Chapter 2

Technical Lemmas

Exercise 2.1

Let $c(x) \in \mathbb{R}^n$ and $P(x) = P^{\mathrm{T}}(x) \in \mathbb{R}^{n\times n}$ depend affinely on x, and $P(x)$ is nonsingular for all x. Find the equivalent LMIs for the following constraints:

$$c^{\mathrm{T}}(x)P^{-1}(x)c(x) < 1, \quad P(x) > 0.$$

Solution. Rewrite the above relations as

$$c^{\mathrm{T}}(x)P^{-1}(x)c(x) - 1 < 0, \quad P(x) > 0.$$

Then, using the Schur complement lemma, these two relations can be shown to be equivalent with the following LMI condition:

$$\begin{bmatrix} -P(x) & c(x) \\ c^{\mathrm{T}}(x) & -1 \end{bmatrix} < 0,$$

or

$$\begin{bmatrix} P(x) & c(x) \\ c^{\mathrm{T}}(x) & 1 \end{bmatrix} > 0.$$

Exercise 2.2

Let $P(x) \in \mathbb{S}^{n\times n}$ and $Q(x) \in \mathbb{R}^{n\times p}$ depend affinely on x. Convert the following constraints

$$\operatorname{trace}\left(Q^{\mathrm{T}}(x)P^{-1}(x)Q(x)\right) < 1, \; P(x) > 0 \tag{s2.1}$$

into a set of LMIs by introducing a new (slack) matrix variable $X \in \mathbb{S}^{p\times p}$.

Solution. According to Lemma 2.13, the first inequality in (s2.1) is equivalent to

$$Q^{\mathrm{T}}(x)P^{-1}(x)Q(x) < X, \tag{s2.2}$$

and

$$\operatorname{trace}(X) < 1. \tag{s2.3}$$

Applying Schur complement lemma to inequality (s2.2), with the condition $P(x) > 0$, yields

$$\begin{bmatrix} -X & Q^{\mathrm{T}}(x) \\ Q(x) & -P(x) \end{bmatrix} < 0.$$

Therefore, (s2.1) is equivalent to the following set of LMIs:

$$\begin{cases} \begin{bmatrix} -X & Q^{\mathrm{T}}(x) \\ Q(x) & -P(x) \end{bmatrix} < 0 \\ \operatorname{trace}(X) < 1 \end{cases}.$$

Exercise 2.3 (Wang and Zhao (2007))

Let

$$\Lambda = \left\{ \alpha = [\alpha_1 \ \ \alpha_2]^{\mathrm{T}} \in \mathbb{R}^2 \middle| \alpha_1 + \alpha_2 = 1, \alpha_1, \alpha_2 \geq 0 \right\}.$$

Show $P(\alpha) = \alpha_1 P_1 + \alpha_2 P_2 > 0$ for any $\alpha \in \Lambda$ if and only if $P_1 > 0$ and $P_2 > 0$.

Solution. *Necessity*. Since

$$P(\alpha) = \alpha_1 P_1 + \alpha_2 P_2 > 0, \ \forall \alpha \in \Lambda,$$

choosing $\alpha_1 = 1, \ \alpha_2 = 0$ gives $P_1 > 0$, while choosing $\alpha_1 = 0, \ \alpha_2 = 1$ gives $P_2 > 0$.

Sufficiency. Since $P_1 > 0$, $P_2 > 0$, we have

$$\alpha_1 P_1 \geq 0, \ \alpha_2 P_2 \geq 0, \ \forall \alpha \in \Lambda. \tag{s2.4}$$

Therefore,

$$P(\alpha) = \alpha_1 P_1 + \alpha_2 P_2 \geq 0.$$

Further, note that $\alpha_1 + \alpha_2 = 1$, the two equalities in (s2.4) do not simultaneously hold. This implies the strict inequality in the above relation.

Exercise 2.4 (Xu and Yang (2000))

Let

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \in \mathbb{R}^{n \times n}, \ \ M_1 \in \mathbb{R}^{m \times m},$$

and M_4 be invertible. Show that $M + M^{\mathrm{T}} < 0$ implies

$$M_1 + M_1^{\mathrm{T}} - M_2 M_4^{-1} M_3 - M_3^{\mathrm{T}} M_4^{-\mathrm{T}} M_2^{\mathrm{T}} < 0.$$

Solution. Since

$$M+M^{\mathrm{T}}=\left[\begin{array}{cc} M_1+M_1^{\mathrm{T}} & M_2+M_3^{\mathrm{T}} \\ M_2^{\mathrm{T}}+M_3 & M_4+M_4^{\mathrm{T}} \end{array}\right]<0, \tag{s2.5}$$

then according to Corollary 2.2, we have

$$M_1+M_1^{\mathrm{T}}<0, \quad M_4+M_4^{\mathrm{T}}<0. \tag{s2.6}$$

With (s2.5) and (s2.6), further applying Schur complement lemma to $M+M^{\mathrm{T}}<0$, yields

$$S_{ch}(M_4+M_4^{\mathrm{T}})<0. \tag{s2.7}$$

According to the definition of Schur complement, we have

$$\begin{aligned} & S_{ch}(M_4+M_4^{\mathrm{T}}) \\ = \ & M_1+M_1^{\mathrm{T}}-\left(M_2+M_3^{\mathrm{T}}\right)\left(M_4+M_4^{\mathrm{T}}\right)^{-1}\left(M_2^{\mathrm{T}}+M_3\right) \\ = \ & M_1+M_1^{\mathrm{T}}+\Phi_1+\Phi_2, \end{aligned} \tag{s2.8}$$

where

$$\begin{aligned} \Phi_1 &= -M_2\left(M_4+M_4^{\mathrm{T}}\right)^{-1}M_2^{\mathrm{T}}-M_3^{\mathrm{T}}\left(M_4+M_4^{\mathrm{T}}\right)^{-\mathrm{T}}M_3, \\ \Phi_2 &= -M_2\left(M_4+M_4^{\mathrm{T}}\right)^{-1}M_3-M_3^{\mathrm{T}}\left(M_4+M_4^{\mathrm{T}}\right)^{-\mathrm{T}}M_2^{\mathrm{T}}, \end{aligned} \tag{s2.9}$$

Furthermore, using Corollary 2.1, that is, the matrix inversion lemma, yields,

$$\begin{aligned} \left(M_4+M_4^{\mathrm{T}}\right)^{-1} &= M_4^{-1}-M_4^{-1}\left(M_4^{-\mathrm{T}}+M_4^{-1}\right)^{-1}M_4^{-1} \\ &= M_4^{-1}+M_4^{-1}HM_4^{-1}, \end{aligned} \tag{s2.10}$$

where

$$H=-\left(M_4^{-\mathrm{T}}+M_4^{-1}\right)^{-1}=-M_4\left(M_4+M_4^{\mathrm{T}}\right)^{-1}M_4^{\mathrm{T}}.$$

Substituting (s2.10) into (s2.9), gives

$$\begin{aligned} \Phi_2 &= -M_2\left(M_4^{-1}+M_4^{-1}HM_4^{-1}\right)M_3 \\ &\quad -M_3^{\mathrm{T}}\left(M_4^{-\mathrm{T}}+M_4^{-\mathrm{T}}HM_4^{-\mathrm{T}}\right)M_2^{\mathrm{T}} \\ &= \Psi_1-\Psi_2, \end{aligned} \tag{s2.11}$$

where

$$\begin{aligned} \Psi_1 &= -M_2M_4^{-1}M_3-M_3^{\mathrm{T}}M_4^{-\mathrm{T}}M_2^{\mathrm{T}}, \\ \Psi_2 &= M_2M_4^{-1}HM_4^{-1}M_3+M_3^{\mathrm{T}}M_4^{-\mathrm{T}}HM_4^{-\mathrm{T}}M_2^{\mathrm{T}}. \end{aligned}$$

On the other hand, considering that M_4 is nonsingular and $M_4 + M_4^{\mathrm{T}} < 0$, we easily observe that $H > 0$. Using Lemma 2.1, it results in that

$$\begin{aligned}
\Psi_2 &\le M_2 M_4^{-1} H M_4^{-\mathrm{T}} M_2^{\mathrm{T}} + M_3^{\mathrm{T}} M_4^{-\mathrm{T}} H M_4^{-1} M_3 \\
&= -M_2 \left(M_4^{\mathrm{T}} + M_4 \right)^{-1} M_2^{\mathrm{T}} - M_3^{\mathrm{T}} \left(M_4^{\mathrm{T}} + M_4 \right)^{-\mathrm{T}} M_3 \\
&= \Phi_1.
\end{aligned}$$

Substituting the above inequality into (s2.8), and using (s2.11), yields

$$\begin{aligned}
S_{ch}(M_4 + M_4^{\mathrm{T}}) &= M_1 + M_1^{\mathrm{T}} + \Phi_1 + \Phi_2 \\
&\ge M_1 + M_1^{\mathrm{T}} + \Psi_2 + \Psi_1 - \Psi_2 \\
&= M_1 + M_1^{\mathrm{T}} + \Psi_1,
\end{aligned}$$

which, together with (s2.7), implies

$$M_1 + M_1^{\mathrm{T}} + \Psi_1 \le S_{ch}(M_4 + M_4^{\mathrm{T}}) < 0.$$

This is the inequality to be shown. The proof is then finished.

Exercise 2.5 (Yu (2002), Page 128)

Let A be an arbitrary square matrix, and Q be some symmetric matrix. Show that there exists a $P > 0$ satisfying

$$A^{\mathrm{T}} P A - P + Q < 0 \tag{s2.12}$$

if and only if there exists an $X > 0$ such that

$$\begin{bmatrix} -X & AX \\ XA^{\mathrm{T}} & -X + XQX \end{bmatrix} < 0. \tag{s2.13}$$

Solution. Let

$$\Phi(X) = \begin{bmatrix} I & 0 \\ 0 & X^{-1} \end{bmatrix} \begin{bmatrix} -X & AX \\ XA^{\mathrm{T}} & -X + XQX \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X^{-1} \end{bmatrix},$$

then (s2.13) holds if and only if $\Phi(X) < 0$. Note that

$$\Phi = \begin{bmatrix} -X & A \\ A^{\mathrm{T}} & -X^{-1} + Q \end{bmatrix},$$

applying Schur complement lemma to the above matrix Φ, we know that there exists an $X > 0$ such that $\Phi(X) < 0$ if and only if there exists an $X > 0$ satisfying

$$-X^{-1} + Q + A^{\mathrm{T}} X^{-1} A < 0.$$

Letting $P = X^{-1}$, the above inequality is turned into (s2.12). Therefore, the conclusion holds true.

Exercise 2.6

Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Work out by hand the following using the matrix inversion lemma and the Schur complement lemma:

1. find out $\det(A)$ and A^{-1} (if exists);
2. judge the negative definiteness of A.

Solutions. Let

$$A_{11} = -2,\ A_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix},$$

then we have

$$\begin{aligned} S_{ch}(A_{11}) &= A_{22} - A_{21}(A_{11})^{-1} A_{12} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

Therefore, it follows from the matrix inversion lemma that

$$\begin{aligned} \det A &= \det A_{11} \det S_{ch}(A_{11}) \\ &= -2 \det \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix} \\ &= -3. \end{aligned}$$

Thus the matrix A is invertible, and A^{-1} is given by

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) \\ -S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & S_{ch}^{-1}(A_{11}) \end{bmatrix}.$$

Since

$$S_{ch}^{-1}(A_{11}) = \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = -\frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix},$$
$$A_{11}^{-1} A_{12} = -\frac{1}{2}\begin{bmatrix} 1 & 0 \end{bmatrix},$$

we have

$$A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) = \frac{1}{6}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$A_{11}^{-1}A_{12}S_{ch}^{-1}(A_{11})A_{21}A_{11}^{-1} = -\frac{1}{6}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{6},$$

thus

$$\begin{aligned} A^{-1} &= \begin{bmatrix} -\frac{1}{2}-\frac{1}{6} & -\frac{1}{3}\begin{bmatrix} 1 & 1 \end{bmatrix} \\ -\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} & -\frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \end{bmatrix} \\ &= -\frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}. \end{aligned}$$

Finally, since

$$\begin{aligned} &A_{11} = -2 < 0, \\ &S_{ch}\left(A_{11}\right) = \begin{bmatrix} -2.5 & 1 \\ 1 & -1 \end{bmatrix} < 0. \end{aligned}$$

It follows from the Schur complement lemma that the matrix A is symmetric negative definite.