Chapter 2

2-1 A particle of mass m moves in the xy-plane and its position vector is given by

$$\vec{r}(t) = a\cos(\omega t)\hat{i} + b\sin(\omega t)\hat{j}$$

where a, b and   are positive constants, and a>b. Show that

(a) the particle moves in an ellipse;

(b) the force acting on the particle is always directed toward the origin;

(c) the total work done by the force in moving the particle once around the ellipse is zero;

(d) the force is conservative.

Solution:

(a) $$\vec{r} = x\hat{i} + y\hat{j} = a\cos(\omega t)\hat{i} + b\sin(\omega t)\hat{j}$$

so $$x = a\cos(\omega t), \qquad y = b\sin(\omega t)$$

and $$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2(\omega t) + \sin^2(\omega t) = 1$$

which is an ellipse with semi-major and semi-minor axes of lengths a and b respectively.

(b) The force acting on the particle is

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

$$= m\frac{d^2}{dt^2}[a\cos(\omega t)\hat{i} + b\sin(\omega t)\hat{j}] = -m\omega^2\vec{r}$$

The minus sign indicates that the force is always directed toward the origin.

(c) Since t goes from 0 to t = 2 /   for a complete circuit around the ellipse, we have

$$W = \int_0^{2\pi/\omega} \vec{F}\cdot d\vec{r} = \int_0^{2\pi/\omega}[-m\omega^2(a\cos\omega t\hat{i} + b\sin\omega t\hat{j})]\cdot[-a\omega\sin\omega t\hat{i} + b\omega\cos\omega t\hat{j}]dt$$

$$= \int_0^{2\pi/\omega} m\omega^3(a^2 - b^2)\sin\omega t\cos\omega t dt = \frac{1}{2}m\omega^3(a^2 - b^2)\sin^2\omega t\Big|_0^{2\pi/\omega} = 0$$

(d) For this part, we write the force in the form (from part b):

$$\vec{F} = -m\omega^2\vec{r} = -m\omega^2(x\hat{i} + y\hat{j}).$$

Then

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -m\omega^2 x & -m\omega^2 y & 0 \end{vmatrix} = 0$$

Hence the force is conservative.

2-2 A constant force F acting on a particle of mass m changes the velocity from v to v in time τ. Prove that $$\vec{F} = m(\vec{v}_2 - \vec{v}_1)/\tau$$

Solution:

By Newton's second law, we have

$$m\frac{d\vec{v}}{dt} = \vec{F}, \qquad or \qquad \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m}$$

For constant force we have on integrating

$$\vec{v} = (\vec{F}/m)t + \vec{c}$$

where c is an integration constant (a vector). At $t = 0$, $\vec{v} = \vec{v}_1$ so that $\vec{c} = \vec{v}_1$, and

$$\vec{v} = (\vec{F}/m)t + \vec{v}_1$$

At $t = \tau$ $\vec{v} = \vec{v}_2$ so that $\vec{v}_2 = (\vec{F}/m)\tau + \vec{v}_1$. Solving for F, we get

$$\vec{F} = m(\vec{v}_2 - \vec{v}_1)/\tau$$

Alternatively, we can rewrite Newton's second law as

$$md\vec{v} = \vec{F}dt$$

Then since $\vec{v} = \vec{v}_1$ at $t = o$ and $\vec{v} = \vec{v}_2$ at $t = \tau$, we have

$$\int_0^{\tau} \vec{F}dt = \int_{\vec{v}_1}^{\vec{v}_2} md\vec{v}, \qquad \text{or} \qquad \vec{F}\tau = m(\vec{v}_2 - \vec{v}_1)$$

2.3 Find the work done in moving an object along a path given by

$$\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$$

and the applied force is $\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$.

Solution:

Work done $= \vec{F}\cdot\vec{r} = (2\hat{i} - \hat{j} - \hat{k})\cdot(3\hat{i} + 2\hat{j} - 5\hat{k}) = 6 - 2 + 5 = 9$

2.4 (a) Show that

$$\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

is a conservative force field.

(b) Find the potential energy V.

(c) Find the work done in moving an object in this force field from (1,-2,1) to (3,1,4).

Solution:

(a)
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = 0$$

Thus F is a conservative force field.

(b)

$$\vec{F} = -\nabla V = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

from which we obtain

$$\partial V/\partial x = -(2xy + z^3), \qquad \partial V/\partial y = -x^2, \qquad \partial V/\partial z = -3xz^2$$

Integrate the first equation with respect to x keeping y and z constant:

$$V = -(x^2y + xz^3) + g_1(y,z) \tag{1}$$

where $g_1(y,z)$ is a function of y and z.

Similarly integrating the second equation with respect to y (keeping x and z constant) and the third equation with respect to z (keeping x and y constant), we obtain

$$V = -x^2y + g_2(x,z) \tag{2}$$

$$V = -xz^3 + g_3(x,y) \tag{3}$$

Equations (1), (2) and (3) yield a common V if we choose

$$g_1(y,z) = c, \qquad g_2(x,z) = -xz^3 + c, \qquad g_3(x,y) = -x^2y + c$$

where c is an arbitrary constant, and it follows that

$$V = -(x^2y + xz^3) + c$$

(c)

$$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} = -\int_{P_1}^{P_2} \nabla V\cdot d\vec{r} = -\int_{P_1}^{P_2} [\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= -\int_{P_1}^{P_2} dV = -(x^2y + xz^3)\Big|_{(1,-2,1)}^{(3,1,4)} = -202.$$

2.5 A particle of mass m moves along the x axis under the influence of a conservative force field having potential V(x). If the particle is located at positions x and x at respective time t and t , prove that if E is the total energy

$$t_2 - t_1 = \sqrt{m/2}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$

Solution:

By conservation of energy, we have

$$\frac{1}{2}m(dx/dt)^2 + V(x) = E$$

from which we obtain (on considering the positive square root)

$$dt = \sqrt{m/2}[dx/\sqrt{E - V(x)}]$$

Hence by integration

$$\int_{t_1}^{t_2} dt = t_2 - t_1 = \sqrt{m/2}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$

2.6 A particle moves in a force field given by $\vec{F} = r^2\vec{r}, where \pm\vec{r}$ is the position vector of the particle. Show that the angular momentum of the particle is conserved.

Solution:

By Equation (48), we have

$$\frac{d\vec{L}}{dt} = \vec{N}^{(e)}$$

which states that if the external torque about a given point is zero, the angular momentum of the system about the same point is conserved (a constant vector in time). Now

$$\vec{N}^{(e)} = \vec{r}\times\vec{F} = \vec{r}\times(r^2\vec{r}) = r^2(\vec{r}\times\vec{r}) = 0$$

Hence the angular momentum is conserved.

2.7 A particle of mass m moves along the path given by

$$x = x_o + at^2, \qquad y = bt^3, \qquad z = ct$$

where x , a, b and c are constants. Find the following quantities at any time t: angular momentum L, force F and torque N on the particle. Verify that they satisfy

$$\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F} = \vec{N}$$

Solution:

(a) The angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = \vec{r} \times m(d\vec{r} / dt) \qquad (1)$$

where
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (x_o + at^2)\hat{i} + bt^3\hat{j} + ct\hat{k} \qquad (2)$$

Now

$$md\vec{r} / dt = m\frac{d}{dt}[(x_o + at^2)\hat{i} + bt^3\hat{j} + ct\hat{k}] = 2mat\hat{i} + 3mbt^2\hat{j} + mc\hat{k} \qquad (3)$$

Substituting Eqs.(2) and (3) into Eq.(1) yields

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_o + at^2 & bt^3 & ct \\ 2mat & 3mbt^2 & mc \end{vmatrix} = (-2mbct^3)\hat{i} + (mact^2 - mx_oc)\hat{j}$$

$$+ (mabt^4 + 3mx_obt^2)\hat{k}$$

(b) The force

From Newton's second law

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d^2}{dt^2}[(x_o + at^2)\hat{i} + bt^3\hat{j} + ct\hat{k}]$$
$$= 2ma\hat{i} + 6mbt\hat{j} + 0\hat{k}$$

(c) The torque

$$\vec{N} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_o + at^2 & bt^3 & ct \\ 2ma & 6mbt & 0 \end{vmatrix} = -6mbct^2\hat{i} + 2mact\hat{j} + (4mabt^3 + 6mx_obt)\hat{k}$$

it is equal to the derivative of $\vec{L}$ with respect to t:

$$d\vec{L} / dt = -6mbct^2\hat{i} + 2mact\hat{j} + (4mabt^3 + 6mx_obt)\hat{k}$$

2.9 Two astronauts A and B, initially at rest in free space, pull on either end of a rope. The maximum force with which A can pull, F_A, is larger than the maximum force with which B can pull, F_B. Their masses are M_A and M_B, the mass of the rope M_r is negligible. Find their motion if each pulls on the other as hard as he can.

Solution:

As shown in Fig. 2.9, the forces exerted by the rope on the astronauts are F_A' and F_B'

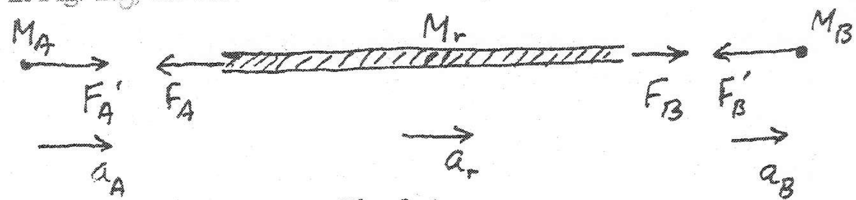

Fig. 2.9

By Newton's third law, we have

$$F_A' = F_A \qquad F_B' = F_B \qquad (1)$$

Only motion along the line of the rope is of interest, and the equation of the motion for the rope is then simply given by

$$F_B - F_A = M_r a_r \qquad (2)$$

Since the mass of the rope is negligible, we take $M_r = 0$, then Eq.(2) gives

$$F_A - F_B = 0, \qquad or \qquad F_A = F_B$$

Thus, the total force on the rope is F_A to the left and F_B to the right. But these forces are

2.9 A block of mass M, resting on a smooth table, is pulled by a string of mass m. If a force F is applied to the string, what is the force that the string transmitted to the block?

Solution:

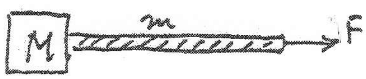

Fig. 2.9a

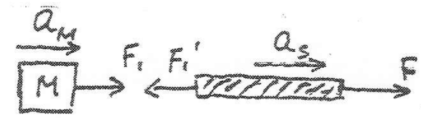

Fig. 2.9b

Figs. 2.9a and 2.9b show the force diagrams. F_1 is the force of the string on the block, and F_1' is the force of the block on the string, a_M is the acceleration of the block and a_S is the acceleration of the string. The equations of motion are

$$F_1 = Ma_M\,, \qquad F - F_1' = ma_S$$

There are two constraints. First, by Newton's third law we have

$$F_1 = F_1'\,.$$

Second, if the string is inextensible, it accelerates at the same rate as the block:

$$a_S = a_M \equiv a\,.$$

Solving for the acceleration, we find that

$$a = \frac{F}{M+m}$$

and the force on the block

$$F_1 = F_1' = \frac{M}{M+m}F$$

The string does not transmit the full applied force to the block.

2.10 Two particles, of masses m_1 and m_2, are connected by a rigid rod lying on a smooth horizontal table. If an impulse I is applied at m_1 in the plane of the table and perpendicular to the rod, find the initial velocities of m_1 and m_2.

Solution:

The center of mass of the system moves in the direction of the impulse and so A and

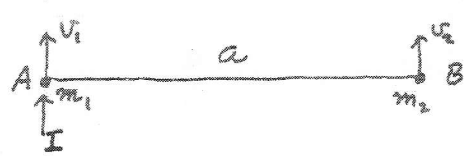

Fig. 2.10

B must also move in the same direction. The momentum of the system is $m_1v_1 + m_2v_2$, and the initial momentum of the system is zero. So the change in momentum is

$$m_1v_1 + m_2v_2 = I \qquad (1)$$

Now the moment of the impulse about B is equal to the gain of angular momentum of the system about B:

$$Ia = m_1v_1a \qquad (2)$$

where a is the length of the rod. Solving Eq.(2) for v_1 we obtain

$$v_1 = I / m_1$$

Then from Eq.(1) we find that $\qquad v_2 = 0.$

2.11 A wheel of radius b is rolling along a muddy road with a speed v_o. Particles of mud attached to the wheel are being continuously thrown off from all points of the wheel. If $v_o^2 > bg$, where g is the gravitational acceleration, show that the maximum height above the road attained by the mud will be

$$b + v_o^2 / 2g + b^2 g / 2v_o^2 .$$

Solution:

In Fig. 2.11a, we see that

$$\vec{r} = \hat{i} b\cos\theta - \hat{j} b\sin\theta, \qquad and$$

$$|\vec{r}| = b, \qquad \theta = \omega t$$

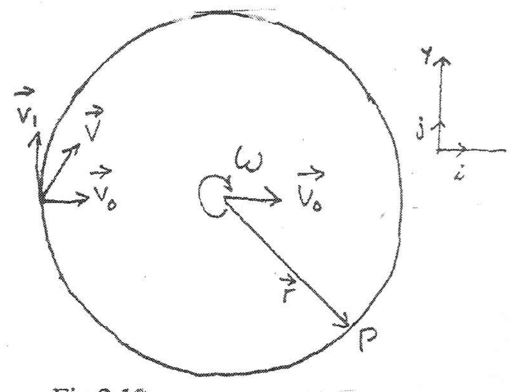

Fig.2.11a

The velocity of P relative to the center of the wheel is

$$\vec{v}_1 = d\vec{r} / dt = -\hat{i} b\omega\sin\theta - \hat{j} b\omega\cos\theta$$

The velocity of the center of the wheel relative to the ground is $\hat{i} v_o$. So the velocity of P relative to the ground is

$$\vec{v} = \hat{i} v_o + [-\hat{i} b\omega\sin\theta - \hat{j} b\omega\cos\theta], \qquad b\omega = v_o$$

$$= \hat{i} v_o(1-\sin\theta) - \hat{j} v_o\cos\theta = \hat{i} v_x + \hat{j} v_y$$

where $\qquad v_x = v_o(1-\sin\theta), \qquad v_y = -v_o\cos\theta$

An object thrown upwards with speed v_y will reach a height of $v_y^2 / 2g$ above the point from which it was thrown. Thus the maximum height h above the ground that a piece of mud will attain is (Fig.2.11b)

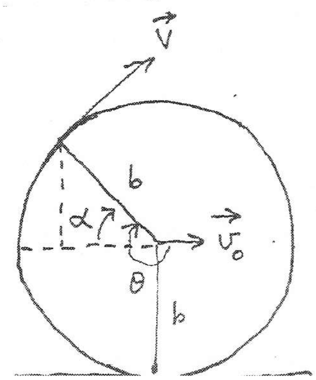

Fig. 2.11b

$$h = b + b\sin\alpha + v_y^2 / 2g$$

$$= b + b\sin(\theta - \pi) + \frac{v_o^2\cos^2\theta}{2g}$$

$$= b + b\sin\theta + \frac{v_o^2\cos^2\theta}{2g}$$

To find the maximum value of h, we set $dh / d\theta = 0$ and obtain

$$\frac{dh}{d\theta} = b\cos\theta - \frac{v_o^2\sin\theta\cos\theta}{g} = 0$$

from which we find

$$\sin\theta = bg / v_o^2, \qquad (v_o^2 > bg)$$

and then $\qquad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{v_o^4 - b^2 g^2} / v_o^2$

Substituting these into Equation (1) yields

$$\sin\theta = bg / v_o^2, \qquad (v_o^2 > bg)$$

and then $$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{v_o^4 - b^2 g^2} / v_o^2$$

Substituting these into Equation (1) yields

$$h = b + \frac{v_o^2}{2g} + \frac{b^2 g}{2v_o^2}, \qquad v_o^2 > bg.$$

2.12 A block, of mass m, is free to slide on the inclined face of a smooth wedge, of mass M and angle A. The wedge is itself free to slide on a smooth horizontal plane. Find the acceleration of the block and the wedge.

Solution:

Fig. 2.12a shows the force diagrams. N is the normal reaction between the block and

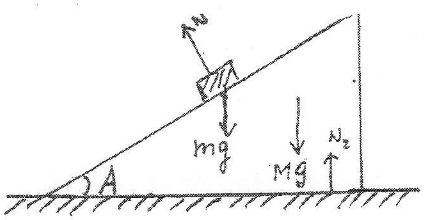

Fig. 2.12a

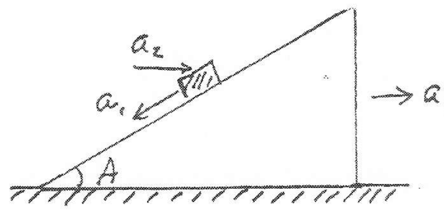

Fig. 2.12b

the wedge and it is perpendicular to wedge face; and N is the normal reaction between the wedge and the plane. In Fig. 2.12b, a_1, down the face of the wedge, is the acceleration of the block relative to the wedge; and a_2 is the acceleration of the wedge relative to the plane

The equation of motion for the wedge, when resolved horizontally and vertically, are

$$N_1 \sin A = Ma_2, \qquad Mg - N_2 + N_1 \cos A = 0$$

and, for the block $$N_1 \sin A = m(a_1 \cos A - a_2), \qquad mg - N_1 \cos A = ma_1 \sin A$$

We can solve these 4 equations for the four unknowns: a_1, a_2, N_1, and N_2. We find that

$$a_1 = \frac{(m+M)g\sin A}{m\sin^2 A + M}, \qquad a_2 = \frac{mg\sin A\cos A}{m\sin^2 A + M}.$$

2.13 A wooden block, of mass M, is resting on a horizontal surface. The coefficient of friction is f. One end of a spring, with spring constant k, is attached to the block; and the other end to a solid wall. The spring is unstretched. A bullet of mass m hits the block and embeds in it. Find the velocity of the bullet before impact in terms of the maximum compression x of the spring amd M, k, g and f.

Solution:

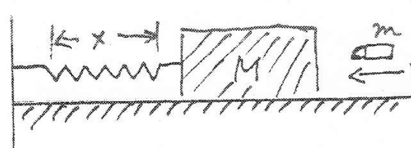

Fig. 2.13

By momentum conservation, we have

$$m \cdot V + M \cdot 0 = (M+m)V_c$$

where V is the velocity of the composite system of the wooden block and the bullet. Solving for V , we get

$$V_c = mV / (m+M) \qquad (1)$$

We now apply the work-energy theorem

$$W = \Delta(KE) + \Delta(PE)$$

We now apply the work-energy theorem

$$W = \Delta(KE) + \Delta(PE)$$

to the composite system of $m + M$:

(a) $$W=\int_a^b \vec{F}\cdot dx = -\int f(m+M)g dx = -f(m+M)gx$$

where $F = F_f = f(M+m)g$, and x is the maximum compression of the spring. (The distance moved by the composite system of m+M should be equal to the maximum compression of the spring.)

(b) $$\Delta(KE)=\frac{1}{2}(m+M)0^2-\frac{1}{2}(m+M)V_c^{\,2}=\frac{1}{2}(m+M)V_c^{\,2}$$

(c) $$\Delta(PE)=\frac{1}{2}kx^2$$

Substituting these into the work-energy theorem yields

$$-f(m+M)gx=-\frac{1}{2}(m+M)V_c^{\,2}+\frac{1}{2}kx^2 \qquad (2)$$

Solving Eqs.(1) and (2) for V, we find that

$$V^2=\frac{(m+M)^2}{m^2}\left[2fgx+\frac{kx^2}{m+M}\right]$$

2.14 Two massless springs S_1 and S_2, with spring constants k_1 and k_2 respectively, are arranged to support a weight A. In case I the springs are coupled in a series and in case II they are in parallel. Determine the extensions of the individual springs in these two cases as a result of the force of gravity on A. Determine also the equivalent spring constant in the two cases.

Solution:

Case I. Fig. 2.14b shows the force diagrams of the two springs in series. At equilibrium, we have

$$Mg = F_2, \qquad F_2 = k_2 x_2$$

hence $x_2 = F_2 / k_2 = Mg / k_2$

Similarly $F_1 = k_1 x_1$

and so $x_1 = F_1/k_1 = F_2/k_1 = Mg/k_1$

The total extension is

$$x = x_1 + x_2 = Mg\left(\frac{1}{k_1}+\frac{1}{k_2}\right)=\frac{Mg}{k_{eq}}$$

and $$\frac{1}{k_{eq}}=\frac{1}{k_1}+\frac{1}{k_2}$$

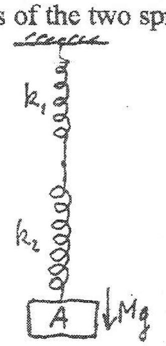

Fig. 2.14a

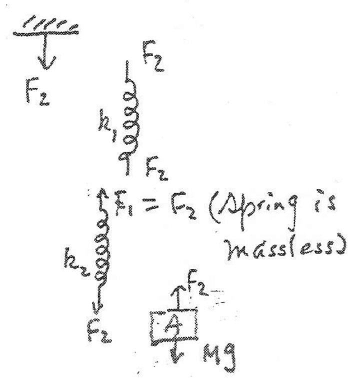

Fig. 2.14b

Case II. In this case the extensions of the two springs should be the same:

$$x_1 = x_2 = x$$

At equilibrium, we have

$$F_1 = k_1 x, \qquad F_2 = k_2 x, \qquad \text{and} \qquad F_1 + F_2 = Mg$$

from which we obtain

$$(k_1+k_2)x = Mg, \quad \text{or} \quad x = Mg/(k_1+k_2) = Mg/k_{eq}$$

where $k_{eq} = k_1 + k_2$.

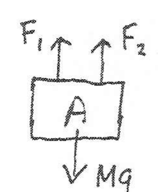

Fig. 2.14c

2.15 Consider a rod of length L. The mass density (mass per unit length) of the rod, ρ, varies as $\rho = \rho_o (s/L)$, where ρ_o is a constant and s is the distance from the end marked 0. Find the center of mass of the rod.

Solution:

The center of mass lies on the rod. Now let the x axis lie along the rod, with the origin at the left end of the rod. Then s = x and the mass in an element of length dx is dm = $\rho dx = \rho_o x dx / L$. The total mass M of the rod is

$$M = \int dm = \int_0^L \rho_o x dx / L = \frac{1}{2}\rho_o L$$

The center of mass is at

$$\vec{R} = \frac{1}{M}\int \vec{r}\rho dm = \frac{2}{\rho_o L}\int_0^L (x\hat{i} + 0\hat{j} + 0\hat{k})\frac{\rho_o x dx}{L} = \frac{2}{3}L\hat{i}$$

2.16 A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at an angle of elvation θ. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble is v_o. Find the final motion of the gun.

Solution:

The final motion of the gun can be found easily by using conservation of momentum. If air resistance is negligible, then there are no horizontal external forces acting on the system (the gun and marble), and the x component of the equation of motion $\vec{F} = d\vec{P} / dt$ is

$$0 = \frac{dP_x}{dt}$$

hence P_x is conserved: $P_{x,initial} = P_{x,final}$

Since the system is initially at rest, so $P_{x,initial} = 0$. It is apparent that $P_{x,final}$ is not zero. We now come to find out what it is. v_o is the velocity of the marble relative to the gun, not to the table. As shown in Fig. 2.16, the horizontal speed of the marble relative to the table is $v_o \cos\theta - V_f$. So we have

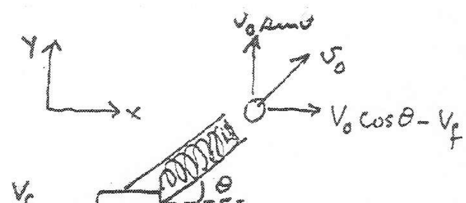

Fig. 2.16

$$0 = m(v_o \cos\theta - V_f) - MV_f$$

or

$$V_f = \frac{mv_o \cos\theta}{m + M}.$$

2.17 Two men of weights W and W are seated in the bow and stern of a boat of weight W at a distance L from each other. Ignoring the water resistance, determine the direction and size of the displacement of the boat if the men change places.

Solution:

We consider the boat and the men in it as one system. The external forces acting on the boat are the four vertical forces W_1, W_2, W, and N, and they are balanced:

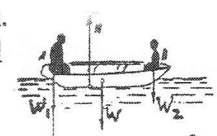

Fig. 2.17a

$$\sum \vec{F}_{ext} = N - (W_1 + W_2 + W) = 0$$

Let the x axis lie along the boat such that the coordinate of W_1 is at x_1, W at x, and W_2 at x_2(Fig. 2.17a). The center of mass of the system, R, is at

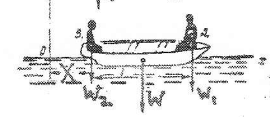

Fig. 2.17b

$$R = (m_1 x_1 + mx + m_2 x_2) / (m_1 + m + m_2) \quad (1)$$

where $m_1 = W_1 / g$, $m = W / g$, $m_2 = W_2 / g$.

Now the men change places and, as a result, the boat is displaced to the right by a distance X (Fig. 2.18b). If our answer give a negatives value, -X, then the boat is displaced to the left. The new coordinate of W_1 is at $(L+X+x_1)$, W at $(X+x)$, W_2 at (x_2-L+X). In terms of the new coordinates, the center of mass of the system can be written as

$$R_{new}=\frac{m_1(L+X+x_1)+m(x+X)+m_2(x_2-L+X)}{m+m_1+m_2} \tag{2}$$

Since the total external force acting on the system is zero, the center of mass of the system should remain unchanged as the men change places. Thus

$$R_{new}=R \tag{3}$$

where R is given by Eq.(1) and R_{new} is given by Eq.(2). Solving Eq.(3) for X we get

$$X=\frac{m_2-m_1}{m+m_1+m_2}L$$

If $m_2>m_1$, then X > 0, and the boat will move to the right; if $m_2<m_1$, then X < 0, and the boat will move to the left. If the men have equal weight, the boat will remain at rest.

2.18 A bar of mass m is placed on a plank of mass M, which rests on a smooth horizontal plane. The coefficient of friction between the surfaces of the bar and the plank is f. The plank is subject to a horizontal force F of the form F = ct, where c is a constant. Find (a) the moment of time t at which the plank starts sliding from under the bar. (b) the acceleration of the bar and of the plank in the process of their motion.

Solution:

(a) Let the x axis lie along the horizontal plane, and F is the friction force between the bar and the plank. (Fig. 2.18). The equations of motion for the bar and the plank are, respectively:

$$ma=F_f,\qquad MA=F-F_f \tag{1}$$

where a and A are accelerations of the bar and the plank respectively.

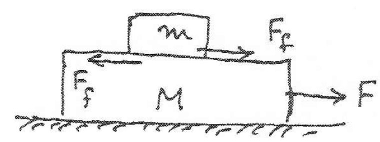

Fig. 2.18

As the force F grows, so does the friction force. But the maximum value of the friction force is fmg. Before this maximum value is reached, both bodies (the bar and the plank) move as a single whole with equal accelerations. But as soon as the friction force reaches the maximum value, the plank starts sliding from under the bar, i.e.

$$A\geq a$$

Substituting the values of a and A taken from Eq.(1) and taking into account that F = fmg we obtain

$$(bt-fmg)/M\geq fg$$

where the equal sign corresponds to the moment t = t .Hence

$$t_o=(m+M)fg/b$$

(b) If $t\leq t_o$, then

$$a = A = bt/(m+M)$$

and if $t\geq t_o$, then

$$a = fg = \text{constant},\qquad A = (bt - fmg)/M$$

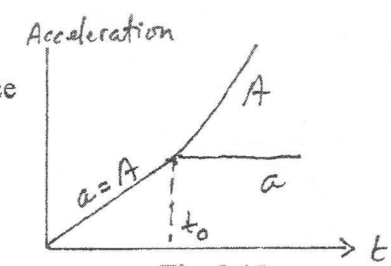

Fig. 2.18a

2.19 A uniform straight rigid bar of mass m and length b is placed in a horizontal position across the top of two identical cylindrical rollers. Axes of the two rollers are 2d apart. If f is the frictional coefficient between the cylinder surface and the bar, show that if the bar is displaced a distance x from its central position, then the net horizontal force on the bar is F = -fmgx/d, and the bar will execute simple harmonic motion with a period of $2\pi\sqrt{d/fg}$.

Solution:

Fig. 2.19 is a force diagram for the system. Displacement of the center of mass of the bar is x, positive to the right, measured from the mid-way point between the rollers. Equations of motion of the bar, in the component forms, are

$$m\ddot{x} = \sum F_x = F_1 - F_2,$$

$$m\ddot{y} = 0 = \sum F_y = N_1 + N_2 - mg$$

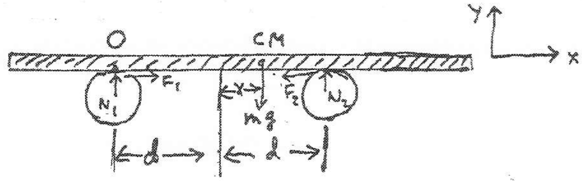

Fig. 2.19

where, as indicated in Fig. 2.20, N and N are normal forces and F and F are the friction forces on the bar. N and N can be determined by using the rotation equilibrium condition:

$$\sum \tau_{total}(total \circ torque) = 0$$

Taking torques about 0 ,we obtain

$$N_2(2d) = mg(x+d), \qquad or \qquad N_2 = \frac{1}{2}mg(1 + x/d) \qquad (3)$$

Then from Eq.(2) we obtain

$$N_1 = mg - N_2 = \frac{1}{2}mg(1 - x/d) \qquad (4)$$

The friction forces can now be determined:

$$F_1 - F_2 = fN_1 - fN_2 = -fmgx/d$$

Substituting this into Equation (1) we obtain

$$\ddot{x} + \frac{fg}{d}x = \ddot{x} + \omega_o^2 x = 0, \qquad \omega_o^2 = fg/d$$

which describes simple harmonic motion at angular frequency ω. The period of oscillation is

$$P = 2\pi/\omega = 2\pi\sqrt{d/fg}\ .$$

2.20 A particle of mass m is attached to the end of a string and moves in a circle of radius r on a horizontal table. The string passes through a frictionless hole in the table and the other end is fixed initially.

(a) If the string is pulled so that the radius of the circular orbit decreases, how does the angular velocity change if it is ω_o when $r = r_o$?

(b) What work is done when the particle is pulled slowly in from a radius r to a radius r /2?

Solution:

(a) When the particle moves in circle of radius r , the magnitude of its angular momentum is

$$L_o = mr_o^2\omega_o, \qquad (v = r_o\omega_o)$$

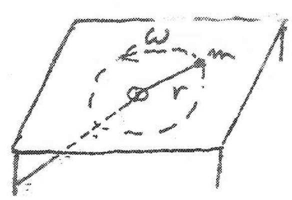

Fig. 2.20

As the radius of the circular orbit is reduced from r_o to r, the angular frequency is correspondingly changed to ω and the angular momentum of the particle becomes

$$L = mr^2\omega$$

If the string is pulled very slowly, the angular momentum of the particle is conserved (adiabatic invarience) and so we have

$$mr^2\omega = mr_o^2\omega_o$$

or

$$\omega = \left(\frac{r_o}{r}\right)^2 \omega_o$$

(b) By using the work-energy principle, we have

$$W = \Delta(KE), \qquad (\Delta(PE) = 0 \text{ for our case})$$

Now

$$\Delta(KE) = \frac{1}{2}mr^2\omega^2 - \frac{1}{2}mr_o^2\omega_o^2$$

where $r = r_o/2$, and $\omega = (r_o/r)^2\omega_o$. Substituting these into the above equation, we find

$$W = \Delta(KE) = -\frac{3}{2}mr_o^2\omega_o^2 .$$

Alternatively the work done on the particle is given by

$$W = \int_{r_o}^{r_o/2} \vec{T}\cdot d\vec{r}$$

where T is the tension in the string which provides the centripetal force required for circular motion,

$$T = F_c = mv^2/r = mr\omega^2 = m\frac{r_o^4}{r^2}\omega_o^2$$

Substituting this into the work-integral we will find the same result given by work-energy principle.

2.2 A bead of mass m slides without friction on a vertical hoop of radius R. The bead moves under a combination of gravity and spring attached to the bottom of the hoop. For simplicity, we assume that the equilibrium length of the spring is zero, so that the force due to the spring is -kr, where r is the instantaneous length of the spring, as shown in Fig. 2.2 . The bead is released at the top of the hoop with negligible speed. How fast is the bead moving at the bottom of the hoop?

Solution:

At the top of the hoop, the gravitational potential energy of the bead is mg(2R) and the potential energy of the spring is $k(2R)^2/2 = 2kR^2$. Hence the initial potential energy of the bead is

$$PE_i = 2mgR + 2kR^2$$

The potential energy at the bottom of the hoop is

$$PE_f = 0$$

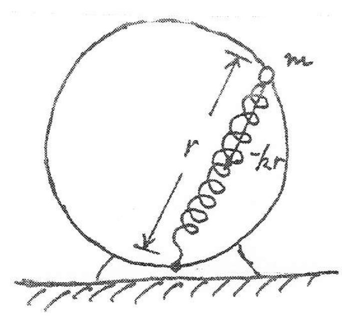

Fig. 2.2

The initial kinetic energy of the bead is zero, $KE_i = 0$; The kinetic energy at the bottom d is $KE_f = mv_f^2/2$.

Since all the forces are conservative, the mechanical energy is conserved and we have

$$KE_i + PE_i = KE_f + PE_f$$

or

$$\frac{1}{2}mv_f^2 + 0 = 0 + (2mgR + 2kR^2)$$

Solving for v_f we get

$$v_f = 2\sqrt{gR + kR^2/m} .$$