Further applications in one dimension

2

2.1 Unsteady diffusion

2.1.1 *Eigenvalues and eigenvectors of the projection matrix*

Confirm the eigenvalues and eigenvectors of the matrix $\mathbf{P}$ are given in (2.1.34) and (2.1.35).

Solution:

We are asked to show that

$$\mathbf{P} \cdot \mathbf{u}^{(m)} = \lambda_m \, \mathbf{u}^{(m)}. \tag{1}$$

In index notation,

$$P_{ij} \, u_j^{(m)} = \lambda_m \, u_i^{(m)}. \tag{2}$$

For simplicity, we omit the superscript (m).

For $i = 1$, we obtain

$$(1 - 2\alpha) \, u_1 + \alpha \, u_2 = \lambda_m \, u_1. \tag{3}$$

Substituting the alleged eigenvalues and eigenvectors, we obtain

$$\begin{aligned} &(1 - 2\,\alpha) \, \sin\Big(\frac{m}{N+1}\,\pi\Big) + \alpha \, \sin\Big(\frac{2m}{N+1}\,\pi\Big) \\ &\qquad = \Big[1 - 4\alpha \, \sin^2\Big(\frac{m}{N+1}\,\frac{\pi}{2}\Big)\Big] \, \sin\Big(\frac{m}{N+1}\,\pi\Big). \end{aligned} \tag{4}$$

Simplifying, we obtain

$$1 - 2\alpha + 2\,\alpha \, \cos\Big(\frac{m}{N+1}\,\pi\Big) = 1 - 4\alpha \, \sin^2\Big(\frac{m}{N+1}\,\frac{\pi}{2}\Big) \tag{5}$$

or

$$\cos\Big(\frac{m}{N+1}\,\pi\Big) = 1 - 2 \, \sin^2\Big(\frac{m}{N+1}\,\frac{\pi}{2}\Big), \tag{6}$$

which is an identity.

For $i = 2, \ldots, N-1$, we obtain

$$\begin{aligned}\alpha \sin\left(\frac{(i-1)m}{N+1}\,\pi\right) + (1-2\alpha)\sin\left(\frac{im}{N+1}\,\pi\right) + \alpha \sin\left(\frac{(i+1)m}{N+1}\,\pi\right)\\ = \left[1 - 4\alpha\sin^2\left(\frac{m}{N+1}\,\frac{\pi}{2}\right)\right]\sin\left(\frac{im}{N+1}\,\pi\right).\end{aligned} \tag{7}$$

Now using the identity

$$\sin A + \sin B = 2\,\sin\frac{A+B}{2}\,\cos\frac{A-B}{2}, \tag{8}$$

we obtain

$$\sin\left(\frac{(i-1)m}{N+1}\,\pi\right) + \sin\left(\frac{(i+1)m}{N+1}\,\pi\right) = 2\,\sin\left(\frac{im}{N+1}\,\pi\right)\cos\left(\frac{m}{N+1}\,\pi\right). \tag{9}$$

Substituting this expression into (7) and simplifying, we obtain

$$\begin{aligned}2\,\alpha\,\sin\left(\frac{im}{N+1}\,\pi\right)\cos\left(\frac{m}{N+1}\,\pi\right) + (1-2\alpha)\sin\left(\frac{im}{N+1}\,\pi\right)\\ = \left[1 - 4\alpha\sin^2\left(\frac{m}{N+1}\,\frac{\pi}{2}\right)\right]\sin\left(\frac{im}{N+1}\,\pi\right).\end{aligned} \tag{10}$$

Simplifying, we obtain

$$2\,\alpha\,\cos\left(\frac{m}{N+1}\,\pi\right) + 1 - 2\alpha = 1 - 4\alpha\sin^2\left(\frac{m}{N+1}\,\frac{\pi}{2}\right) \tag{11}$$

or

$$\cos\left(\frac{m}{N+1}\pi\right) = 1 - 2\,\sin^2\left(\frac{m}{N+1}\,\frac{\pi}{2}\right), \tag{12}$$

which is an identity.

For $i = N$, we obtain

$$\alpha u_{N-1} + (1-2\alpha)\,u_N = \lambda_m\,u_N. \tag{13}$$

Substituting the alleged eigenvalues and eigenvectors, we obtain

$$\begin{aligned}\alpha\,\sin\left(\frac{(N-1)m}{N+1}\,\pi\right) + (1-2\alpha)\sin\left(\frac{Nm}{N+1}\,\pi\right)\\ = \left[1 - 4\alpha\sin^2\left(\frac{m}{N+1}\,\frac{\pi}{2}\right)\right]\sin\left(\frac{Nm}{N+1}\,\pi\right),\end{aligned} \tag{14}$$

which simplifies to

$$\sin\left(\frac{(N-1)m}{N+1}\,\pi\right) = 2\left[1 - 2\,\sin^2\left(\frac{m}{N+1}\,\frac{\pi}{2}\right)\right]\sin\left(\frac{Nm}{N+1}\,\pi\right) \tag{15}$$

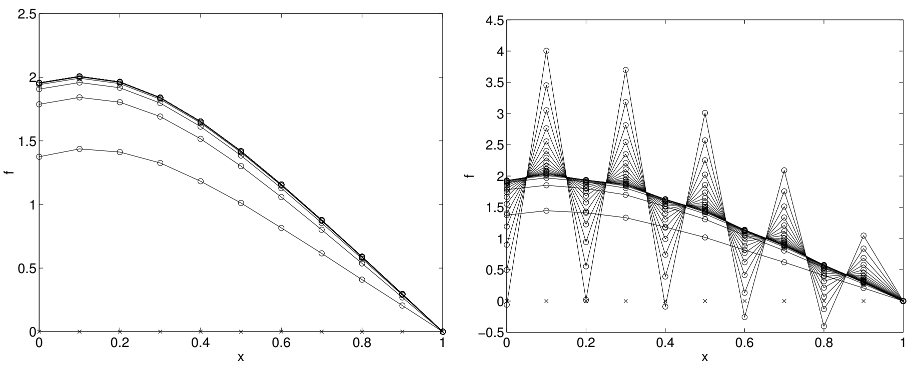

FIGURE 2.1.2-1 Graphics output of the finite element code `udl` with mass lumping and time step corresponding to diffusion numbers $\alpha = 0.5$ (left) and 0.503 (right). Theoretical analysis shows that the threshold for numerical stability is $\alpha = 0.5$.

or

$$\sin\Big(\frac{(N-1)m}{N+1}\,\pi\Big) = 2\,\cos\Big(\frac{m}{N+1}\,\pi\Big)\,\sin\Big(\frac{Nm}{N+1}\,\pi\Big). \tag{16}$$

Next, we note that

$$\sin\Big(\frac{(N-1)m}{N+1}\,\pi\Big) = \sin\Big(\frac{(N+1-2)m}{N+1}\,\pi\Big) = \sin\Big(\frac{2m}{N+1}\,\pi\Big) \tag{17}$$

and

$$\sin\Big(\frac{Nm}{N+1}\,\pi\Big) = \sin\Big(\frac{(N+1-1)m}{N+1}\,\pi\Big] = \sin\Big(\frac{m}{N+1}\,\pi\Big). \tag{18}$$

Substituting (17) and (18) into (16), we obtain an identity.

2.1.2 *Code with mass lumping*

FSELIB function *udl_sys_lump*, not listed in the text, implements mass lumping. Run the code *udl* with the function *udl_sys_lump* and discuss the onset of numerical instability with reference to the theoretical predictions.

Solution:

The finite element solution for 10 evenly spaced elements of size $h = 0.1$, $\kappa = 1$, and time step $\Delta t = 0.0050$ and 0.00503, corresponding to diffusion numbers $\alpha = 0.5$ and 0.503, is shown in Figure 2.1.2-1. A graph is generated every 100 time steps.

The analysis in the text predicts that the threshold for numerical stability is $\alpha = 0.5$. Consistent with the theoretical prediction, the first numerical solution

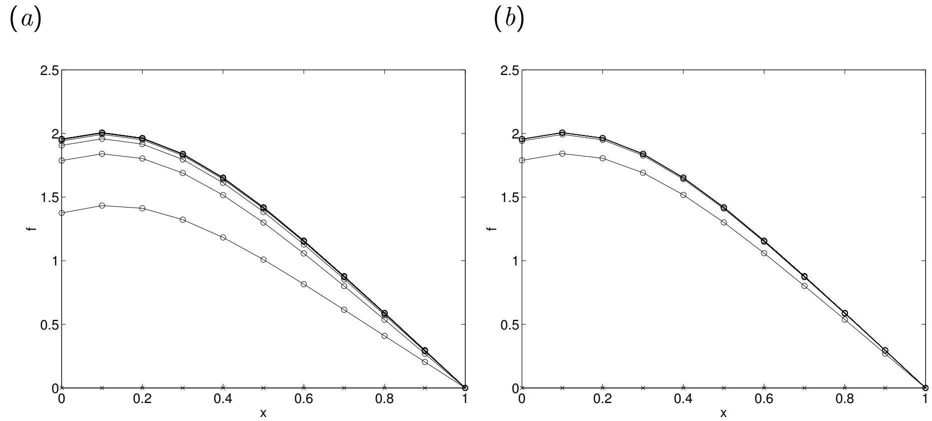

FIGURE 2.1.3-1 Graphics output generated by the finite element code *udl* with the Crank-Nicolson method for a time step corresponding to diffusion numbers $\alpha =$ 0.5 and 1.0. Theoretical analysis shows that the method is unconditionally stable.

shown in Figure 2.1.2-1 smoothly tends to the steady-steady solution, whereas the second solution develops an unphysical numerical oscillation dominated by a saw-tooth wave in spite of the smallness of the time step.

2.1.3 *Code for the Crank-Nicolson method*

Run the code *udl* with the Crank–Nicolson method and confirm that the algorithm is stable regardless of the size of the time step.

Solution:

The solution for 10 evenly spaced elements of size $h = 0.1$, $\kappa = 1$, and time steps $\Delta t = 0.0050$ and 0.00503, corresponding to diffusion numbers $\alpha = 0.5$ and 0.503 is shown in Figure 2.1.3-1. A graph is generated every 100 time steps. Consistent with the theoretical predictions regarding numerical stability, the numerical solution is stable for any size of the time step.

2.1.4 *Consistency analysis and hyperdiffusivity*

A consistency analysis is carried out working backwards from an algebraic difference equation to a modified differential equation (MDE). If the MDE reduces to the governing partial differential equation as the time step and element size tend to zero independently, then the algebraic difference equation is consistent.

To carry out the consistency analysis of the difference equation (2.1.23), we express all discrete variables in terms of the value at the ith node at time level n.

Omitting the hats for simplicity, we obtain, for example,

$$\varphi_i^{n+1} = \varphi_i^n + \left(\frac{\partial \varphi}{\partial t}\right)_i^n \Delta t + \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial t^2}\right)_i^n \Delta t^2 + \cdots \tag{1}$$

and

$$\varphi_{i+1}^{n} = \varphi_i^n + \left(\frac{\partial \varphi}{\partial x}\right)_i^n \Delta x + \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial x^2}\right)_i^n \Delta x^2 + \cdots . \tag{2}$$

Substituting these expressions into equation (2.1.23) and simplifying, we obtain

$$\begin{aligned}\left(\frac{\partial \varphi}{\partial t}\right)_i^n \Delta t + \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial t^2}\right)_i^n \Delta t^2 + \cdots \\ = \alpha\left(\frac{\partial^2 \varphi}{\partial x^2}\right)_i^n \Delta x^2 + \frac{\alpha}{12}\left(\frac{\partial^4 \varphi}{\partial x^4}\right)_i^n \Delta x^4 + \cdots,\end{aligned} \tag{3}$$

which can be rearranged into

$$\left(\frac{\partial \varphi}{\partial t}\right)_i^n = \kappa\left(\frac{\partial^2 \varphi}{\partial x^2}\right)_i^n - \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial t^2}\right)_i^n \Delta t + \frac{\kappa}{12}\,\Delta x^2\left(\frac{\partial^4 \varphi}{\partial x^4}\right)_i^n + \cdots . \tag{4}$$

Because as $\Delta t \to 0$ and $\Delta x \to 0$ independently the modified differential equation (3) reduces to the unsteady diffusion equation, the numerical method is consistent.

The underlying continuous function, φ, satisfies the unsteady heat conduction equation in the absence of a source,

$$\boxed{\frac{\partial \varphi}{\partial t} = \kappa\,\frac{\partial^2 \varphi}{\partial x^2}.} \tag{5}$$

Differentiating with respect to time, we find that the solution also satisfies the high-order equation

$$\boxed{\frac{\partial^2 \varphi}{\partial t^2} = \kappa^2\,\frac{\partial^4 \varphi}{\partial x^4}.} \tag{6}$$

Using this equation to eliminate the second derivative with respect to t on the right-hand side of (2.1.56), and rearranging, we obtain

$$\left(\frac{\partial \varphi}{\partial t}\right)_i^n = \kappa\left(\frac{\partial^2 \varphi}{\partial x^2}\right)_i^n - \kappa_4\left(\frac{\partial^4 \varphi}{\partial x^4}\right)_i^n + \cdots, \tag{7}$$

where the coefficient

$$\kappa_4 \equiv \frac{1}{2}\,\kappa\,\Delta x^2\,(\frac{1}{6} - \alpha) = \frac{1}{2}\,\kappa\,(\frac{1}{6}\,\Delta x^2 - \kappa\,\Delta t) \tag{8}$$

is called the *hyperdiffusivity.* Derive the hyperdiffusivity of the consistent formulation expressed by (2.1.38).

Solution:

The consistent formulation is based on the difference equation

$$\frac{1}{6}\,\varphi_{i-1}^{n+1} + \frac{2}{3}\,\varphi_i^{n+1} + \frac{1}{6}\,\varphi_{i+1}^{n+1} \tag{9}$$
$$= (\frac{1}{6} + \alpha)\,\varphi_{i-1}^{n} + (\frac{2}{3} - 2\,\alpha)\,\varphi_i^{n} + (\frac{1}{6} + \alpha)\,\varphi_{i+1}^{n}.$$

Expanding the left-hand side in a Taylor series about f_i^{n+1} and judiciously rearranging the right-hand side, we obtain

$$\varphi_i^{n+1} + \frac{1}{6}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)_i^{n+1}\Delta x^2 + \frac{1}{72}\Big(\frac{\partial^4\varphi}{\partial x^4}\Big)_i^{n+1}\Delta x^4 + \cdots \tag{10}$$
$$= \alpha\,\varphi_{i-1}^{n} + (1 - 2\,\alpha)\,\varphi_i^{n} + \alpha\,\varphi_{i+1}^{n} + \big(\frac{1}{6}\,\varphi_{i-1}^{n} - \frac{1}{3}\,\varphi_i^{n} + \frac{1}{6}\,\varphi_{i+1}^{n}\big).$$

Expanding the terms inside the large parentheses on the right-hand side in a Taylor series about φ_i^n, we obtain

$$\varphi_i^{n+1} + \frac{1}{6}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)_i^{n+1}\Delta x^2 + \frac{1}{72}\Big(\frac{\partial^4\varphi}{\partial x^4}\Big)_i^{n+1}\Delta x^4 + \cdots \tag{11}$$
$$= \alpha\,\varphi_{i-1}^{n} + (1 - 2\,\alpha)\,\varphi_i^{n} + \alpha\,\varphi_{i+1}^{n} + \frac{1}{6}\Big(\frac{\partial^2\varphi}{\partial x^2}\Big)_i^{n}\Delta x^2 + \frac{1}{72}\Big(\frac{\partial^4\varphi}{\partial x^4}\Big)_i^{n}\Delta x^4 + \cdots.$$

Expanding the derivatives on the left-hand side in a Taylor series about φ_i^n and simplifying, we obtain

$$\varphi_i^{n+1} + \frac{1}{6}\Big(\frac{\partial^3\varphi}{\partial x^2\partial t}\Big)_i^{n}\Delta x^2\,\Delta t + \cdots = \alpha\,\varphi_{i-1}^{n} + (1 - 2\,\alpha)\,\varphi_i^{n} + \alpha\,\varphi_{i+1}^{n}. \tag{12}$$

Recalling that φ satisfies (6), we obtain

$$\varphi_i^{n+1} = \alpha\,f_{i-1}^{n} + (1 - 2\,\alpha)\,\varphi_i^{n} + \alpha\,\varphi_{i+1}^{n} - \frac{1}{6}\Big(\frac{\partial^4\varphi}{\partial x^4}\Big)_i^{n}\kappa\,\Delta x^2\,\Delta t + \cdots. \tag{13}$$

Using (2.1.60), we find that the hyperdiffusivity of the consistent formulation is

$$\boxed{\kappa_4 = \frac{1}{2}\,\kappa\,(\frac{1}{6}\,\Delta x^2 - \kappa\,\Delta t) + \frac{1}{6}\,\kappa\,\Delta x^2 = \frac{1}{2}\,\kappa\,(\frac{1}{2}\,\Delta x^2 - \kappa\,\Delta t).} \tag{14}$$

2.2 Convection

2.2.1 *Quadratic elements*

Display the structure of the global advection matrix for linear convection with the quadratic elements discussed in Section 1.5.

Solution:

Using (1.5.30), we find that the global advection matrix is given by

$$\mathbf{N} \equiv \frac{1}{6}\,U \begin{bmatrix} -3 & 4 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 4 & -1 & \dots & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & \dots & 0 & 1 & -4 & 0 & 4 & -1 \\ 0 & 0 & \dots & \dots & 0 & 0 & 0 & -4 & 0 & -4 \\ 0 & 0 & \dots & \dots & 0 & 0 & 0 & 1 & -4 & 3 \end{bmatrix}. \tag{1}$$

2.2.2 *Dispersion with mass lumping*

Show that the second fraction on the right-hand sides of (2.2.28)–(2.2.32) is replaced by unity in the lumped mass approximation.

Solution:

This follows readily by substituting (2.2.24) into the mass-lumped equation

$$\frac{\mathrm{d}f_i}{\mathrm{d}t} + \frac{U}{2h}\,(f_{i+1} - f_{i-1}) = 0. \tag{1}$$

2.3 Convection–diffusion

2.3.1 *Convection dominated transport*

(a) Run the FSELIB code *scdl* with the Robin boundary condition at the left end of the solution domain, parameter values $L = 1.0$, $k = 1.0$, $h_T = 1.0$, $f_\infty = 0.0$, $f_L = 0.0$, $\rho = 1.0$, $c_p = 1.0$, $U = 100$, and $N_E = 2, 4, 8, 16, 32$, and 64, evenly spaced elements. Discuss the behavior of the numerical solution.

(b) Repeat (a) with 16 elements and investigate the effect of element clustering near the right end of the computation domain where sharp gradients may arise.

Solution:

(a) This is a straightforward exercise.

(b) This is a straightforward exercise.

2.3.2 *Neumann boundary condition*

Repeat Problem 2.3.1(a, b) with the Neumann boundary condition at the left end.

Solution:

This is a straightforward exercise.

2.4 Beam bending

2.4.1 *Principal moment of inertia*

Present expressions for the principal moment of inertia of three beams with cross-sectional shapes of your choice.

Solution:

For a beam with a rectangular cross-section with side lengths $2a$ and $2b$, $I = 4ab^3/3$. For a beam with a circular cross-section of radius a, $I = \pi a^4/4$. For a beam with an elliptical cross-section with axis a and b, $I = \pi b a^3/4$.

2.5 Finite element methods for beam bending

2.5.1 *Element stiffness matrix*

Confirm that the determinant of the element stiffness matrix shown in equation (2.5.33) is zero, which indicates that the matrix is singular.

Solution:

We observe that the sum of the first and third columns is zero. The sum of the first and third rows is also zero.

2.5.2 *Two-element cantilever beam*

Consider the two-element discretization of a cantilever beam subject to nodal load, as shown in Figure 2.5.6. The length of the first element is αL, where α is a dimensionless coefficient taking values in the interval $(0, 1)$. Compile a 6×6 linear system arising from the finite element formulation in terms of E, I, L, α, F_2, F_3, $Q(0)$, and $M(0)$, before, and after the implementation of the boundary conditions (2.5.36).

Solution:

In this case, $h_1 = \alpha L$ and $h_2 = (1 - \alpha)L$. The assembled stiffness matrix takes the form

$$\mathbf{K} = \begin{bmatrix} H_{11}^{(1)} & H_{12}^{(1)} & H_{13}^{(l)} & H_{14}^{(1)} & 0 & 0 \\ H_{21}^{(1)} & H_{22}^{(1)} & H_{23}^{(l)} & H_{24}^{(1)} & 0 & 0 \\ H_{31}^{(1)} & H_{32}^{(1)} & H_{33}^{(1)} + H_{11}^{(2)} & H_{34}^{(1)} + H_{12}^{(2)} & H_{13}^{(2)} & H_{14}^{(2)} \\ H_{41}^{(1)} & H_{42}^{(1)} & H_{43}^{(1)} + H_{21}^{(2)} & H_{44}^{(1)} + H_{22}^{(2)} & H_{23}^{(2)} & H_{24}^{(2)} \\ 0 & 0 & H_{31}^{(2)} & H_{32}^{(2)} & H_{33}^{(2)} + H_{11}^{(3)} & H_{34}^{(2)} + H_{12}^{(3)} \\ 0 & 0 & H_{41}^{(2)} & H_{42}^{(2)} & H_{43}^{(2)} + H_{21}^{(3)} & H_{44}^{(2)} + H_{22}^{(3)} \end{bmatrix}. \tag{1}$$

The six-dimensional vector of nodal deflections and slopes is

$$\mathbf{u} \equiv \begin{bmatrix} v_1 \\ v_1' \\ v_2 \\ v_2' \\ v_3 \\ v_3' \end{bmatrix}. \tag{2}$$

The vector $\mathbf{b}$ on the right-hand side of the linear system (2.5.17) is given by

$$\mathbf{b} = \begin{bmatrix} -Q(0) \\ M(0) \\ -F_2 \\ 0 \\ -F_3 \\ 0 \end{bmatrix}. \tag{3}$$

To implement the boundary conditions, we replace the first two equations in the linear system by $v_1 = 0$ and $v_1' = 0$.

2.5.3 *Code beam*

Consider a cantilever beam where the load is zero at all nodes, except at the last node located at the free end. Modify accordingly the FSELIB code *beam* and run the modified code for several discretization levels. Compare and discuss the finite element solutions.

Solution:

This requires a straightforward modification of the first `for` loop in the code *beam*, so that $Fg(i) = 0$ for $i = 1, \ldots, ng - 1$, and $Fg(ng) = 1$.

2.6 Beam buckling

2.6.1 *Element stiffness matrix*

Confirm that the determinant of the geometric element stiffness matrix displayed in (2.6.10) is zero, indicating that the matrix is singular.

Solution:

We observe that the sum of the first and third column is zero. The sum of the first and third row is also zero.

2.6.2 *Buckling of a simply supported beam*

Derive analytical expressions for the eigenvalues and eigensolutions of (2.6.21) for a beam that is simply supported at both ends.

Solution:

For a beam of length L that is simply supported at both ends,

$$v(x) = A \sin(n\pi x/L), \tag{1}$$

where A is an unspecified amplitude and n is an integer. Substituting this expression into (2.6.21), we find that

$$EI \left(\frac{n\pi}{L}\right)^4 - P \left(\frac{n\pi}{L}\right)^2 = 0, \tag{2}$$

which yields the critical loads (2.6.29).

2.6.3 *Buckling of a beam with different types of support*

A beam that is simply supported at both ends is said to have a pin-pin support. Other types of support are the fixed-pin support, the fixed-fixed support, and the fixed-free support. Solving (2.6.21) by elementary analytical methods, we find that the eigenvalues are given by

$$\hat{P}_n = \left(\frac{n\pi}{\alpha}\right)^2, \tag{1}$$

where n is an integer and the coefficient α depends on the boundary conditions. For the pin-pin support, $\alpha = 1$, as discussed in the text.

(a) Show that, for a beam with a fixed-pin support, $\alpha = 0.7$ for $n = 1$. Derive and solve the one-element eigenvalue problem.

(b) Show that, for a beam with a fixed-fixed support, $\alpha = 0.5$. Derive and solve the one-element eigenvalue problem.

(c) Show that, for a beam with a fixed-free support, $\alpha = 2$ for $n = 1$; Derive and solve the one-element eigenvalue problem.

Solution:

(a) The boundary conditions require that

$$v = 0, \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 0, \qquad \text{at } x = 0, \tag{2}$$

and

$$v = 0, \qquad \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0, \qquad \text{at } x = L. \tag{3}$$

The beam deflection is described by the equation

$$v = A\left(\sin kx - kx - B\,(\cos kx - 1)\right). \tag{4}$$

This expression satisfies the boundary conditions (1). To also satisfy the boundary conditions (2), we compute

$$\begin{aligned} v(L) &= A\,(\sin kL - kL - B\ \cos kL + B), \\ v''(L) &= A\,k^2\,(-\sin kL + B\ \cos kL), \end{aligned} \tag{5}$$

where a prime denotes a derivative with respect to x. These will be zero if

$$B = kL, \qquad \tan kL = kL. \tag{6}$$

The smallest root of the second equation is $kL = 1.4303\,\pi$. Substituting (3) into the governing equation (2.6.21), we find

$$EIk^4 - Pk^2 = 0 \tag{7}$$

or

$$\widehat{P} = (kL)^2. \tag{8}$$

For the smallest root, $n = 1$,

$$\widehat{P}_1 = (1.4303\pi)^2 = \left(\frac{\pi}{0.7}\right)^2. \tag{9}$$

(b) The boundary conditions require that

$$v = 0, \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 0, \qquad \text{at } x = 0, L. \tag{10}$$

The deflection in this case is described by

$$u(x) = A\left(\cos\frac{2n\pi x}{L} - 1\right). \tag{11}$$

Substituting this expression into (2.6.21), we derive the desired result.

(c) The boundary conditions require that

$$v = 0, \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 0, \qquad \text{at } x = 0, \tag{12}$$

and

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0, \qquad Q = -P\,\frac{\mathrm{d}v}{\mathrm{d}x} - EI\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} = 0, \qquad \text{at } x = L. \tag{13}$$

The beam deflection is described by the equation

$$v = A\,(\cos kx - 1). \tag{14}$$

This expression satisfies the boundary conditions (11). To also satisfy the boundary condition (12), we require that

$$v''(L) = -A\,\cos kL = 0, \tag{15}$$

and

$$Q(L) = (-P\,A\,k + EIA\,k^3)\,\sin kL = 0. \tag{16}$$

These are satisfied if

$$k = (n - \frac{1}{2})\,\pi, \qquad \hat{P} = k^2 L^2, \tag{17}$$

where n is an integer. For $n = 1$, we find that $k = \pi/2$ and $\alpha = 2$.