}\right)^j}{\sqrt{5}}$$

as desired.

9. We can write this sequence as

$$\left[\left(1 + \frac{1}{j^2}\right)^{j^2}\right]^{1/j} .$$

Of course the expression inside the brackets tends to e. So the entire sequence tends to the same limit as $e^{1/j}$, which is 1.

10. Draw the graph of $y = 1/x$ and superimpose on that the boxes corresponding to height $1/j$ at position j. Then you can easily see that the given sequence converges.