2

Free Fall and Harmonic Oscillators

1. Find all the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\dfrac{dy}{dx} = \dfrac{e^x}{2y}$.

Using separation of variables, we have

$$\begin{aligned} 2\int y\,dy &= \int e^x\,dx \\ y^2(x) &= e^x + C. \end{aligned}$$

This is an implicit solution.

b. $\dfrac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

Using separation of variables, we have

$$\begin{aligned} \int \frac{dy}{y^2} &= \int (1+t^2)\,dt \\ -\frac{1}{y} &= t + \frac{1}{3}t^3 + C. \end{aligned}$$

The initial condition, $y(0) = 1$, implies $C = -1$, giving the solution

$$y(t) = \frac{1}{1 - t - \frac{1}{3}t^3} = \frac{3}{3 - 3t - t^3}.$$

c. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$.

Using separation of variables, we have

$$\begin{aligned} \int \frac{dy}{\sqrt{1-y^2}} &= \int \frac{dx}{x} \\ \sin^{-1} y &= \ln|x| + C \\ y(x) &= \sin(\ln|x| + C) \end{aligned}$$

d. $xy' = y(1-2y), \quad y(1) = 2$.

Using separation of variables and partial fraction decomposition, we have

$$\begin{aligned}
\int \frac{dx}{x} &= \int \frac{dy}{y(1-2y)} \\
\ln|x| + C &= \int \left(\frac{1}{y} + \frac{2}{1-2y} \right) dy \\
&= \ln \left| \frac{y}{1-2y} \right|
\end{aligned}$$

Exponentiating,

$$\left| \frac{y}{1-2y} \right| = e^{\ln|x|+C} \equiv A|x|.$$

Using the initial condition, $y(1) = 2$, we find that $A = \frac{2}{3}$. For solutions near $y = 2$ and $x = 1$, this gives

$$\begin{aligned}
\frac{y}{2y-1} &= \frac{2}{3}x \\
3y &= 2x(2y-1) \\
(3-4x)y &= -2x \\
y(x) &= \frac{2x}{4x-3}.
\end{aligned}$$

e. $y' - (\sin x)y = \sin x$.

This is a linear first order differential equation. The integrating factor is

$$\mu(x) = \exp\left(-\int \sin x \, dx \right) = e^{\cos x}.$$

This gives

$$\begin{aligned}
(e^{\cos x} y(x))' &= \sin x e^{\cos x} \\
e^{\cos x} y(x) &= \int \sin x e^{\cos x} \, dx + C \\
&= -e^{\cos x} + C \\
y(x) &= Ce^{\cos x} - 1.
\end{aligned}$$

f. $xy' - 2y = x^2, y(1) = 1$.

This is a linear first order differential equation. The integrating factor is

$$\mu(x) = \exp\left(-2 \int \frac{dx}{x} \right) = e^{-2\ln x} = \frac{1}{x^2}.$$

This gives

$$\begin{aligned}
\left(\frac{1}{x^2} y(x) \right)' &= \frac{1}{x} \\
y(x) &= x^2(\ln|x| + C).
\end{aligned}$$

The initial condition, $y(1) = 1$, gives $C = 1$, or

$$y(x) = x^2(\ln|x| + 1).$$

g. $\dfrac{ds}{dt} + 2s = st^2, \quad s(0) = 1.$

This is a linear first order differential equation, however it is also separable. Rewriting the problem in separable form, we have

$$\begin{aligned} \frac{ds}{dt} &= s(t^2 - 2) \\ \int \frac{ds}{s} &= \int (t^2 - 2)\, dt \\ \ln |s| &= \frac{1}{3}t^3 - 2t + C. \end{aligned}$$

Using the initial condition, $s(0) = 1$, $C = 0$. The solution satisfying this initial condition is then

$$s(t) = e^{\frac{1}{3}t^3 - 2t}.$$

h. $x' - 2x = te^{2t}$.

This is a linear first order differential equation. The integrating factor is $\mu(t) = e^{-2t}$. This gives

$$\begin{aligned} (xe^{-2t})' &= t \\ xe^{-2t} &= \frac{1}{2}t^2 + C \\ x(t) &= \left(\frac{1}{2}t^2 + C\right) e^{2t}. \end{aligned}$$

i. $\dfrac{dy}{dx} + y = \sin x$, $y(0) = 0$.

This is a linear first order differential equation. The integrating factor is $\mu(x) = e^x$. This gives

$$\begin{aligned} (y(x)e^x)' &= e^x \sin x \\ y(x)e^x &= \int e^x \sin x \, dx + C \\ &= \frac{1}{2}e^x(\sin x - \cos x) + C. \\ y(x) &= \frac{1}{2}(\sin x - \cos x) + Ce^{-x}. \end{aligned}$$

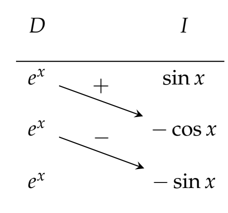


Figure 2.1: Tabular Method for Problem 1i for computing $\int e^x \sin x \, dx$.

The initial condition gives $C = \frac{1}{2}$. Thus,

$$y(x) = \frac{1}{2}(\sin x - \cos x + e^{-x}).$$

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3$, $y(1) = 4$.

The integrating factor is

$$\mu(x) = \exp\left(-3 \int \frac{dx}{x}\right) = e^{-3\ln x} = x^{-3}.$$

This gives

$$\begin{aligned} (x^{-3}y(x))' &= 1 \\ x^{-3}y(x) &= x + C \\ y(x) &= x^3(x + C). \end{aligned}$$

For $y(1) = 4$, $C = 3$. Thus, $y(x) = x^4 + 3x^3$.

2. Find all the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0$.

This is a second order, constant coefficient differential equation. The characteristic equation is $0 = r^2 - 9r + 20 = (r-4)(r-5)$. Thus, the roots are $r = 4, 5$ and the general solution is

$$y(x) = c_1 e^{4x} + c_2 e^{5x}.$$

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$.

The characteristic equation is $0 = r^2 - 3r + 4$. The roots are found as $r = \frac{3 \pm \sqrt{7}i}{2}$ and the general solution is

$$y(x) = e^{3x/2}\left[c_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right].$$

The initial conditions are $y(0) = 0$, $y'(0) = 1$. The first condition gives $c_1 = 0$. Thus, $y(x) = c_2 e^{3x/2} \sin\left(\frac{\sqrt{7}}{2}x\right)$. Noting that

$$y'(x) = c_2 e^{3x/2}\left[\frac{3}{2}\sin\left(\frac{\sqrt{7}}{2}x\right) + \frac{\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}x\right)\right],$$

we have $y'(0) = \frac{\sqrt{7}}{2}c_2 = 1$. This gives $c_2 = \frac{2}{\sqrt{7}}$ and the particular solution is given as

$$y(x) = \frac{2\sqrt{7}}{7}e^{3x/2}\sin\left(\frac{\sqrt{7}}{2}x\right).$$

c. $x^2y'' + 5xy' + 4y = 0, \quad x > 0$.

This is a second order, constant coefficient differential equation. This is a Cauchy-Euler type of differential equation. The characteristic equation is $0 = r(r-1) + 5r + 4 = r^2 + 4r + 4$. It has one real root $r = -2$. The general solution is $y(x) = x^{-2}(c_1 + c_2 \ln|x|)$.

d. $x^2y'' - 2xy' + 3y = 0, \quad x > 0$.

This is a Cauchy-Euler type of differential equation. The characteristic equation is $0 = r(r-1) - 2r + 3 = r^2 - 3r + 3$. It has complex conjugate roots $r = \frac{3}{2} \pm \frac{\sqrt{3}}{2}i$. The general solution is

$$y(x) = x^{3/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right].$$

3. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

First, we note that this equation is separable,

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y} = \frac{x}{y(1+y)}.$$

Separating variables, we find

$$\begin{aligned} \int (y + y^2)\, dy &= \int x\, dx \\ \frac{1}{2}y^2 + \frac{1}{3}y^3 &= \frac{1}{2}x^2 + C, \end{aligned}$$

or $3y^2 + 2y^3 = 3x^2 + k$.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

Inserting $x = 0$ into the implicit solution, we find $k = 5$. Yes, $3y^2 + 2y^3 = 3x^2 + 5$ is a member of the 1-parameter family.

4. The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in this chapter. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ that can be solved. Use this substitution to solve the initial value problem for $y(x)$.

Let $y(x) = xz(x)$. Then, $y' = z + xz'$. From the original equation, we have $y' = z^2 + z$. Equating these expressions, we obtain $xz' = z^2$. This is separable and can be solved,

$$\begin{aligned} \int \frac{dz}{z^2} &= \int \frac{dx}{x} \\ -\frac{1}{z} &= \ln|x| + C, \\ z(x) &= \frac{-1}{\ln|x| + C}. \end{aligned}$$

This gives,

$$y(x) = xz(x) = \frac{-x}{\ln|x| + C}.$$

Using the initial condition, $C = -1$, the particular solution becomes

$$y(x) = \frac{x}{1 - \ln|x|}.$$

5. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

The characteristic equation is $0 = r^2 - 3r + 2 = (r-1)(r-2)$. The roots are $r = 1, 2$. This gives the solution of the homogeneous equation as

$$x_h(t) = c_1 e^t + c_2 e^{2t}.$$

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

We seek the particular solution satisfying $x_p'' - 3x_p' + 2x_p = 6e^{3t}$. We guess $x_p(t) = Ae^{3t}$. Inserting this into the differential equation, $(9A - 9A + 2A)e^{3t} = 6e^{3t}$, or $A = 3$. Therefore, $x_p(t) = 3e^{3t}$.

c. Use your answers in the previous parts to write the general solution for this problem.

The general solution is

$$x(t) = x_h(t) + x_p(t) = c_1e^t + c_2e^{2t} + 3e^{3t}.$$

6. Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.

The solution to the homogeneous problem is of the same form as in the last problem, $y_h(x) = c_1e^x + c_2e^{2x}$. We obtain the particular solution of the nonhomogeneous problem from the guess $y_p(x) = A$. Inserting this guess, we obtain $A = 5$. This gives the general solution as $y(x) = c_1e^x + c_2e^{2x} + 5$.

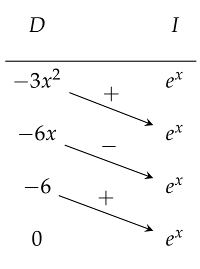


Figure 2.2: Tabular Method for Problem 6b for computing $-3\int x^2e^x\,dx$.

b. $y'' + y' = 3x^2$. Variation of Parameters.

We first solve the homogeneous problem, $y_h'' + y_h' = 0$. The characteristic equation is $0 = r^2 + r = r(r+1)$. The roots are $r = 0, -1$, giving $y_h(x) = c_1 + c_2e^{-x}$.

In order to apply the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x) + c_2(x)e^{-x}.$$

The unknown coefficients satisfy

$$\begin{aligned} c_1' + c_2'e^{-x} &= 0 \\ -c_2'e^{-x} &= 3x^2. \end{aligned}$$

Solving the second equation for c_2,

$$c_2 = -\int 3x^2e^x\,dx = (-3x^2 + 6x - 6)e^x + k_2.$$

Using the second equation for c_2' in the first equation, c_1 can be found as

$$c_1 = \int 3x^2\,dx = x^3 + k_1.$$

Inserting these results into the form for $y_p(x)$,

$$\begin{aligned} y_p(x) &= c_1(x) + c_2(x)e^{-x} \\ &= (x^3 + k_1) + (-3x^2 + 6x - 6)e^x + k_2)e^{-x} \\ &= k_1 + k_2e^{-x} + x^3 - 3x^2 + 6x - 6. \end{aligned}$$

k_1 and k_2 can be anything. Setting them to zero gives one solution $y_p(x) = x^3 - 3x^2 + 6x - 6$. Note that leaving the k's arbitrary takes care of the homogeneous part of the solution. In fact, the constant term can be absorbed into k_1, giving the general solution to the original problem as

$$y(x) = k_1 + k_2 e^{-x} + x^3 - 3x^2 + 6x.$$

7. Find the general solution of each differential equation. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

The solution to the homogeneous problem is $y_h(x) = c_1 e^x + c_2 e^{2x}$. The particular solution is found using the guess $y_p(x) = Ae^{-2x}$. Inserting this guess into the equation gives $A = \frac{5}{3}$. So, the general solution is

$$y(x) = c_1 e^x + c_2 e^{2x} + \frac{5}{3} e^{-2x}.$$

Inserting the initial conditions, we have

$$\begin{aligned} 0 &= c_1 + c_2 + \frac{5}{3}, \\ 6 &= c_1 + 2c_2 - \frac{10}{3}. \end{aligned}$$

Solving these equations, we obtain $c_1 = -\frac{38}{3}$, $c_2 = 11$. Thus, the solution to the initial value problem is

$$y(x) = -\frac{38}{3} e^x + 11 e^{2x} + \frac{5}{3} e^{-2x}.$$

b. $y'' + y = 2 \sin 3x.$

The solution to the homogeneous problem is $y_h(x) = c_1 \cos x + c_2 \sin x$. We guess a particular solution of the form $y_p(x) = A \sin 3x$ since there is no first derivative term. Then, $A = 5$. The general solution is then $y(x) = c_1 \cos x + c_2 \sin x + 5 \sin 3x$.

c. $y'' + y = 1 + 2 \cos x.$

The solution to the homogeneous problem is $y_h(x) = c_1 \cos x + c_2 \sin x$. In this problem the forcing term, $1 + 2 \cos x$, involves a solution to the homogeneous problem. Therefore, we need to use a modification of the Method of Undetermined Coefficients by making the guess $y_p(x) = A + Bx \sin x$. Computing the derivatives,

$$\begin{aligned} y_p'(x) &= B \sin x + Bx \cos x, \\ y_p''(x) &= 2B \cos x - Bx \sin x. \end{aligned}$$

Then,

$$y'' + y = A + 2B \cos x = 1 + 2 \cos x.$$

This gives $A = B = 1$. As a result, the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + 1 + x \sin x.$$

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad x > 0.$

This is a Cauchy-Euler type of differential equation. The characteristic equation is

$$0 = r(r-1) - 2r + 2 = r^2 - 3r + 2 = (r-1)(r-2).$$

The roots are $r = 1, 2$. Thus, the solution to the homogeneous problem is

$$y_h(x) = c_1 x + c_2 x^2.$$

There are two ways this could be solved. We will first use the Method of Variation of Parameters and then use a modification of the Method of Undetermined Coefficients since the forcing terms are solutions of the homogeneous problem.

Variation of Parameters

We assume

$$y_p(x) = c_1(x)x + c_2(x)x^2.$$

The unknown coefficients satisfy

$$\begin{aligned} c_1' x + c_2' x^2 &= 0 \\ c_1' + 2c_2' x &= \frac{3x^2 - x}{x^2}. \end{aligned}$$

Multiplying the second equation by x, the system becomes

$$\begin{aligned} c_1' x + c_2' x^2 &= 0 \\ c_1' x + 2c_2' x^2 &= 3x - 1. \end{aligned}$$

Eliminating c_1' from the new system, we have $c_2' = \frac{3}{x} - \frac{1}{x^2}$. Integrating, we find $c_2 = 3 \ln|x| + \frac{1}{x}$. This gives

$$c_1' = -c_2' x = -3 + \frac{1}{x}.$$

Therefore, $c_1 = -3x + \ln|x|$.

Using these results, we find

$$\begin{aligned} y_p(x) &= c_1(x)x + c_2(x)x^2 \\ &= (-3x + \ln|x|)x + (3\ln|x| + \frac{1}{x})x^2 \\ &= -3x^2 + x\ln|x| + 3x^2 \ln|x| + x. \end{aligned}$$

The first and last terms are solution of the homogeneous problem, so we can take

$$y_p(x) = x \ln|x| + 3x^2 \ln|x|$$

and write the general solution as

$$y(x) = c_1 x + c_2 x^2 + x \ln|x| + 3x^2 \ln|x|.$$

Undetermined Coefficients

The solution using the Method of Variation of Parameters suggests an approach to the modified Method of Undetermined Coefficients for Cauchy-Euler equations. We can prove this in general. Consider the problem

$$ax^2y'' + bxy' + cy = Cx^p,$$

where x^p is a solution to the homogeneous problem. Then,

$$ap(p-1) + bp + c = 0.$$

We assume a particular solution of the form $y_p(x) = Ax^p \ln|x|$. Inserting this guess into the differential equation, we obtain

$$(ap(p-1) + bp + c)Ax^p \ln|x| + (2ap - a + b)Ax^p = Cx^p.$$

Since, $ap(p-1) + bp + c = 0$, we can solve for A if $2ap - a + b \neq 0$,

$$A = \frac{C}{2ap - a + b}.$$

Now, consider the problem at hand. There are two forcing terms and due to linearity, we can make the guess $y_p(x) = (Ax + Bx^2)\ln|x|$. According to the theory just developed, we have (for $a = 1$, $b = -2$, and $C = -1, 3$ with $p = 1, 2$, respectively)

$$\begin{aligned} A &= \frac{-1}{2(1)(1) - 1 + (-2)} = 1, \\ B &= \frac{3}{2(1)(2) - 1 + (-2)} = 3. \end{aligned}$$

So, $y_p(x) = (x + 3x^2)\ln|x|$. This is the same solution as we had obtained using Variation of Parameters.

8. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4.$

Verification is simple,

$$x^2y'' - 2xy' - 4y = x^2(12x^2) - 2x(4x^3) - 4x^4 = 0.$$

For Reduction of Order, we let $y_2(x) = v(x)y_1(x)$ and determine $v(x)$. First, compute the derivatives,

$$\begin{aligned} y_2 &= x^4 v, \\ y_2' &= 4x^3 v + x^4 v', \\ y_2'' &= 12x^2 v + 8x^3 v' + x^4 v''. \end{aligned}$$

Inserting these into the differential equation, we find

$$6x^5 v' + x^6 v'' = 0.$$

This is a separable equation for $v'(x)$. Defining $z(x) = v'(x)$, we obtain a first order, separable equation,

$$6x^5 z + x^6 z' = 0.$$

Solving for z, we have

$$z = \exp\left[-6\int \frac{dx}{x}\right] = e^{-6\ln|x|} = x^{-6}, \quad x > 0.$$

A further integration gives $v(x) = -\frac{1}{5}x^{-5}$.

Finally, we see that $y_2(x) = x^4 v(x) = x^{-1}$ (up to a multiplicative constant).

b. $xy'' - y' + 4x^3 y = 0, \quad y_1(x) = \sin(x^2)$.

Let $y_2 = v\sin(x^2)$. The derivatives are given by

$$\begin{aligned} y_2' &= v'\sin(x^2) + 2xv\cos(x^2), \\ y_2'' &= v''\sin(x^2) + 4xv'\cos(x^2) - 4x^2 v\sin(x^2) + 2v\cos(x^2). \end{aligned}$$

Inserting these into the differential equation, we find

$$(x\sin(x^2))v'' + (4x^2\cos(x^2) - \sin(x^2))v' = 0.$$

Let $z = v'$. Then,

$$(x\sin(x^2))z' + (4x^2\cos(x^2) - \sin(x^2))z = 0.$$

This is a separable first order differential equation.

$$\begin{aligned} \frac{z'}{z} &= \frac{4x^2\cos(x^2) - \sin(x^2)}{x\sin(x^2)} \\ \ln z &= \int \left(\frac{1}{x} - 4x\cot(x^2)\right) dx \\ &= \ln|x| - 2\ln|\sin(x^2)| \end{aligned}$$

$$z = x(\sin(x^2))^{-2}$$

Since $z = v'$, one further integration gives $v(x)$,

$$v(x) = \int x(\sin(x^2))^{-2}\,dx = \frac{1}{2}\int \csc^2 u\,du = -\frac{1}{2}\cot^2(x^2).$$

So, up to a multiplicative constant, we have

$$y_2(x) = \sin(x^2)v(x) = \sin(x^2)\frac{\cos(x^2)}{\sin(x^2)} = \cos(x^2).$$

9. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

The linearly independent solutions of the homogeneous problem are $y(x) = \cos x, \sin x$. So, we consider $y_p(x) = c_1(x)\cos x + c_2(x)\sin x$. The coefficients satisfy the equations

$$\begin{aligned} c_1' \cos x + c_2' \sin x &= 0, \\ -c_1' \sin x + c_2' \cos x &= \tan x. \end{aligned}$$

Multiplying the first equation by $\sin x$, the second by $\cos x$, and adding the equations, we find $c_2' = \sin x$. Thus, $c_2(x) = -\cos x$.

Substituting this result into the first equation, we have $c_1' = -\tan x \sin x$. So,

$$\begin{aligned} c_1(x) &= -\int \tan x \sin x \, dx \\ &= -\int \frac{\sin^2 x}{\cos x} \, dx \\ &= -\int \frac{1 - \cos^2 x}{\cos x} \, dx \\ &= -\int (\sec x - \cos x) \, dx \\ &= -\ln|\sec x + \tan x| + \sin x. \end{aligned}$$

Thus,

$$\begin{aligned} y_p(x) &= c_1(x)\cos x + c_2(x)\sin x \\ &= (-\ln|\sec x + \tan x| + \sin x)\cos x - \cos x \sin x \\ &= -\ln|\sec x + \tan x|. \end{aligned}$$

Therefore, the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \ln|\sec x + \tan x|.$$

b. $y'' - 4y' + 4y = 6xe^{2x}$.

The homogeneous equation is a constant coefficient equation, $y'' - 4y' + 4y = 0$. The characteristic equation is $r^2 - 4r + 4 = (r-2)^2 = 0$. There is one root, $r = 2$, therefore the general solution of the homogeneous problem is

$$y_h(x) = (c_1 + c_2 x)e^{2x}.$$

We seek a particular solution of the form

$$y_p(x) = c_1(x)e^{2x} + c_2(x)xe^{2x}.$$

The coefficients satisfy the system of equations

$$\begin{aligned} c_1' e^{2x} + c_2' x e^{2x} &= 0, \\ 2c_1' e^{2x} + c_2'(1+2x)e^{2x} &= 6xe^{2x}. \end{aligned}$$

This system of equations can be solved for c_1' and c_2' using Cramer's Rule (from next chapter):

$$\begin{aligned} c_1' &= \frac{\begin{vmatrix} 0 & xe^{2x} \\ 6xe^{2x} & (1+2x)e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}} = \frac{-6x^2e^{4x}}{(1+2x)e^{4x} - 2xe^{4x}} = -6x^2, \\ c_2' &= \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 6xe^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}} = \frac{6xe^{4x}}{(1+2x)e^{4x} - 2xe^{4x}} = 6x. \end{aligned}$$

Integrating these results, gives $c_1 = -2x^3$ and $c_2 = 3x^2$. Therefore,

$$y_p(x) = -2x^3e^{2x} + (3x^2)xe^{2x} = x^3e^{2x}$$

and the general solution is

$$y(x) = (c_1 + c_2x)e^{2x} + x^3e^{2x}.$$

10. Instead of assuming that $c_1'y_1 + c_2'y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1'y_1 + c_2'y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

We begin with the nonhomogeneous equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x).$$

The solution of the homogeneous equation can be written in terms of two linearly independent solutions,

$$y_h(x) = c_1y_1(x) + c_2y_2(x).$$

Let the particular solution be

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x).$$

At this point we assume that $c_1'(x)y_1(x) + c_2'(x)y_2(x) = h(x)$, where $h(x)$ is an arbitrary function. This gives

$$\begin{aligned} y_p'(x) &= c_1(x)y_1'(x) + c_2(x)y_2'(x) + h(x) \\ y_p''(x) &= c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + h'(x). \end{aligned}$$

Now, we insert these expressions into the differential equation and use the fact that $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous problem.

Then, we have

$$\begin{aligned}
f(x) &= a(x)y_p''(x)+b(x)y_p'(x)+c(x)y_p(x)\\
&= a\left[c_1y_1''+c_2y_2''+c_1'y_1'+c_2'y_2'+h'\right]\\
&\quad +b\left[c_1y_1'+c_2y_2'+h\right]+c\left[c_1y_1+c_2y_2\right].\\
&= c_1\left[ay_1''+by_1'+cy_1\right]+c_2\left[ay_2''+by_2'+cy_2\right]\\
&\quad +a\left[c_1'y_1'+c_2'y_2'+h'\right]+b\left[h\right]\\
&= a\left[c_1'y_1'+c_2'y_2'+h'\right]+bh.
\end{aligned}$$

Thus, we have the equations

$$\begin{aligned}
c_1'y_1+c_2'y_2 &= h,\\
c_1'y_1'+c_2'y_2' &= \frac{f-bh}{a}-h'.
\end{aligned}$$

This system of equations can be solved for c_1' and c_2' using Cramer's Rule (from the next chapter):

$$\begin{aligned}
c_1' &= \frac{\begin{vmatrix} h & y_2\\ \frac{f-bh}{a}-h' & y_2'\end{vmatrix}}{\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}} = \frac{hy_2'-(\frac{f-bh}{a}-h')y_2}{y_1y_2'-y_1'y_2}\\
&= -\frac{fy_2}{aW}+\frac{bhy_2}{aW}+\frac{(hy_2)'}{aW}\\
c_2' &= \frac{\begin{vmatrix} y_1 & h\\ y_1' & \frac{f-bh}{a}-h'\end{vmatrix}}{\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}} = \frac{y_1(\frac{f-bh}{a}-h')-y_1'h}{y_1y_2'-y_1'y_2}\\
&= \frac{fy_1}{aW}-\frac{bhy_1}{aW}-\frac{(hy_1)'}{aW}
\end{aligned}$$

$$\begin{aligned}
y_p(x) &= y_1(x)\int^x\left(-\frac{fy_2}{aW}+\frac{bhy_2}{aW}+\frac{(hy_2)'}{aW}\right)d\xi\\
&\quad +y_2(x)\int^x\left(\frac{fy_1}{aW}-\frac{bhy_1}{aW}-\frac{(hy_1)'}{aW}\right)d\xi\\
&= y_2(x)\int^x\frac{f(\xi)y_1(\xi)}{a(\xi)W(\xi)}\,d\xi-y_1(x)\int^x\frac{f(\xi)y_2(\xi)}{a(\xi)W(\xi)}\,d\xi+H(x),
\end{aligned}$$

where

$$H(x)=y_1(x)\int^x\left(\frac{bhy_2}{aW}+\frac{(hy_2)'}{aW}\right)d\xi-y_2(x)\int^x\left(\frac{bhy_1}{aW}+\frac{(hy_1)'}{aW}\right)d\xi.$$

$H(x)$ depends on the arbitrary function $h(x)$ and, more importantly, is independent of $f(x)$. Therefore, when $f(x)=0$, $y_p(x)=H(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$, $H(x)=k_1y_1(x)+k_2y_2(x)$. So, we can absorb $H(x)$ into the homogeneous equation and we obtain the same solution as we had for the derivation of the Method of Variation of Parameters.

$$y(x)=c_1y_1(x)+c_2y_2(x)+y_2(x)\int^x\frac{f(\xi)y_1(\xi)}{a(\xi)W(\xi)}\,d\xi-y_1(x)\int^x\frac{f(\xi)y_2(\xi)}{a(\xi)W(\xi)}\,d\xi.$$

11. Find the solution of each initial value problem using the appropriate initial value Green's function.

For these problems we determine the initial value Green's function

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a(\xi)W(\xi)}$$

and then find the solution of the initial value problem using the integral form

$$y(x) = y_h(x) + \int_0^x G(x,\xi)f(\xi)\,d\xi,$$

where $y_1(x)$, $y_2(x)$, and $y_h(x)$ are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0, y_h(0) = y_0, y_h'(0) = v_0.$$

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

First one finds $y_1(x)$ and $y_2(x)$. The general solution to the homogeneous problem, $y'' - 3y' + 2y = 0$, is $y_h(x) = c_1e^x + c_2e^{2x}$. Requiring $y_1(0) = 0$ and $y_2'(0) = 0$, we obtain

$$y_1(x) = e^x - e^{2x} \quad \text{and} \quad y_2(x) = e^{2x} - 2e^x.$$

The Wronskian is given by

$$\begin{aligned} W(y_1,y_2) &= y_1y_2' - y_1'y_2 \\ &= (e^x - e^{2x})(2e^{2x} - 2e^x) - (e^x - 2e^{2x})(e^{2x} - 2e^x) \\ &= -e^{3x}. \end{aligned}$$

and $a(x) = 1$.

We also need the solution of the homogeneous problem, $y_h(x)$, which satisfies the given initial conditions, $y(0) = 0$, $y'(0) = 6$. The first condition gives $y_h(x) = c_1(e^x - e^{2x})$. For the second condition, we have

$$y_h'(0) = c_1(e^0 - 2e^{2(0)}) = 6.$$

Therefore, $c_1 = -6$ and $y_h(x) = 6(e^{2x} - e^x)$.

We construct the Green's function,

$$\begin{aligned} G(x,\xi) &= \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a(\xi)W(\xi)} \\ &= .\frac{(e^\xi - e^{2\xi})(e^{2x} - 2e^x) - (e^x - e^{2x})(e^{2\xi} - 2e^\xi)}{-e^{3x}} \\ &= e^{2(x-\xi)} - e^{x-\xi}. \end{aligned}$$

Finally, we obtain the particular solution through integration:

$$\begin{aligned} y_p(x) &= \int_0^x G(x,\xi)f(\xi)\,d\xi \\ &= \int_0^x [e^{2(x-\xi)} - e^{x-\xi}](20e^{-2\xi})\,d\xi \\ &= 20\int_0^x (e^{2x}e^{-4\xi} - e^xe^{-3\xi})\,d\xi \\ &= \frac{5}{3}e^{-2x} + 5e^{2x} - \frac{20}{3}e^x. \end{aligned}$$

The solution to the original problem is

$$\begin{aligned} y(x) &= 6(e^{2x} - e^x) + 5e^{2x} - \frac{20}{3}e^x + \frac{5}{3}e^{-2x} \\ &= 11e^{2x} - \frac{38}{3}e^x + \frac{5}{3}e^{-2x}. \end{aligned}$$

b. $y'' + y = 2\sin 3x, \quad y(0) = 5, \quad y'(0) = 0.$

First one finds $y_1(x)$ and $y_2(x)$. The general solution to the homogeneous problem, $y'' + y = 0$, is $y_h(x) = c_1 \cos x + c_2 \sin x$. Requiring $y_1(0) = 0$ and $y_2'(0) = 0$, we obtain $y_1(x) = \cos x$ and $y_2(x) = \sin x$. The Wronskian is given by

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = -1$$

and $a(x) = 1$.

We also need $y_h(x)$ to satisfy the initial conditions, $y(0) = 5, y'(0) = 0$. The solution is easily found as $y_h(x) = 5\cos x$.

We construct the Green's function,

$$\begin{aligned} G(x,\xi) &= \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a(\xi)W(\xi)} \\ &= \cos\xi \sin x - \cos x \sin\xi = \sin(x - \xi). \end{aligned}$$

Finally, we obtain the particular solution through integration:

$$\begin{aligned} y_p(x) &= \int_0^x G(x,\xi) f(\xi)\, d\xi \\ &= \int_0^x [\cos\xi \sin x - \cos x \sin\xi](2\sin 3\xi)\, d\xi \\ &= 2\int_0^x [\cos\xi \sin x - \cos x \sin\xi][3\sin\xi - 4\sin^3\xi]\, d\xi \\ &= (3\sin^2 x - 2\sin^4 x)\sin x - 2\sin^3 x \cos^2 x \\ &= \sin^3 x. \end{aligned}$$

The solution to the original problem is $y(x) = 5\cos x + \sin^3 x$.

c. $y'' + y = 1 + 2\cos x, \quad y(0) = 2, \quad y'(0) = 0.$

This problem is similar to the last problem. The Green's function is

$$G(x,\xi) = \cos\xi \sin x - \cos x \sin\xi = \sin(x - \xi).$$

The particular solution is then

$$\begin{aligned} y_p(x) &= \int_0^x G(x,\xi) f(\xi)\, d\xi \\ &= \int_0^x [\cos\xi \sin x - \cos x \sin\xi](1 + 2\cos\xi)\, d\xi \\ &= \int_0^x [\cos\xi \sin x - \cos x \sin\xi]\, d\xi \\ &\quad +2\int_0^x [\cos^2\xi \sin x - \cos x \sin\xi \cos\xi]\, d\xi \end{aligned}$$

$$
\begin{aligned}
&= \left[\sin\xi \sin x + \cos\xi \cos x + (\xi + \frac{\sin 2\xi}{2}) \sin x + \cos^2\xi \cos x\right]_0^x \\
&= (1 + x\sin x + \sin^2 x \cos x + \cos^3 x) - 2\cos x \\
&= 1 + x\sin x - \cos x.
\end{aligned}
$$

The solution to the homogeneous problem $[y'' + y = 0, y(0) = 2, y'(0) = 0]$ is given by $y_h(x) = 2\cos x$. This gives the solution of the nonhomogeneous initial value problem as $y(x) = 1 + \cos x + x\sin x$.

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad y(1) = \pi, \quad y'(1) = 0.$

We first determine the solution of the Cauchy-Euler equation, $x^2y_h'' - 2xy_h' + 2y_h = 0$. The characteristic equation is

$$0 = r(r-1) - 2r + 2 = r^2 - 3r + 2 = (r-2)(r-1).$$

Thus, $y_h(x) = c_1x + c_2x^2$. We need $y_h(x)$ to satisfy the given initial conditions, $y(1) = \pi, y'(1) = 0$. Therefore,

$$
\begin{aligned}
\pi &= c_1 + c_2 \\
0 &= c_1 + 2c_2.
\end{aligned}
$$

Subtracting equations, we find $c_2 = -\pi = -c_1/2$. So,

$$y_h(x) = \pi(2x - x^2).$$

The solution $y_1(x) = x^2 - x$ satisfies $y_1(1) = 0$. For $y_2'(1) = 0$, we consider $y_2(x) = c_1x + c_2x^2$. Then, $y_2'(x) = c_1 + 2c_2x$ and this gives $y_2'(1) = c_1 + 2c_2 = 0$. Choosing $c_2 = 1$ and $c_1 = -2$, we obtain $y_2(x) = x^2 - 2x$.

The Wronskian is given by

$$
\begin{aligned}
W(y_1, y_2) &= y_1y_2' - y_1'y_2 \\
&= (x^2 - x)(2x - 2) - (2x - 1)(x^2 - 2x) \\
&= x^2.
\end{aligned}
$$

and $a(x) = x^2$.

We construct the Green's function,

$$
\begin{aligned}
G(x,\xi) &= \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a(\xi)W(\xi)} \\
&= \frac{(\xi^2 - \xi)(x^2 - 2x) - (x^2 - x)(\xi^2 - 2\xi)}{\xi^4}. \\
&= \frac{\xi x^2 - x\xi^2}{\xi^4} = \frac{x(x - \xi)}{\xi^3}
\end{aligned}
$$

Finally, we obtain the particular solution through integration:

$$
\begin{aligned}
y_p(x) &= \int_1^x G(x,\xi) f(\xi)\, d\xi \\
&= \int_1^x \left[\frac{x(x - \xi)}{\xi^3}\right] (3\xi^2 - \xi)\, d\xi
\end{aligned}
$$

$$\begin{aligned}
&= x^2 \int_1^x \left(\frac{3}{\xi} - \frac{1}{\xi^2}\right) d\xi - x \int_1^x \left(3 - \frac{1}{\xi}\right) d\xi \\
&= \left[\left(3\ln|\xi| + \frac{1}{\xi}\right) x^2 - (3\xi - \ln|\xi|)\, x\right]_1^x \\
&= 4x - 4x^2 + (3x^2 + x)\ln|x|.
\end{aligned}$$

The solution to the original problem is

$$y(x) = \pi(2x - x^2) + 4(x - x^2) + (3x^2 + x)\ln|x|.$$

12. Use the initial value Green's function for $x'' + x = f(t)$, $x(0) = 4$, $x'(0) = 0$, to solve the following problems.

As noted in Problem 11b, the initial value Green's function is given by

$$G(t, \tau) = \cos\tau \sin t - \cos t \sin\tau.$$

The corresponding solution of the homogeneous problem is $y_h(t) = 4\cos t$. For the problems below we only need to find the particular solutions.

a. $x'' + x = 5t^2$.

$$\begin{aligned}
y_p(t) &= \int_0^t G(t,\tau) f(\tau)\, d\tau \\
&= \int_0^t (\cos\tau \sin t - \cos t \sin\tau)(5\tau^2)\, d\tau \\
&= 5[(\tau^2 \sin\tau - 2\sin\tau + 2\tau\cos\tau)\sin t]_0^t \\
&\quad -[(-\tau^2\cos\tau + 2\cos\tau + 2\tau\sin\tau)\cos t]_0^t \\
&= 10\cos t - 10 + 5t^2.
\end{aligned}$$

The solution is $y(t) = 14\cos t - 10 + 5t^2$.

b. $x'' + x = 2\tan t$.

$$\begin{aligned}
y_p(t) &= \int_0^t G(t,\tau) f(\tau)\, d\tau \\
&= \int_0^t (\cos\tau \sin t - \cos t \sin\tau)(2\tan\tau)\, d\tau \\
&= 2\int_0^t \left(\sin\tau \sin t - \frac{\sin^2\tau}{\cos\tau}\cos t\right) d\tau \\
&= 2\int_0^t (\sin\tau\sin t - (\sec\tau - \cos\tau)\cos t)\ d\tau \\
&= 2\left[-\cos\tau\sin t - (\ln|\sec\tau + \tan\tau| - \sin\tau)\cos t\right]_0^t \\
&= -2\ln|\sec t + \tan t|\cos t + 2\sin t
\end{aligned}$$

The solution to the nonhomogeneous initial value problem is
$y(t) = 4\cos t + 2\sin t - 2\cos t \ln|\sec t - \tan t|$.

13. For the problem $y'' - k^2 y = f(x)$, $y(0) = 0$, $y'(0) = 1$,

The general solution to the homogeneous problem, $y'' - k^2 y = 0$, is $y_h(x) = c_1 e^{kx} + c_2 e^{-kx}$. The solution satisfying the given initial conditions has to satisfy

$$c_1 + c_2 = 0, \quad k(c_1 - c_2) = 1.$$

Therefore $c_1 = -c_2 = 1/2k$. So, $y_h(x) = \frac{1}{2k}(e^{kx} - e^{-kx}) = \frac{1}{k}\sinh kx$.

a. Find the initial value Green's function.

The linearly independent solutions needed for construction of the Green's function are $y_1(x) = \sinh kx$ and $y_2(x) = \cosh kx$. The Wronskian of these solutions is given by

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_1' y_2 \\ &= (\sinh kx)(k \sinh kx) - (k \cosh kx)(\cosh kx) \\ &= k(\sinh^2 kx - \cosh^2 kx) = -k. \end{aligned}$$

This gives the Green's function as

$$\begin{aligned} G(x, \xi) &= \frac{y_1(\xi) y_2(x) - y_1(x) y_2(\xi)}{a(\xi) W(\xi)} \\ &= \frac{(\sinh k\xi)(\cosh kx) - (\sinh kx)(\cosh k\xi)}{-k} = \frac{\sinh k(x - \xi)}{k}. \end{aligned}$$

b. Use the Green's function to solve $y'' - y = e^{-x}$.

For this problem, $k = 1$ and $y_h(x) = \sinh x$. The particular solution is

$$\begin{aligned} y_p(x) &= \int_0^x G(x, \xi) f(\xi)\, d\xi \\ &= \int_0^x \sinh(x - \xi) e^{-\xi}\, d\xi \\ &= \frac{1}{2} \int_0^x (e^{(x-\xi)} - e^{(\xi - x)}) e^{-\xi}\, d\xi \\ &= \frac{1}{2} \int_0^x (e^{(x-2\xi)} - e^{-x})\, d\xi \\ &= \frac{1}{2} \left(\frac{e^{(x-2\xi)}}{-2} - \xi e^{-x} \right) \Bigg|_0^x \\ &= \frac{1}{2} \left(\frac{e^{-x}}{-2} - x e^{-x} + \frac{e^x}{2} \right) \\ &= \frac{1}{2} \sinh x - \frac{1}{2} x e^{-x}. \end{aligned}$$

Then, the general solution is $y(x) = \frac{3}{2} - \frac{1}{2} x e^{-x}$.

c. Use the Green's function to solve $y'' - 4y = e^{2x}$.

For this problem, $k = 2$ and $y_h(x) = \frac{1}{2}\sinh 2x$. The particular solution is

$$y_p(x) = \int_0^x G(x, \xi) f(\xi)\, d\xi$$

$$\begin{aligned}
&= \frac{1}{2}\int_0^x \sinh 2(x-\xi)e^{2\xi}\,d\xi \\
&= \frac{1}{4}\int_0^x (e^{2(x-\xi)} - e^{2(\xi-x)})e^{2\xi}\,d\xi \\
&= \frac{1}{4}\int_0^x (e^{2x} - e^{4\xi-2x})\,d\xi \\
&= \frac{1}{4}\left(\xi e^{2x} - \frac{e^{4\xi-2x}}{4}\right)_0^x \\
&= \frac{1}{4}\left(xe^{2x} - \frac{e^{2x}}{4} + \frac{e^{-2x}}{4}\right) \\
&= \frac{1}{4}xe^{2x} - \frac{1}{8}\sinh 2x.
\end{aligned}$$

Then, the general solution is $y(x) = \frac{1}{4}xe^{2x} + \frac{3}{8}\sinh 2x$.

14. Find and use the initial value Green's function to solve

$$x^2y'' + 3xy' - 15y = x^4e^x,$$

$y(1) = 1$, $y'(1) = 0$.

We first need the solution of $x^2y_h'' + 3xy_h' - 15y_h = 0$. The characteristic equation is

$$0 = r(r-1) + 3r - 15 = r^2 + 2r - 15 = (r+5)(r-3).$$

Thus, $y_h(x) = c_1x^3 + c_2x^{-5}$. We need $y_h(x)$ to satisfy the given initial conditions, $y(1) = 1$ and $y'(1) = 0$. Therefore,

$$\begin{aligned}
1 &= c_1 + c_2 \\
0 &= 3c_1 - 5c_2.
\end{aligned}$$

Since $c_2 = \frac{3}{5}c_1$,

$$1 = c_1 + c_2 = \frac{8}{5}c_1,$$

or $c_1 = \frac{5}{8}$ and $c_2 = \frac{3}{8}$. So,

$$y_h(x) = \frac{5}{8}x^3 + \frac{3}{8}x^{-5}.$$

Next, we construct the Green's function. The solution $y_1(x) = x^3 - x^{-5}$ satisfies $y_1(1) = 0$. For $y_2'(1) = 0$, we consider $y_2(x) = c_1x^3 + c_2x^{-5}$. Then, $y_2'(x) = 3c_1x^2 - 5c_2x^{-6}$ and this gives $y_2'(x) = 3c_1 - 5c_2 = 0$ Choosing $c_2 = 3$ and $c_1 = 5$, we obtain $y_2(x) = 5x^3 + 3x^{-5}$.

The Wronskian is given by

$$\begin{aligned}
W(y_1, y_2) &= y_1y_2' - y_1'y_2 \\
&= (x^3 - x^{-5})(15x^2 - 15x^{-6}) - (3x^2 + 5x^{-6})(5x^3 + 3x^{-5}) \\
&= -64x^{-3}.
\end{aligned}$$

and $a(x) = x^2$.

D		I
ξ^8	$+$	e^ξ
$8\xi^7$	$-$	e^ξ
$56\xi^6$	$+$	e^ξ
$336\xi^5$	$-$	e^ξ
$1680\xi^4$	$+$	e^ξ
$6720\xi^3$	$-$	e^ξ
$20160\xi^2$	$+$	e^ξ
40320ξ	$-$	e^ξ
40320	$+$	e^ξ
0		e^ξ

Figure 2.3: Tabular Method for the integral $\int \xi^8 e^\xi\, d\xi$ in Problem 14.

The Green's function can now be found. We have

$$\begin{aligned} G(x,\xi) &= \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a(\xi)W(\xi)} \\ &= \frac{(\xi^3 - \xi^{-5})(5x^3 + 3x^{-5}) - (x^3 - x^{-5})(5\xi^3 + 3\xi^{-5})}{\xi^2(-64\xi^{-3})}. \\ &= \frac{1}{8}\left(\frac{x^3}{\xi^4} - \frac{\xi^4}{x^5}\right). \end{aligned}$$

Finally, we obtain the particular solution through integration:

$$\begin{aligned} y_p(x) &= \int_1^x G(x,\xi) f(\xi)\, d\xi \\ &= \frac{1}{8}\int_1^x \left(\frac{x^3}{\xi^4} - \frac{\xi^4}{x^5}\right)\xi^4 e^\xi\, d\xi \\ &= \frac{1}{8}\int_1^x \left(x^3 - \frac{\xi^8}{x^5}\right) e^\xi\, d\xi \\ &= \frac{1}{8}x^3 e^\xi\Big|_1^x - \frac{1}{8x^5}\int_1^x \xi^8 e^\xi\, d\xi. \end{aligned}$$

From Table 2.3 we have

$$\begin{aligned} \int_1^x \xi^8 e^\xi\, d\xi &= (40320 - 40320\xi + 20160\xi^2 - 6720\xi^3 \\ &\quad +1680\xi^4 - 336\xi^5 + 56\xi^6 - 8\xi^7 + \xi^8)e^\xi\Big|_1^x \\ &= (40320 - 40320x + 20160x^2 - 6720x^3 \\ &\quad +1680x^4 - 336x^5 + 56x^6 - 8x^7 + x^8)e^x - 14833e. \end{aligned}$$

Then, $y_p(x)$

$$\begin{aligned} &= \frac{1}{8}x^3 e^\xi\Big|_1^x - \frac{1}{8x^5}\int_1^x \xi^8 e^\xi\, d\xi \\ &= \frac{1}{8}x^3(e^x - e) - \frac{1}{8x^5}[(40320 - 40320x + 20160x^2 - 6720x^3 \\ &\quad +1680x^4 - 336x^5 + 56x^6 - 8x^7 + x^8)e^x - 14833e] \\ &= \frac{14833e}{8x^5} - \frac{e}{8}x^3 + (-5040x^{-5} + 5040x^{-4} - 2520x^{-3} + 840x^{-2} \\ &\quad -210x^{-1} + 42 - 7x + x^2)e^x. \end{aligned}$$

The solution to the original problem is

$$\begin{aligned} y(x) &= \frac{5}{8}x^3 + \frac{3}{8}x^{-5} + y_p(x) \\ &= \frac{3 + 14833e}{8x^5} + \frac{5-e}{8}x^3 + (-5040x^{-5} + 5040x^{-4} - 2520x^{-3} \\ &\quad +840x^{-2} - 210x^{-1} + 42 - 7x + x^2)e^x. \end{aligned}$$

15. A ball is thrown upward with an initial velocity of 49 m/s from 539 m high. How high does the ball get, and how long does in take before it hits the ground? [Use results from the simple free fall problem, $y'' = -g$.]

Starting with $y(t) = -4.9t^2 + 49t + 539$ m, one can compute the time to reach maximum height ($v = 0$)

$$0 = \frac{dy}{dt} = -9.8t + 49,$$

or $t = 5.0$ s. The height at this time is $y(5.0) = 760$ m.

The time in flight is twice the time it takes to reach maximum height. Thus, the ball returns in 10 s.

16. Consider the case of free fall with a damping force proportional to the velocity, $f_D = \pm kv$ with $k = 0.1$ kg/s.

a. Using the correct sign, consider a 50 kg mass falling from rest at a height of 100 m. Find the velocity as a function of time. Does the mass reach terminal velocity?

We start with $\dot{v} = -g - \alpha v$, where $\alpha = \frac{k}{m} > 0$. This equation is separable, leading to

$$\begin{aligned} \int \frac{dv}{g + \alpha v} &= -t + C, \\ \frac{1}{\alpha} \ln |g + \alpha v| &= -t + C, \\ g + \alpha v &= Ae^{-\alpha t}, \quad A = e^C, \end{aligned}$$

Using the initial condition $v(0) = 0$, we find $A = g$. This gives the solution

$$v(t) = -\frac{g}{\alpha}(1 - e^{-\alpha t}).$$

As $t \to \infty$, $v \to -\frac{g}{\alpha}$.

Using the numbers in the problem, we have

$$v(t) = -(98\text{m/s})(1 - e^{-0.002t})$$

and $v_{\text{term}} = -98$ m/s.

b. Let the mass be thrown upward from the ground with an initial speed of 50 m/s. Find the velocity as a function of time as it travels upward and then falls to the ground. How high does the mass get? What is its speed when it returns to the ground?

As with the first part of the problem, we start with $\dot{v} = -g - \alpha v$, where $\alpha = \frac{k}{m} > 0$. After integrating, we found

$$g + \alpha v = Ae^{-\alpha t}.$$

In this case we have $v(0) = v_0$. Thus, $A = g + \alpha v_0$ and we can solve for the velocity as a function of time,

$$v(t) = -\frac{g}{\alpha} + \left(\frac{g}{\alpha} + v_0\right) e^{-\alpha t}.$$

Further integration gives the position as a function of time,

$$y(t) = y_0 - \frac{g}{\alpha} - \frac{1}{\alpha}(\frac{g}{\alpha} + v_0)e^{-\alpha t}.$$

For $y(0) = 0$, $y_0 = \frac{g+\alpha v_0}{\alpha^2}$ and

$$y(t) = \frac{g+\alpha v_0}{\alpha^2}(1-e^{-\alpha t}) - \frac{gt}{\alpha}.$$

The maximum height occurs for $v(t) = 0$. Solving for T, we find

$$\begin{aligned} -\frac{g}{\alpha} + \left(\frac{g}{\alpha} + v_0\right) e^{-\alpha T} &= 0, \\ \left(\frac{g}{\alpha} + v_0\right) e^{-\alpha T} &= \frac{g}{\alpha}, \\ e^{\alpha T} &= \left(1 + \frac{\alpha v_0}{g}\right), \\ T &= \frac{1}{\alpha} \ln \left(1 + \frac{\alpha v_0}{g}\right). \end{aligned}$$

Inserting this into $y(t)$, we have

$$y(T) = \frac{v_0}{\alpha} - \frac{g}{\alpha^2} \ln \left(1 + \frac{\alpha v_0}{g}\right).$$

Using the values in the problem, we can determine the time to the maximum as $T = 5.076$ s leading to a height $y(5.076) = 127$ m.

We also need the time it takes to return to the ground. Thus, we seek $y(\tau) = 0$. This equation is transcendental and one needs technology to solve for this. The value obtained is $\tau = 10.17$ s. The speed at this time is then $v(10.17) = -49.7$ m/s.

17. A piece of a satellite falls to the ground from a height of 10,000 m. Ignoring air resistance, find the height as a function of time. [Hint: For free fall from large distances,

$$\ddot{h} = -\frac{GM}{(R+h)^2}.$$

Multiplying both sides by $\dot{h}$, show that

$$\frac{d}{dt}\left(\frac{1}{2}\dot{h}^2\right) = \frac{d}{dt}\left(\frac{GM}{R+h}\right).$$

Integrate and solve for $\dot{h}$. Further integrating gives $h(t)$.]

We begin with

$$\ddot{h} = -\frac{GM}{(R+h)^2}$$

and multiply both sides by $\dot{h}$,

$$\begin{aligned} \ddot{h}\dot{h} &= -\frac{GM}{(R+h)^2}\dot{h} \\ \frac{d}{dt}\left(\frac{1}{2}\dot{h}^2\right) &= \frac{d}{dt}\left(\frac{GM}{R+h}\right). \end{aligned}$$

We can integrate and solve for $\dot{h}$.

$$\frac{1}{2}\dot{h}^2(t) - \cancel{\frac{1}{2}\dot{h}^2(0)}^{=0} = \frac{GM}{R+h}.$$

$$\dot{h} = -\sqrt{\frac{2GM}{R+h}}.$$

Here we note that the velocity is negative for falling bodies.

This equation is separable and can be integrated.

$$\begin{aligned} \int \sqrt{R+h}\, dh &= -\sqrt{2GM} \int dt \\ \frac{2}{3}(R+h)^{3/2} &= -\sqrt{2GM}t + C. \end{aligned}$$

At $t = 0$, $h(0) = h_0 = 10,000$ m. So, $C = \frac{2}{3}(R+h_0)^{3/2}$. Thus,

$$\frac{2}{3}(R+h)^{3/2} = \frac{2}{3}(R+h_0)^{3/2} - \sqrt{2GM}t,$$

or

$$h(t) = \left[(R+h_0)^{3/2} - \frac{3}{2}\sqrt{2GM}t\right]^{2/3} - R,$$

18. The problem of growth and decay is stated as follows: The rate of change of a quantity is proportional to the quantity. The differential equation for such a problem is

$$\frac{dy}{dt} = \pm ky.$$

The solution of this growth and decay problem is $y(t) = y_0 e^{\pm kt}$. Use this solution to answer the following questions if 40 percent of a radioactive substance disappears in 100 years.

Before answering the questions, one can find the decay constant. Since 60% is left after 100 years, $0.6y_0 = y_0 e^{-100k}$. Therefore, $k = -\frac{\ln 0.6}{100} = .0051$.

a. What is the half-life of the substance?

The half-life is the time, τ, for which 50% of the initial substance decays. Thus, $0.5y_0 = y_0 e^{-k\tau}$. Therefore, $-k\tau = \ln 0.5$, or

$$\tau = -\frac{\ln 0.5}{k} = \frac{ln 2}{k} = 136 \text{ yr}.$$

b. After how many years will 90% be gone?

For 90% gone, there is 10% left. Therefore, we need to solve $0.1y_0 = y_0 e^{-kt}$ for t. The result is $t = -\frac{\ln 0.1}{k} = 451$ yr.

19. A spring fixed at its upper end is stretched 6 inches by a 10-pound weight attached at its lower end. The spring-mass system is suspended in a viscous medium so that the system is subjected to a damping force of $5\frac{dx}{dt}$ lbs. Describe the motion of the system if the weight is drawn down an additional 4 inches and released. What would happen if you changed the coefficient "5" to "4"? [You may need to consult your introductory physics text.]

The key equation governing the oscillation of the mass on the spring is

$$m\ddot{x} + b\dot{x} + kx = 0.$$

We need to determine the constants in the equation.

First, we note that adding the block to the spring allows one to determine the spring constant from the equilibrium equation

$$mg = kx.$$

In this problem one needs to be careful with units. The mass is not 10-pounds. That is mg. So,

$$k = \frac{mg}{x} = \frac{10\text{lb}}{0.5ft} = 20\text{lb/ft}.$$

The mass is given by $m = \frac{mg}{g}$. In these units the mass is in slugs and $g = 32.2$ m/s^2. [Note that if one takes $g = 32$ m/s^2, then the qualitative answer will be off for the first part of the problem.] So, $m = 0.311$ slugs.

When the damping force is $5\frac{dx}{dt}$ lbs, we have

$$b^2 - 4km = 5^2 - 4(20)(.311) > 0.$$

Therefore, the system is overdamped and the mass's motion will decay monotonically to zero.

When the damping force is $4\frac{dx}{dt}$ lbs, we have

$$b^2 - 4km = 4^2 - 4(20)(.311) < 0.$$

Therefore, the system is underdamped and the mass will oscillate with a decreasing amplitude.

20. Consider an LRC circuit with $L = 1.00$ H, $R = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ F, and $V = 1.00 \times 10^3$ V. Suppose that no charge is present and no current is flowing at time $t = 0$ when a battery of voltage V is inserted. Find the current and the charge on the capacitor as functions of time. Describe how the system behaves over time.

In this problem we need to solve the equation

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

The given values lead to the initial value problem

$$\ddot{q} + 100\dot{q} + 10000q = 1000, \quad q(0) = \dot{q}(0) = 0.$$

The solution of the homogeneous problem: The roots of the characteristic equation are

$$r = \frac{-100 \pm \sqrt{10^4 - 4 \times 10^4}}{2} = -50 \pm 50\sqrt{3}i.$$

Therefore, the solution is given by

$$q_h(t) = (c_1 \cos 50\sqrt{3}t + c_2 \sin 50\sqrt{3}t)e^{-50t}.$$

The particular solution is relatively simple using the Method of Undetermined Coefficients. Let $q_p(t) = A$. Then, we find $y_p(t) = 0.10$. This gives the general solution to the original problem as

$$q(t) = (c_1 \cos 50\sqrt{3}t + c_2 \sin 50\sqrt{3}t)e^{-50t} + \frac{1}{10}.$$

Requiring $q(0) = 0$, gives $C_1 = -0.1$ C. Noting that

$$\dot{q}(t) = 50\left[\sqrt{3}(c_2 \cos 50\sqrt{3}t - c_1 \sin 50\sqrt{3}t) - (c_1 \cos 50\sqrt{3}t + c_2 \sin 50\sqrt{3}t)\right]e^{-50t}.$$

and

$$\dot{q}(0) = 50(\sqrt{3}(c_2) - (c_1)) = 0,$$

we find $c_2 = c_1/sqrt3 = -\sqrt{3}/30$.

So,

$$\begin{aligned} q(t) &= \frac{1}{10} - (\frac{1}{10}\cos 50\sqrt{3}t + \frac{\sqrt{3}}{10}\sin 50\sqrt{3}t)e^{-50t}. \\ I(t) &= \frac{20\sqrt{3}}{3}e^{-50t}\sin 50\sqrt{3}t. \end{aligned}$$

In Figure 2.4 we show the behavior of these solutions. We see that as $t \to \infty$, $q(t) \to 0.1$ C and $i(t) \to 0$ A.

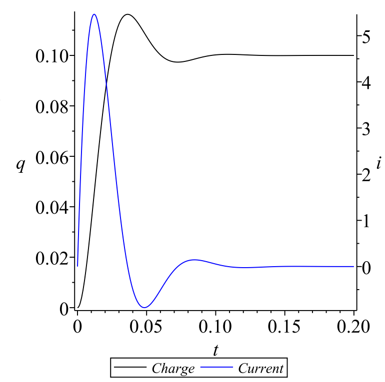


Figure 2.4: Plots of the charge and the current as functions of time for Problem 20.

21. Consider the problem of forced oscillations as described in Section 2.7.2.

a. Derive the general solution in Equation (2.77).

The problem is given by $\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$. In the text we solved the $\omega \neq \omega_0$ case. So, we consider the problem

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t.$$

The solution to the homogeneous problem was found as

$$x_h(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

Because the driving term is a solution of the homogeneous problem, we need to use the Modified Method of Undetermined Coefficients. The guess $x_p(t)$ and its derivatives are given by

$$\begin{aligned} x_p(t) &= t(A\cos\omega_0 t + B\sin\omega_0 t), \\ \dot{x}_p(t) &= (A\cos\omega_0 t + B\sin\omega_0 t) + \omega_0 t(-A\sin\omega_0 t + B\cos\omega_0 t), \\ \ddot{x}_p(t) &= 2\omega_0(-A\sin\omega_0 t + B\cos\omega_0 t) - \omega_0^2 t(A\cos\omega_0 t + B\sin\omega_0 t). \end{aligned}$$

Then,

$$\ddot{x_p} + \omega_0^2 x_p = 2\omega_0(-A\sin\omega_0 t + B\cos\omega_0 t) = \frac{F_0}{m}\cos\omega t.$$

From this result we see that $A = 0$ and $B = \frac{F_0}{2m\omega_0}$. Therefore, the solution for this case is

$$x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0}{2m\omega_0} t\sin\omega_0 t.$$

b. Plot the solutions in Equation (2.77) for the following cases: Let $c_1 = 0.5$, $c_2 = 0$, $F_0 = 1.0$ N, and $m = 1.0$ kg for $t \in [0, 100]$.

i. $\omega_0 = 2.0$ rad/s, $\omega = 0.1$ rad/s.

ii. $\omega_0 = 2.0$ rad/s, $\omega = 0.5$ rad/s.

iii. $\omega_0 = 2.0$ rad/s, $\omega = 1.5$ rad/s.

iv. $\omega_0 = 2.0$ rad/s, $\omega = 2.2$ rad/s.

v. $\omega_0 = 1.0$ rad/s, $\omega = 1.2$ rad/s.

vi. $\omega_0 = 1.5$ rad/s, $\omega = 1.5$ rad/s.

In this problem we plot the solution

$$x(t) = 0.5 \cos \omega_0 t + \begin{cases} \dfrac{1}{(\omega_0^2 - \omega^2)} \cos \omega t, & \omega \neq \omega_0, \\ \dfrac{1}{2\omega_0} t \sin \omega_0 t, & \omega = \omega_0. \end{cases}$$

For the different cases we plot the solutions in Figures 2.5-2.10

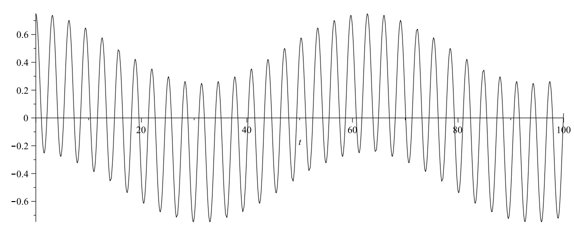


Figure 2.5: Plot from Problem 21b [i.] $\omega_0 = 2$ rad/s, $\omega = 0.1$ rad/s.

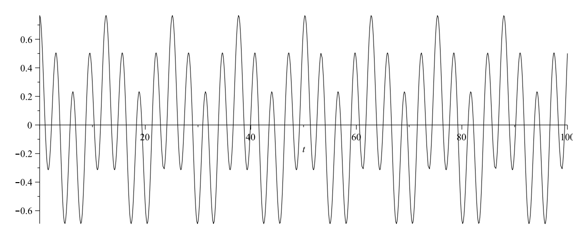


Figure 2.6: Plot from Problem 21b [ii.] $\omega_0 = 2$ rad/s, $\omega = 0.5$ rad/s.

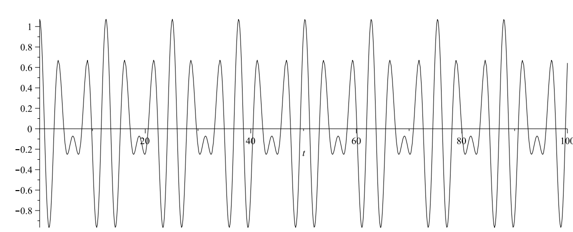


Figure 2.7: Plot from Problem 21b [iii.] $\omega_0 = 2$ rad/s, $\omega = 1.5$ rad/s.

c. Derive the form in Equation (2.78).

We consider the case that $\omega \neq \omega_0$, and choose initial conditions such that $c_1 = -F_0/(m(\omega_0^2 - \omega^2))$, $c_2 = 0$.

$$\begin{aligned} x(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t - \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega_0 t \\ &= \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \end{aligned}$$

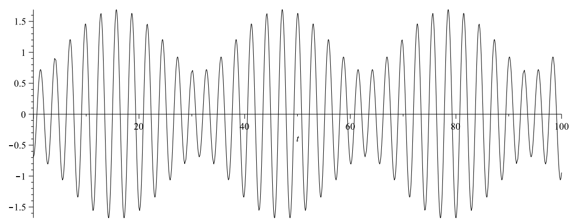


Figure 2.8: Plot from Problem 21b [iv.] $\omega_0 = 2$ rad/s, $\omega = 2.2$ rad/s.

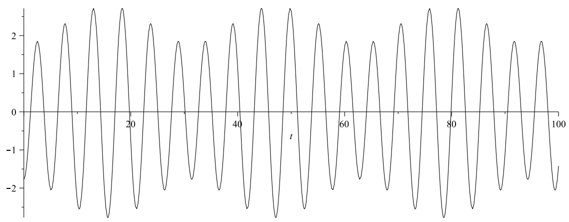


Figure 2.9: Plot from Problem 21b [v.] $\omega_0 = 1$ rad/s, $\omega = 1.2$ rad/s.

$$
\begin{aligned}
&= \frac{F_0}{m(\omega_0^2-\omega^2)}\left[\cos\left(\frac{(\omega_0+\omega)t}{2}-\frac{(\omega_0-\omega)t}{2}\right)-\cos\left(\frac{(\omega_0-\omega)t}{2}+\frac{(\omega_0+\omega)t}{2}\right)\right]\\
&= \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}\sin\frac{(\omega_0+\omega)t}{2}.
\end{aligned}
$$

d. Confirm that the solution in Equation (2.78) is the same as the solution in Equation (2.77) for $F_0 = 2.0$ N, $m = 10.0$ kg, $\omega_0 = 1.5$ rad/s, and $\omega = 1.25$ rad/s, by plotting both solutions for $t \in [0, 100]$.

Using these values, the functions obtained are

$$x(t) = 0.581\sin(0.125t)\sin(1.375t) \quad \text{and} \quad x(t) = 0.290(\cos(1.25t) - \cos(1.5t)).$$

These solutions are the same and plots of these solutions give the plot as in Figure **??**.

22. A certain model of the motion of a light plastic ball tossed into the air is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here m is the mass of the ball, g=9.8 m/s^2 is the acceleration due to gravity and c is a measure of the damping. Since there is no x term, we can write this as a first order equation for the velocity $v(t) = x'(t)$:

$$mv' + cv + mg = 0.$$

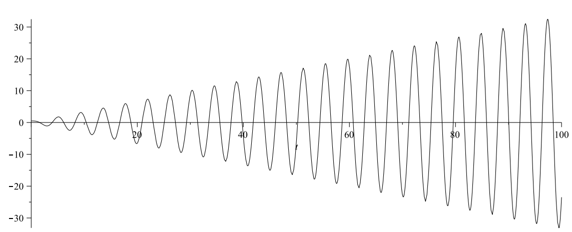
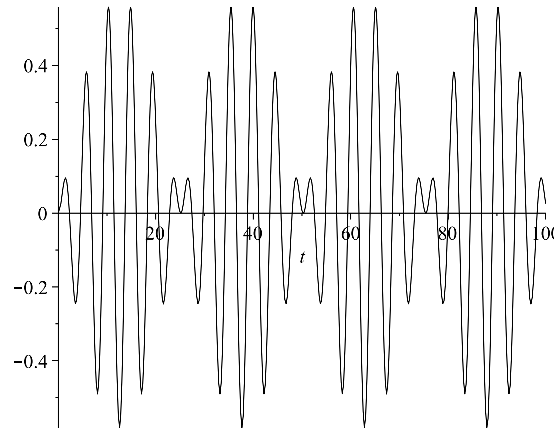


Figure 2.10: Plot from Problem 21b [vi.] $\omega_0 = 1.5$ rad/s, $\omega = 1.5$ rad/s.

Figure 2.11: Plot from Problem 21d.



??

This problem was essentially solved in Problem 16 with $\alpha = \frac{c}{m} > 0$. However, care needs to be exercised in this problem since the ball moves upward and then downward. The resistive force is $f_d = -cv$, where $c > 0$. When the ball is traveling upward, $v > 0$ and $f_d < 0$. When the ball is traveling downward, $v < 0$ and $f_d > 0$. Therefore, the resistive force acts in the correct direction.

a. Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.

$$v(t) = -\frac{g}{\alpha} + \left(\frac{g}{\alpha} + v_0\right) e^{-\alpha t}.$$

b. Use the solution of part a to find the general solution for the position $x(t)$.

$$x(t) = \frac{g + \alpha v_0}{\alpha^2}(1 - e^{-\alpha t}) - \frac{gt}{\alpha}.$$

c. Find an expression to determine how long it takes for the ball to reach it's maximum height?

$$T = \frac{1}{\alpha} \ln\left(1 + \frac{\alpha v_0}{g}\right).$$

d. Assume that $\alpha = c/m = 5\ \mathrm{s}^{-1}$. For $v_0 = 5, 10, 15, 20$ m/s, plot the solution, $x(t)$, versus the time.
The solutions are shown in Figure 2.12.

e. From your plots and the expression in part c, determine the rise time. Do these answers agree?
The rise times are

$v_0(m/s)$	$T(s)$
5	0.2534
10	0.3617
15	0.4316
20	0.4833

.

The values $t_{rise} = T$ are confirmed in Figure 2.12.

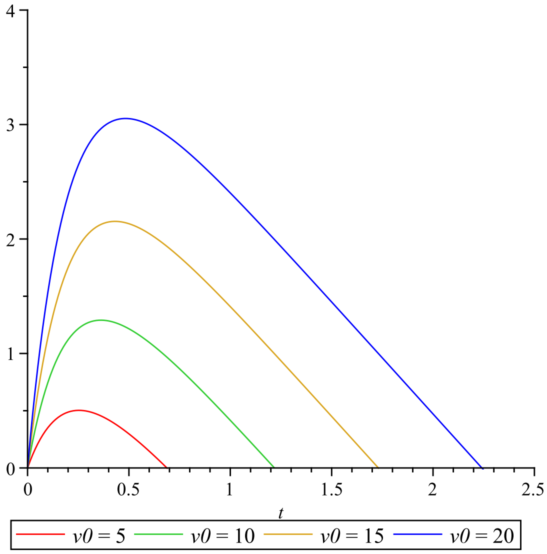


Figure 2.12: Plot of Position vs. Time for the plastic ball in Problem 22 for $\alpha = 5$ s^{-1} and $v_0 = 5, 10, 15, 20$ m/s.

f. What can you say about the time it takes for the ball to fall as compared to the rise time?

From the plots we see that the fall time is longer than the rise time. These can be determined using the total flight time. The total flight time is found by numerically solving $x(t) = 0$ for t_{final}. Then, we compute $t_{fall} = t_{final} - t_{rise}$ using the previous values for the rise time. The values of these times are indicated below.

$v_0(m/s)$	$t_{final}(s)$	$t_{rise}(s)$	$t_{fall}(s)$
5	0.6874	0.2534	0.4340
10	1.2176	0.3617	0.8559
15	1.7303	0.4316	1.2987
20	2.2408	0.4833	1.7575

.

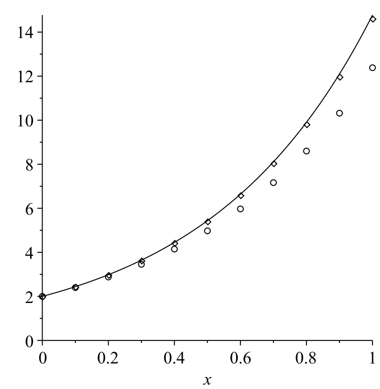


Figure 2.13: Plot from Problem 23a.

23. Use i) Euler's Method and ii) the Midpoint Method to determine the given value of y for the following problems.

For each problem the Euler and Midpoint Methods were executed and the results plotted in the Figures 2.13-2.15 with circles designating Euler's Method and diamonds the Midpoint Method. Plots of the exact solutions are given by the solid curves. The sought values for each problem are provided below.

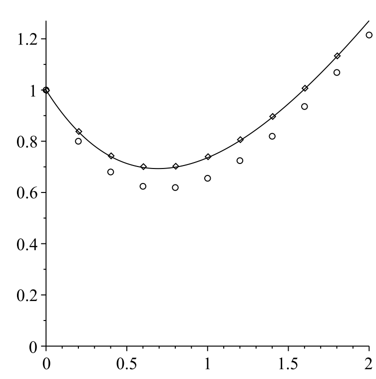


Figure 2.14: Plot from Problem 23b.

a. $\dfrac{dy}{dx} = 2y$, $y(0) = 2$. Find $y(1)$ with $h = 0.1$.
$y(x) = 2e^{2x}$,
$y_{exact} = 14.78$, $y_{Euler} = 12.38$, $y_{Midpoint} = 14.61$.

b. $\dfrac{dy}{dx} = x - y$, $y(0) = 1$. Find $y(2)$ with $h = 0.2$.
$y(x) = x - 1 + 2e^{-x}$,
$y_{exact} = 1.271$, $y_{Euler} = 1.215$, $y_{Midpoint} = 1.275$.

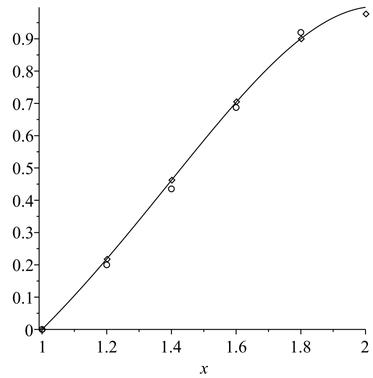


Figure 2.15: Plot from Problem 23c.

c. $\dfrac{dy}{dx} = x\sqrt{1-y^2}$, $y(1) = 0$. Find $y(2)$ with $h = 0.2$.
$y(x) = \sin\left(\frac{x^2-1}{2}\right)$,
$y_{exact} = 0.998$, $y_{Euler} = 1.061$, $y_{Midpoint} = 0.978$.

24. Numerically solve the nonlinear pendulum problem using the Euler-Cromer method for a pendulum with length $L = 0.5$ m using initial angles of $\theta_0 = 10^o$, and $\theta_0 = 70^o$. In each case run the routines long enough and with an appropriate h such that you can determine the period in each case. Compare your results with the linear pendulum period.

As shown in the Figure 2.16 there is little difference for 10^o. The period of the linear pendulum for this problem gives $T = 1.4192$ s. From the plots in Figure 2.17 we find five cycles at $t = 7.096$ s for the linear pendulum and $t = 7.815$ s for the linear pendulum. This gives $T = 1.4192$ s for the linear pendulum and $t = 1.563$ s for the linear pendulum.

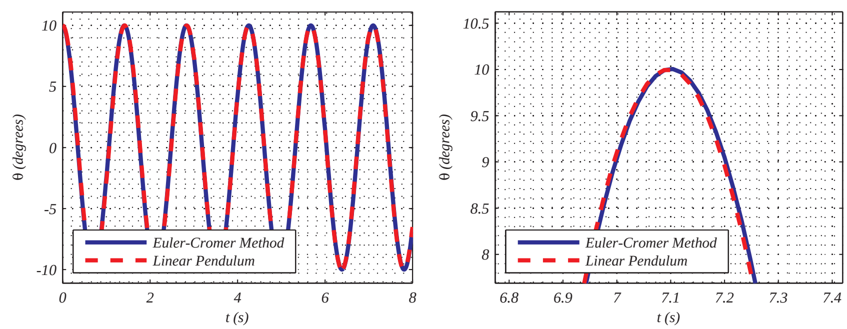


Figure 2.16: Plot comparing the nonlinear and linear pendulum for $\theta_0 = 10^o$ in Problem 24.

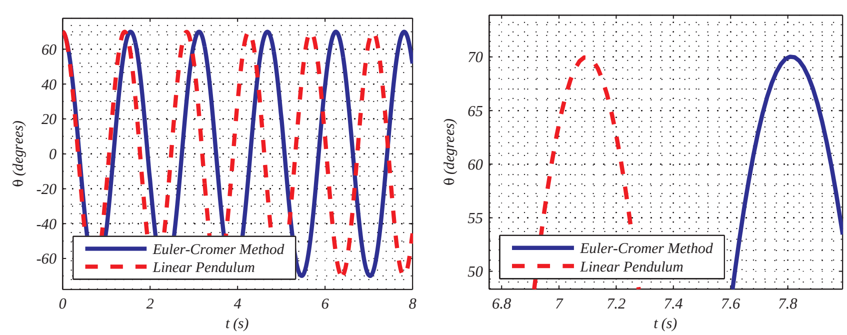


Figure 2.17: Plot comparing the nonlinear and linear pendulum for $\theta_0 = 70^o$ in Problem 24.

25. For the Baumgartner sky dive we had obtained the results for his position as a function of time. There are other questions that could be asked.

a. Find the velocity as a function of time for the model developed in the text.
See Figure 2.18.

b. Find the velocity as a function of altitude for the model developed in the text.
See Figure 2.18.

c. What maximum speed is obtained in the model? At what time and position?

The maximum speed is about 390 m/s at an altitude of 27 km and time of 52 s.

d. Does the model indicate that terminal velocity was reached?

No, it does not.

e. What speed is predicted for the point at which the parachute opened?

It is roughly 64.01 m/s at 238.8 s, or 1585 m.

f. How do these numbers compare with reported data?

The data was released in February, 2013 at `http://www.redbullsratos.com/science/scientific-data-review/`. The maximum vertical speed was reported as 1,357.6 kmh. The jump altitude was adjusted as 38,969.4 m and he experienced free fall 36,402.6 m. He reached up to 560 kmh before the parachute opened. At 34s, he was traveling 1115 kmh at 33,446 m. Maximum speed was reached at 50 s, with speed 1,357,6 kmh at 27,833 m.

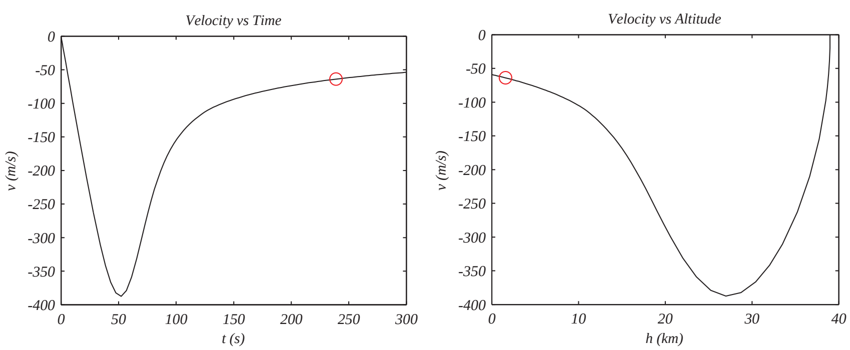


Figure 2.18: The velocity as a function of time and altitude in Problem 25.

26. Consider the flight of a golf ball with mass 46 g and a diameter of 42.7 mm. Assume it is projected at 30^o with a speed of 36 m/s and no spin.

We begin with the equations

$$\begin{aligned}\frac{dv_x}{dt} &= -\alpha(C_D v_x + C_L v_z)(v_x^2+v_z^2)^{1/2},\\ \frac{dv_z}{dt} &= -g - \alpha(C_D v_z - C_L v_x)(v_x^2+v_z^2)^{1/2}.\end{aligned}$$

a. Ignoring air resistance, analytically find the path of the ball and determine the range, maximum height, and time of flight for it to land at the height that the ball had started.

In this case,

$$\begin{aligned}\frac{dv_x}{dt} &= 0,\\ \frac{dv_z}{dt} &= -g.\end{aligned}$$

Integrating this system yields the projectile motion equations encountered in introductory physics. $v_x = v_{x0}$, $v_z = v_{z0} - gt$,

$$x(t) = x_0 + v_{x0}t, \quad z(t) = z_0 + v_{z0}t - \frac{1}{2}gt^2.$$

The maximum height occurs for $v_z(t) = 0$ which occurs when $t = \frac{v_{z0}}{g}$. This gives a height, $H = z(t) - z_0$, of

$$\begin{aligned} H &= v_{z0}\left(\frac{v_{z0}}{g}\right) - \frac{1}{2}g\left(\frac{v_{z0}}{g}\right)^2 \\ &= \frac{v_{z0}^2}{2g}. \end{aligned}$$

The time of flight is twice this time, $t_f = \frac{2v_{z0}}{g}$ and the range is given by

$$R = v_{x0}(\frac{2v_{z0}}{g}) = \frac{2v_{x0}v_{z0}}{g} = \frac{v_0^2\sin(2\theta)}{g}.$$

For the numbers in the problem, we have $R = 115$ m, $H = 16.5$ m, and $t_f = 3.673$ s.

b. Now consider a drag force $f_D = \frac{1}{2}C_D\rho\pi r^2 v^2$, with $C_D = 0.42$ and $\rho = 1.21$ kg/m^3. Determine the range, maximum height, and time of flight for the ball to land at the height that it had started.

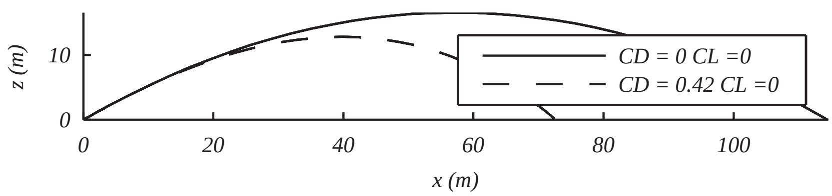


Figure 2.19: The flight of the golf ball in Problem 26.

Flight with drag needs to be solved numerically. In Figure 2.19 are plots for this problem and the previous part of the problem. Reading from the plot we have $R = 115$ m and $H = 16.5$ m.

$t_f = 3.673$ s.

c. Plot the Reynolds number as a function of time. [Take the kinematic viscosity of air, $\nu = 1.47 \times 10^{-5}$.]

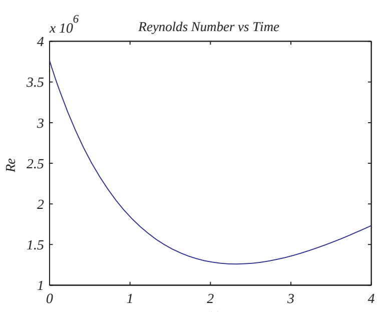


Figure 2.20: The plot of the Reynolds number as a function of time for the flight of the golf ball in Problem 26.

Change Part 26d.

d. Based on the plot in part c, create a model to incorporate the change in Reynolds number and repeat part b. Compare the results from parts a, b, and d.

The Reynolds number is too high to make any difference.

27. Consider the flight of a tennis ball with mass 57 g and a diameter of 66.0 mm. Assume the ball is served 6.40 m from the net at a speed of 50.0 m/s down the center line from a height of 2.8 m. It needs to just clear the net (0.914 m).

a. Ignoring air resistance and spin, analytically find the path of the ball assuming it just clears the net. Determine the angle to clear the net and the time of flight.

The path of the ball just clearing the net is shown in Figure 2.21. In order to find the angle and time for the ball to clear the net, we use the projectile motion equations from the last problem with $v_{x0} = v_0 \cos\theta$ and $v_{z0} = v_0 \sin\theta$:

$$x(t) = x_0 + (v_0 \cos\theta)t, \quad z(t) = z_0 + (v_0 \sin\theta)t - \frac{1}{2}gt^2.$$

For the ball to barely clear the net, we set $x(t) = 6.40$ m and $z(t) = 0.914$ m. Then,

$$\begin{aligned} 6.40 &= (50\cos\theta)t \\ 0.914 &= 2.8 + (50\sin\theta)t - \frac{1}{2}gt^2. \end{aligned}$$

A solution gives $t = 0.1330$ s and $\theta = -0.27408$ rad.

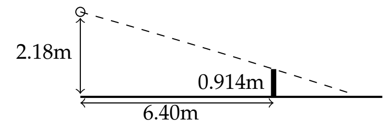


Figure 2.21: In Problem 2.27 a ball is served 6.40 meters from the net from a height of 2.8 m.

b. Find the angle to clear the net assuming the tennis ball is given a topspin with $\omega = 50$ rad/s.

We turn to the system of equations for two dimensional motion incorporating both drag and lift. These are given by

$$\begin{aligned} \frac{dv_x}{dt} &= -\alpha(C_D v_x + C_L v_z)(v_x^2 + v_z^2)^{1/2}, \\ \frac{dv_z}{dt} &= -g - \alpha(C_D v_z - C_L v_x)(v_x^2 + v_z^2)^{1/2}. \end{aligned}$$

For this part we set $C_D = 0$. One can obtain C_L from ω using

$$C_L = \frac{1}{2 + \frac{v}{v_{spin}}},$$

where $v_{spin} = r\omega$. However, $v = \sqrt{v_{x0}^2 + (v_{y0} - gt)^2}$ varies from 50 to 50.5 m/s over the short time the ball is in the air. Then,

$$C_L = \frac{1}{2 + \frac{v}{r\omega}}$$

varies from 0.0310 to 0.0307 for this range of speeds. So, taking $C_L = -0.031$ for top spin, we numerically solve the system.

In Figures 2.22-2.23 we indicate the paths for topspin and bottom spin for an angle $\theta = -0.27408$. Numerically changing the initial angle one can have the ball with spin just clear the net. For the topspin in this part of the problem, we find $\theta = -0.2702$ radians.

c. Repeat part b assuming the tennis ball is given a bottom spin with $\omega = 50$ rad/s.

The only change from part b is letting $C_L = 0.031$. For the bottom spin in this part of the problem, we find $\theta = -0.2778$ radians. This is shown in the plots in Figure 2.23.

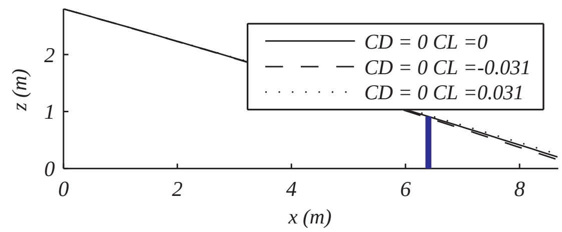


Figure 2.22: The flight of the tennis ball in Problem 27 as it clears the net for $C_D = 0$.

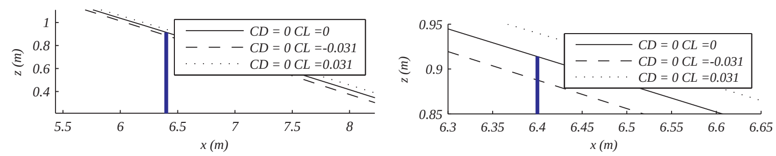


Figure 2.23: The (zoomed in) flight of the tennis ball in Problem 27 as it clears the net for $C_D = 0$.

d. Repeat parts a, b, and c with a drag force, taking $C_D = 0.55$.

In Figures 2.24-2.25 we indicate the paths for topspin and bottom spin for an angle $\theta = -0.27408$ and $C_D = 0.55$. This angle no longer works without spin. Numerically changing the initial angle one can have the ball with drag just clear the net.

For the topspin in the problem, we find $\theta = -0.2692$ radians. For no spin in the part of the problem, we find $\theta = -0.2728$ radians. For the bottom spin in the problem, we find $\theta = -0.2766$ radians.

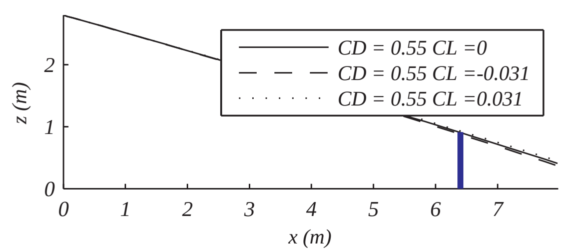


Figure 2.24: The flight of the tennis ball in Problem 27 as it clears the net for $C_D = 0.55$.

28. In Example 2.32 $a(t)$ was determined for a curved universe with nonrelativistic matter for $\Omega_0 > 1$. Derive the parametric equations for $\Omega_0 < 1$,

$$\begin{aligned} a &= \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\eta - 1), \\ t &= \frac{\Omega_0}{2H_0(1-\Omega_0)^{3/2}}\left(\sinh\eta - \eta\right), \end{aligned}$$

for $\eta \geq 0$.

We begin with the Friedman equation in the form

$$\dot{a} = \pm H_0\sqrt{\frac{\Omega_0}{a} + (1-\Omega_0)}.$$

Let $\alpha = \frac{1-\Omega_0}{\Omega_0}$. We will see that only the positive sign applies in this case, so

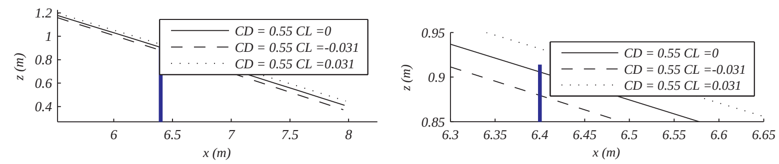


Figure 2.25: The (zoomed in) flight of the tennis ball in Problem 27 as it clears the net for $C_D = 0.55$

the equation becomes

$$\dot{a} = H_0\sqrt{\frac{\Omega_0}{a}}\sqrt{1+\alpha a}.$$

This differential equation is separable and can be integrated with a hyperbolic function substitution. We let

$$\alpha a = \sinh^2 u, \quad \alpha\, du = 2\sinh u \cosh u\, du.$$

Then, we have

$$\begin{aligned} H_0\sqrt{\Omega_0}\,dt &= \frac{\sqrt{a}}{\sqrt{1+\alpha a}} \\ &= \frac{\sinh u(2\sinh u\cosh u\, du)}{\alpha\sqrt{\alpha}\cosh u} \\ &= 2\alpha^{-3/2}\sinh^2 u\, du \\ &= \alpha^{-3/2}[\cosh 2u - 1]. \end{aligned}$$

Integrating,

$$H_0\sqrt{\Omega_0}t + C = \frac{1}{2}\sinh u - u.$$

For $t = 0$, $a(0) = 0$, therefore $u(0) = 0$.

Defining $\eta = 2u$, we have the parametric form for t :

$$\begin{aligned} t &= \frac{1}{2H_0\sqrt{\Omega_0}\alpha^{3/2}}(\sinh\eta - \eta) \\ &= \frac{\omega_0}{2H_0(1-\Omega_0)^{3/2}}(\sinh\eta - \eta). \end{aligned}$$

the parametric form for a can be obtained from the original hyperbolic substitution:

$$\begin{aligned} a &= \frac{1}{\alpha}\sinh^2\frac{\eta}{2} \\ &= \frac{\omega_0}{2(1-\Omega_0)}(\cosh\eta - 1). \end{aligned}$$

In Figure 2.30 we give both the exact and the numerical solution for $\Omega_0 = 0.8, 1.1$.

29. Find numerical solutions for other models of the universe.

a. A flat universe with nonrelativistic matter only with $\Omega_{m,0} = 1$.

In Figure 2.26 we show the numerical solution for this problem. The exact solution to this problem was found as

$$a(t) = \left(\frac{t}{\frac{2}{3}H_0}\right)^{2/3}.$$

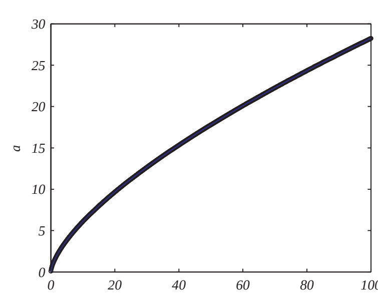


Figure 2.26: A plot of $a(t)$ vs t for a flat universe with nonrelativistic matter only in Problem 2.29a.

b. A curved universe with radiation only with curvature of different types.

In Figure 2.27 we show the numerical solution for this problem. The exact solution can also be determined.

We insert $\Omega_{\Lambda,0} = 0 = \Omega_{m,0}$ and $\Omega_{r,0} = \Omega_0$, into the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}\right].$$

and obtain

$$\dot{a} = \pm H_0 \sqrt{\frac{\Omega_0}{a^2} + 1 - \Omega_0}.$$

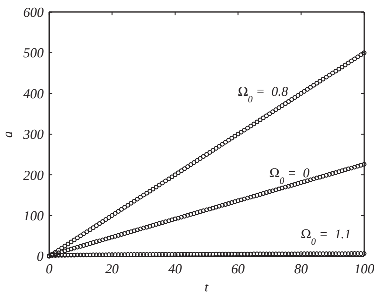


Figure 2.27: A plot of $a(t)$ vs t for a curved universe with radiation only in Problem 2.29b.

This equation is separable and can be integrated,

$$H_0 t = \pm \int \frac{a\,da}{\sqrt{\Omega_0 + (1-\Omega_0)a^2}}.$$

For the case $\Omega_0 < 1$, we have ($\dot{a} > 0$)

$$\begin{aligned} H_0 t &= \frac{1}{\sqrt{\Omega_0}} \int \frac{a\,da}{\sqrt{1+\alpha a^2}}, \\ &= \frac{1}{\alpha\sqrt{\Omega_0}} \sqrt{1+\alpha a^2} + C, \end{aligned}$$

where $\alpha = \frac{1-\Omega_0}{\Omega_0} > 0$. For $a(0) = 0$,

$$H_0 t = \frac{1}{1-\Omega_0}\left[\sqrt{1+\alpha a^2} - 1\right].$$

For the case $\Omega_0 > 1$, we have the possibility that $\dot{a}$ changes sign. Thus, we have

$$a_{max} = \sqrt{\frac{\Omega_0}{\Omega_0 - 1}}$$

and for $a < a_{max}$,

$$\begin{aligned} H_0 t &= \frac{1}{\sqrt{\Omega_0}} \int_0^a \frac{a\,da}{\sqrt{1 - \frac{a^2}{a_{max}^2}}}, \\ &= \frac{a_{max}^2}{\sqrt{\Omega_0}} \left[1 - \sqrt{1 - \frac{a^2}{a_{max}^2}}\right] \\ &= \frac{1}{\sqrt{\Omega_0 - 1}} [a_m ax - \sqrt{a_{max}^2 - a^2}]. \end{aligned}$$

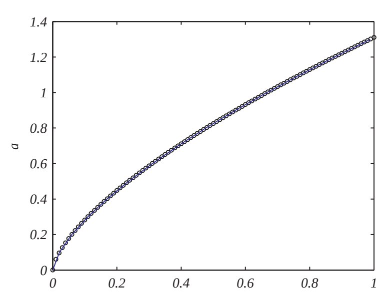


Figure 2.28: A plot of $a(t)$ vs t for a flat universe with nonrelativistic matter and radiation in Problem 2.29c.

c. A flat universe with nonrelativistic matter and radiation with several values of $\Omega_{m,0}$ and $\Omega_{r,0} + \Omega_{m,0} = 1$.

In Figure 2.28 we show the numerical solution for this problem with the exact solution superimposed. The exact solution can be found in Ryden's book, *Introduction to Cosmology* (2003).

We begin with the Friedmann equation in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}\right].$$

For this problem we have $\Omega_{\Lambda,0} = 0$ and $\Omega_0 = 1$. Therefore,

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}\right].$$

Defining $\alpha = \frac{\Omega_{r,0}}{\Omega_{m,0}}$, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\frac{\Omega_{r,0}}{a^4}\left[1+\frac{a}{\alpha}\right].$$

Since the differential equation is separable, we write

$$\begin{aligned} H_0\,dt &= \frac{a\,da}{\sqrt{\Omega_{r,0}}\sqrt{1+\frac{a}{\alpha}}}, \quad u = 1+\frac{a}{\alpha}, \\ &= \frac{\alpha^2}{\sqrt{\Omega_{r,0}}}\frac{u-1}{\sqrt{u}}\,du. \end{aligned}$$

Integrating with the condition $a(0) = 0$, we have

$$\begin{aligned} H_0 t &= \frac{\alpha^2}{\sqrt{\Omega_{r,0}}}\int_1^{1+a/\alpha}\frac{u-1}{\sqrt{u}}\,du. \\ &= \frac{\alpha^2}{\sqrt{\Omega_{r,0}}}\left[\frac{2}{3}\left(1+\frac{a}{\alpha}\right)^{3/2} - 2\left(1+\frac{a}{\alpha}\right)^{1/2} + \frac{4}{3}\right] \\ &= \frac{4\alpha^2}{3\sqrt{\Omega_{r,0}}}\left[1+\left(1+\frac{a}{\alpha}\right)^{1/2}\left(\frac{a}{2\alpha}-1\right)\right]. \end{aligned}$$

d. Look up the current values of $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, and κ. Use these values to predict future values of $a(t)$.

   In Figure 2.29 we show the numerical solution for $\Omega_0 = 1$, $\Omega_{r,0} = 8.4\times 10^{-5}$, $\Omega_{m,0} = 0.30$, and $\Omega_{\Lambda,0} = \Omega_0 - (\Omega_{r,0} + \Omega_{m,0})$.

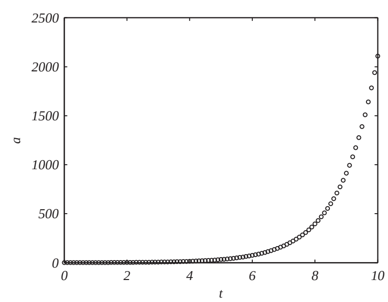


Figure 2.29: A plot of $a(t)$ vs t for the numerical solution for $\Omega_0 = 1$, $\Omega_{r,0} = 8.4\times 10^{-5}$, $\Omega_{m,0} = 0.30$ in Problem 2.29d.

e. Investigate other types of universes of your choice, but different from the previous problems and examples.

   In Figure 2.30 we show the numerical solution for a simple case of nonrelativistic matter plus curvature. The solutions were presented in the text with the $\Omega_0 < 1$ case derived in the Problem 28. In Figure 2.30 we also show the exact solution superimposed.

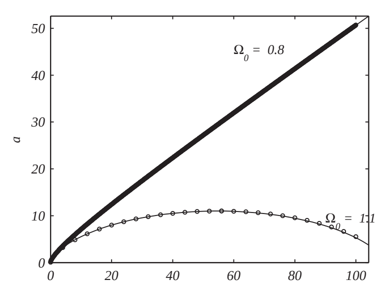


Figure 2.30: A plot of $a(t)$ vs t for a flat universe with nonrelativistic matter and curvature for Problem 2.29d.

30. Consider the system

$$\begin{aligned} x' &= -4x - y, \\ y' &= x - 2y. \end{aligned}$$

a. Determine the second order differential equation satisfied by $x(t)$.

Differentiate the first equation with respect to t and then replace $y' = x - 2y$ and $y = -x' - 4x$ to find

$$\begin{aligned} x'' &= -4x' - y' \\ &= -4x' - (x - 2y) \\ &= -4x' - (x - 2(-x' - 4x)) \\ &= -6x' - 9x. \end{aligned}$$

Therefore, $x'' + 6x' + 9 = 0$.

b. Solve the differential equation for $x(t)$.

The characteristic equation is $r^2 + 6r + 9 = 0$, whose solution is $r = -3$. This has the general solution

$$x(t) = (c_1 + c_2 t)e^{-3t}.$$

c. Using this solution, find $y(t)$.

Using the first equation of the system, we find $y(t)$ as

$$\begin{aligned} y(t) &= -x' - 4x \\ &= -(c_2 - 3(c_1 + c_2 t))e^{-3t} - 4(c_1 + c_2 t)e^{-3t} \\ &= (-c_1 - c_2 - c_2 t)e^{-3t}. \end{aligned}$$

So, $y(t) = (-c_1 - c_2 - c_2 t)e^{-3t}$.

d. Verify your solutions for $x(t)$ and $y(t)$.

Insert $x(t) = (c_1 + c_2 t)e^{-3t}$ and $y(t) = (-c_1 - c_2 - c_2 t)e^{-3t}$ into the system of differential equations. Then we have

$$\begin{aligned} x' &= (c_2 - 3(c_1 + c_2 t))e^{-3t} \\ &= (3c_1 - 3c_2 t + c_2)e^{-3t}. \\ -4x - y &= -4(c_1 + c_2 t)e^{-3t} - (-c_1 - c_2 - c_2 t)e^{-3t} \\ &= (3c_1 - 3c_2 t + c_2)e^{-3t}. \end{aligned}$$

$$\begin{aligned} y' &= (2c_2 - 3(-c_1 - c_2 - c_2 t))e^{-3t} \\ &= (3c_1 + 2c_2 + 3c_2 t)e^{-3t}. \\ -x - 2y &= -(c_1 + c_2 t)e^{-3t} - 2(-c_1 - c_2 - c_2 t)e^{-3t} \\ &= (3c_1 + 2c_2 + 3c_2 t)e^{-3t}. \end{aligned}$$

e. Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

From $x(t) = (c_1 + c_2 t)e^{-3t}$, $x(0) = c_1 = 1$.

From $y(t) = (-c_1 - c_2 - c_2 t)e^{-3t}$,

$$y(0) = -c_1 - c_2 = -1 - c_2 = 0.$$

Therefore, $c_2 = -1$. Thus, the particular solution to the system is $x(t) = (1 - t)e^{-3t}$ and $y(t) = te^{-3t}$.

31. Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.).

a.

$$\begin{aligned} x' &= 3x, \\ y' &= -2y. \end{aligned}$$

This is an uncoupled systems and is easily solved: $x(t) = c_1 e^{3t}$, $y(t) = c_2 e^{-2t}$. For initial conditions in which $c_1 = 0$, solutions tend to the origin, otherwise they tend to infinity. Thus, this system has an unstable saddle as seen in Figure 2.31.

One could also obtain a family solution curves

$$\frac{dy}{dx} = \frac{y'}{x'} = -\frac{2}{3}\frac{y}{x}.$$

Integrating this separable first order equation,

$$\begin{aligned} \int \frac{dy}{y} &= -\frac{2}{3}\int \frac{dx}{x}, \\ \ln|y| &= -\frac{2}{3}\ln|x| + C, \end{aligned}$$

we find $y = \frac{A}{x^{2/3}}$. Sketching these curves also leads to a saddle.

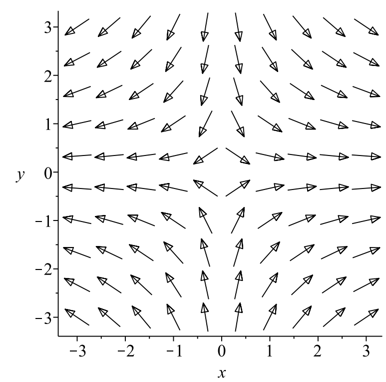


Figure 2.31: The direction field for the system in Problem 31a.

b.

$$\begin{aligned} x' &= -y, \\ y' &= -5x. \end{aligned}$$

Differentiating the first equation, $x'' = -y' = 5x$. The roots of the characteristic equation are $r = \pm\sqrt{5}$. Thus, the solution for x is

$$x(t) = c_1 e^{\sqrt{5}t} + c_2 e^{-\sqrt{5}t}.$$

Inserting this solution for $x(t)$ back into the system, gives

$$y(t) = -\sqrt{5}c_1 e^{\sqrt{5}t} - c_2 e^{-\sqrt{5}t}.$$

Based upon the solutions, or that the roots are $r = \pm\sqrt{5}$, we see that this system has an unstable saddle. This can also be seen in Figure 2.32

One could also obtain a family solution curves

$$\frac{dy}{dx} = \frac{y'}{x'} = 5\frac{x}{y}.$$

Integrating this separable first order equation,

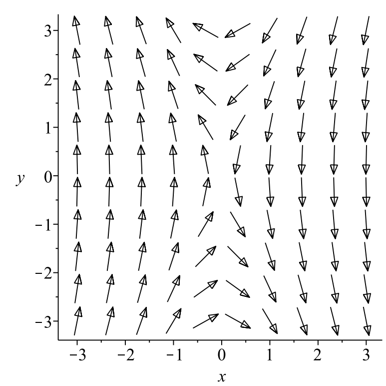


Figure 2.32: The direction field for the system in Problem 31b.

$$\begin{aligned} \int y\,dy &= 5\int x\,dx, \\ y^2 + C &= 5x^2, \end{aligned} \tag{2.1}$$

we find $5x^2 - y^2 = C$. This is a family of hyperbolae for real C, showing that the equilibrium point is a saddle point.

c.

$$\begin{aligned} x' &= 2y, \\ y' &= -3x. \end{aligned}$$

Differentiating the first equation, $x'' = 2y' = -6x$. The roots of the characteristic equation $r^2 + 6 = 0$, are $r = \pm i\sqrt{6}$. Thus, the solution for x is

$$x(t) = c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t.$$

Inserting this solution into the system of differential equations, gives

$$y(t) = \frac{\sqrt{5}}{2}\left(c_2 \cos\sqrt{6}t - c_1 \sin\sqrt{6}t\right).$$

Thus, this system has solutions that follow elliptical paths.

One could also obtain a family solution curves

$$\frac{dy}{dx} = \frac{y'}{x'} = -\frac{3}{2}\frac{x}{y}.$$

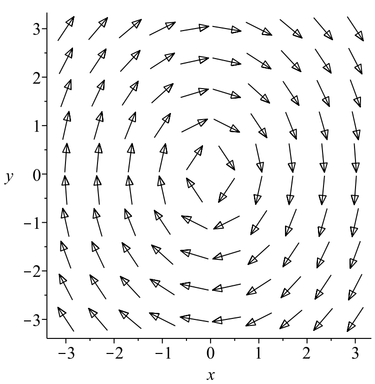


Figure 2.33: The direction field for the system in Problem 31c.

Integrating this separable first order equation, we find $3x^2 + 2y^2 = C$. This is a family of ellipses for $C > 0$ showing that the equilibrium point is a center as seen in Figure 2.33.

d.

$$\begin{aligned} x' &= x - y, \\ y' &= y. \end{aligned}$$

The second equation can be solved directly. $y(t) = c_1 e^t$. Inserting this into the first equation,

$$x' - x = -c_1 e^t.$$

This can be solved using the integrating factor $\mu = e^{-t}$. Then, $(e^{-t}x)' = -c_1$. Thus,

$$x(t) = (c_2 - c_1 t)e^t.$$

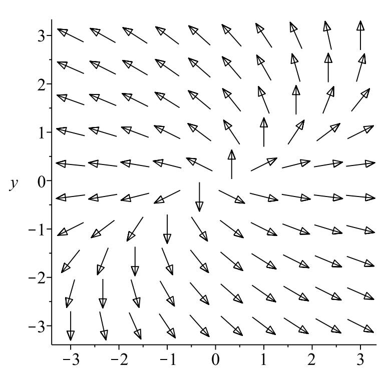


Figure 2.34: The direction field for the system in Problem 31d.

In Figure 2.34 the direction field for this system is shown. This system is similar to Example 2.43, which has a line of unstable equlbria.

e.

$$\begin{aligned} x' &= 2x + 3y, \\ y' &= -3x + 2y. \end{aligned}$$

Differentiating the first equation,

$$\begin{aligned} x'' &= 2x' + 3y' = 2x' + 3(-3x + 2y) \\ &= 2x' - 9x + 2(x' - 2x) \\ &= 4x' - 13x \end{aligned}$$

So, we need to solve $x'' - 4x' + 13x = 0$. The roots of the characteristic equation are $r = 2 \pm 3i$. So,

$$x(t) = (c_1 \cos 3t + c_2 \sin 3t)e^{2t}.$$

Inserting this into the system, we find

$$y(t) = (c_2 \cos 3t - c_1 \sin 3t)e^{2t}.$$

The direction field plot in Figure 2.35 as well as the solution indicate that the orbits are spirals.

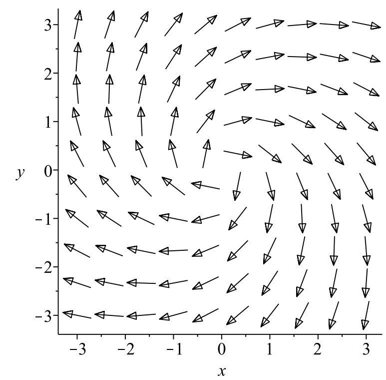


Figure 2.35: The direction field for the system in Problem 31e.

This problem could also be approached in polar coordinates. Recall that

$$\begin{aligned} r' &= \frac{xx' + yy'}{r}, \\ \theta' &= \frac{xy' - yx'}{r^2}. \end{aligned}$$

The radial equation is obtained by multiplying the first equation by x, the second equation by y,

$$\begin{aligned} xx' &= 2x^2 + 3xy, \\ yy' &= -3xy + 2y^2, \end{aligned}$$

and then adding these expressions to obtain

$$rr' = xx' + yy' = 2(x^2 + y^2) = 2r^2.$$

Similarly, the equation for θ is obtained by multiplying the first equation by y, the second equation by x,

$$\begin{aligned} xy' &= -3x^2 + 2xy, \\ yx' &= 2xy + 3y^2, \end{aligned}$$

and subtracting, to find

$$r^2\theta' = xy'yx' = -3(x^2 + y^2) = -3r^2.$$

As a result we have the system in polar as $r' = 2r$ and $\theta' = -3$.

The second equation indicates that the spirals move in a clockwise direction as time increases. The radial equation indicates that the spiral is exponentially growing since the solution is $r = r_0 e^{2t}$. This is consistent with the previous analysis of the problem.

32. Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right].$$

We begin with $\tan\theta = \frac{y}{x}$. Differentiating this expression with respect to time,

$$\sec^2\theta\,\theta' = \frac{y'x - yx'}{x^2}.$$

Note that

$$\sec^2\theta = 1 + \tan^2\theta = \frac{x^2+y^2}{x^2}.$$

This gives

$$\theta' = \frac{y'x - yx'}{x^2}\left(\frac{x^2}{x^2+y^2}\right) = \frac{y'x - yx'}{r^2}.$$

33. Consider the system of equations in Example 2.46.

a. Derive the polar form of the system.

The system of equation under consideration is

$$\begin{aligned} x' &= -y + x(1-x^2-y^2) \\ y' &= x + y(1-x^2-y^2). \end{aligned}$$

We compute the forms

$$\begin{aligned} xx' + yy' &= -xy + x^2(1-x^2-y^2) + (xy + y^2(1-x^2-y^2)) \\ &= (x^2+y^2)(1-x^2-y^2). \\ xy' - yx' &= x^2 + xy(1-x^2-y^2) - (-y^2 + xy(1-x^2-y^2)) \\ &= (x^2+y^2). \end{aligned}$$

These reduce to

$$\begin{aligned} rr' &= r^2(1-r^2), \\ r^2\theta' &= r^2, \end{aligned}$$

or

$$\begin{aligned} r' &= r(1-r^2), \\ \theta' &= 1, \end{aligned}$$

b. Solve the radial equation, $r' = r(1-r^2)$, for the initial values $r(0) = 0, 0.5, 1.0, 2.0$.

Next, we solve the radial problem. Note that $r(t) = 0, 1$ are equilibrium solutions ($r' = 0$.) So, we consider solutions with $r(0) = r_0 \neq 0, 1$.

The equation is separable, leading to

$$t + C = \int \frac{dr}{r(1-r^2)}.$$

The integration is done using a partial fraction decomposition. Let

$$\frac{1}{r(1-r^2)} = \frac{A}{r} + \frac{B}{1-r} + \frac{C}{1+r} = \frac{A(1-r^2) + Br(1+r) + Cr(1-r)}{r(1-r^2)}.$$

Solving for the constants, we find $A = 1$, $B = -C = \frac{1}{2}$.
Finishing the integration,

$$\begin{aligned} t + C &= \ln r + \frac{1}{2}\ln(1+r) - \frac{1}{2}\ln|1-r| \\ &= \ln \frac{r}{\sqrt{|1-r^2|}} \\ Ae^t &= \frac{r}{\sqrt{|1-r^2|}} \\ A^2e^{2t} &= \frac{r^2}{|1-r^2|}. \end{aligned}$$

Solving for $r^2(t)$, we find that for $0 < r < 1$,

$$r^2(t) = \frac{1}{1+Ce^{-2t}},$$

and for $r > 1$,

$$r^2(t) = \frac{1}{1-Ce^{-2t}}.$$

Using the initial conditions, one can obtain C in terms of r_0. We find that for $0 < r < 1$,

$$r(t) = \frac{1}{\sqrt{1+(\frac{1}{r_0^2}-1)e^{-2t}}} = \frac{r_0}{\sqrt{r_0^2+(1-r_0^2)e^{-2t}}},$$

and for $r > 1$,

$$r(t) = \frac{1}{\sqrt{1-(1-\frac{1}{r_0^2})e^{-2t}}} = \frac{r_0}{\sqrt{r_0^2-(r_0^2-1)e^{-2t}}}.$$

Solutions for the initial values $r(0) = 0.5, 1.0, 2.0$ are shown in the figure.

c. Based upon these solutions, plot and describe the behavior of all solutions to the original system in Cartesian coordinates.

Typical solutions are shown in Figure 2.36. These indicate that the origin, $r = 0$, is unstable and the unit circle, $r = 1$, is stable. The unit circle is a stable limit cycle with nearby orbits spiraling in towards the orbit.

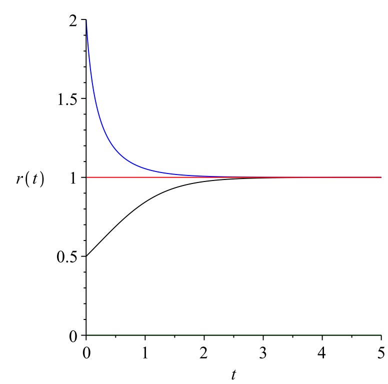


Figure 2.36: Solutions for the initial conditions in Problem 33.