Chapter 2 problems

2.9.2

$p(x) \geq 0$ for all $x = 1, 2, \cdots$ and $\sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{\infty} \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots = 1 + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{3}) + \cdots = 1$

Thus, $p(x)$ is a legitimate probability mass function.

2.9.4

Let A denotes the event that team A wins and B denotes the event that team B wins. Let $P(A) = p$ Let the series ends in k games where $k = 4, 5, 6, 7$. Conditional on the event A wins the series (or equivalently, the last game), A should have won 3 games and B should have won $(k-4)$ games before the last game. Roles of A and B are reversed, given that B wins the series. Hence, if X denotes the number of games in the series, then

$$\begin{aligned} P(X = k) &= P(\text{A wins the last game})P(X = k|\text{A wins the last game}) \\ &\quad + P(\text{B wins the last game})P(X = k|\text{B wins the last game}) \\ &= p \cdot \binom{k-1}{3} p^3 (1-p)^{k-4} + (1-p) \cdot \binom{k-1}{3} (1-p)^3 p^{k-4} \qquad \text{for } k = 4, 5, 6, 7 \end{aligned}$$

Plugging in the values of k and p, we get

$$\begin{aligned} P(X = 4) &= (0.6)\binom{3}{3}(0.6)^3(0.4)^0 + (0.4)\binom{3}{3}(0.4)^3(0.6)^0 = 0.1552 \\ P(X = 5) &= (0.6)\binom{4}{3}(0.6)^3(0.4) + (0.4)\binom{4}{3}(0.4)^3(0.6) = 0.2688 \\ P(X = 6) &= (0.6)\binom{5}{3}(0.6)^3(0.4)^2 + (0.4)\binom{5}{3}(0.4)^3(0.6)^2 = 0.29952 \\ P(X = 7) &= (0.6)\binom{6}{3}(0.6)^3(0.4)^3 + (0.4)\binom{6}{3}(0.4)^3(0.6)^3 = 0.27648 \end{aligned}$$

Therefore, the expected number of games is

$$E(X) = \sum_{k=4}^{7} kP(X = k) = 5.69728.$$

2.9.6

Of the $5! = 120$ permutations of the integers $1, 2, 3, 4, 5$ the following will yield system failure after 2-components failure.

(a) $1, 2, \_, \_, \_$: $3! = 6$ ways.

(b) $2, 1, \_, \_, \_$: 6 ways.

(c) $4, 5, \_, \_, \_$: 6 ways.

(d) $5, 4, \_, \_, \_$: 6 ways.

i.e., 24 ways to fail upon 2-component failure. Hence, $P(X = 2) = \frac{24}{120} = \frac{1}{5}$.

$P(X = 3) = 1 - P(X = 2) - P(X = 4)$, since system can not fail after 1 failure, and can not survive after 4 failures, $P(X = 1) = 0 = P(X = 5)$.

$P(X = 4)$: Failure at the 4-th component failure will occur for orderings

(a) $1, 3, 4, \_, \_$: $3!.2!$ ways $= 12$ ways

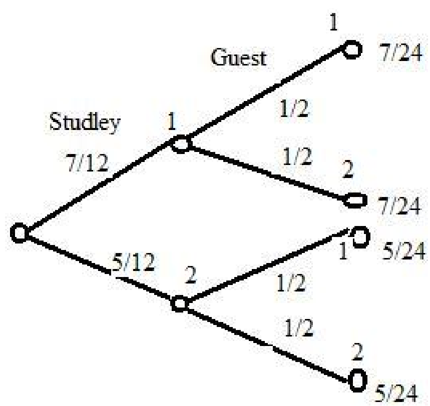
(b) $2, 3, 5, \_, \_$: 12 ways

i.e., 24 ways. Hence, $P(X=4)=\frac{24}{120}=\frac{1}{5}$. So, distribution of X is

$$\begin{array}{cccccc} X & 1 & 2 & 3 & 4 & 5 \\ P(X=x) & 0 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 \end{array}$$

2.9.8

The following tree describes the problem schematically. Let F be the total number of fingers and W be the winning



amount for Studley. Then their distributions are given by

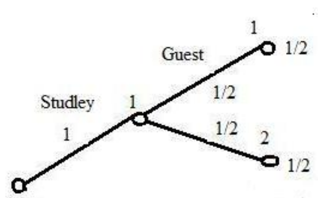
F	2	3	4
P(F=f)	$\frac{7}{24}$	$\frac{1}{2}$	$\frac{5}{24}$

and

W	-5	4	6
P(W=w)	$\frac{1}{2}$	$\frac{5}{24}$	$\frac{7}{24}$

Expected value of Studley's winning under this strategy $E(W)=(-5).\frac{1}{2}+4.\frac{5}{24}+6.\frac{7}{24}=\frac{1}{12}$.

If he always sticks one finger out, then



which means

W	-5	6
P(W=w)	$\frac{1}{2}$	$\frac{1}{2}$

Since $EW=(-5).\frac{1}{2}+6.\frac{1}{2}=\frac{1}{2}$, sticking out one finger always favors Studley.

2.9.10

The expected payoff (without any charge) is

$EP=5.\frac{1}{10}+2.\frac{3}{10}+1.\frac{2}{10}+0.\frac{4}{10}=\frac{13}{10}$ or \$1.30. In order to make a profit of \$0.20, \$1.30 + \$0.20 = \$1.50 is to be charged each time the game is played.

2.9.12

(a) $P(X=k) = \frac{\binom{3}{k}\binom{7}{3-k}}{\binom{10}{3}}$ for $k = 0, 1, 2, 3$

Therefore, $P(X=0) = \frac{\binom{3}{0}\binom{7}{3}}{\binom{10}{3}} = \frac{\frac{7.6.5}{6}}{\frac{10.9.8}{6}} = \frac{7}{24} = 0.2917$

$P(X=1) = \frac{21}{40} = 0.525$

$P(X=2) = \frac{7}{40} = 0.175$

$P(X=3) = \frac{1}{120} = 0.008333$

(b) $E(X) = 0 \times (0.2917) + 1 \times (0.525) + 2 \times (0.175) + 3 \times (0.008333) = 0.9$

2.9.14

By definition $E(X) = \sum_{n=0}^{\infty} nP(X=n) = \sum_{n=1}^{\infty} nP(X=n)$

$= P(X=1)$

$+P(X=2) + P(X=2)$

$+P(X=3) + P(X=3) + P(X=3)$

$+\cdots$

$= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \cdots = \sum_{n=1}^{\infty} P(X \geq n)$

2.9.16

Let $f(c) = E(X-c)^2 = E(X^2) - 2cEX + c^2$. Then $f'(c) = -2EX + 2c$.

$f'(c) = 0$ gives $c = EX = \mu$. $f''(c) = 2 > 0$. So $E(X-c)^2$ is minimized when $c = \mu$.

2.9.18

Since the probability of obtaining $X = x$ is the weighted sum of $p_i(x)$ where the weights are given by the probability of taking the value X_i, the p.m.f. of X is given by $p(x) = \sum_{i=1}^{k} a_i p_i(x)$. Alternatively,

$P(X=x) = \sum_{i=1}^{k} P(X=X_i)P(X_i = x|X = X_i) = \sum_{i=1}^{k} a_i p_i(x)$

$EX = \sum_{\text{all } x} xp(x) = \sum_{\text{all } x} x \sum_{i=1}^{k} a_i p_i(x) = \sum_{i=1}^{k} a_i \sum_{\text{all } x} xp_i(x) = \sum_{i=1}^{k} a_i EX_i$

2.9.20

$X \sim HG(N, n, r)$. So $EX = \frac{nr}{N}$. Since the randomly observed x is likely to be reasonably close to EX, we assume that $x \approx \frac{nr}{N}$, from which we obtain $N^* = \frac{nr}{x}$ as a guess for N.

2.9.22

Let W be the number of sets (out of five) that Roger wins. $P(\text{Roger wins set}) = 0.4$. Then,

$P(W \geq 3) = 1 - P(W < 3) = 1 - P(W \leq 2) = 1 - \sum_{w=0}^{2} \binom{5}{w} (0.4)^w (0.6)^{5-w} = 0.31744$

2.9.24

$\frac{P(X=x)}{P(X=x-1)} = \frac{\binom{r}{x}\binom{N-r}{n-x}/\binom{N}{n}}{\binom{r}{x-1}\binom{N-r}{n-x+1}/\binom{N}{n}} = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{r}{x-1}\binom{N-r}{n-x+1}}$

$= \frac{\frac{r!}{x!(r-x)!}\frac{(N-r)!}{(n-x)!(N-r-n+x)!}}{\frac{r!}{(x-1)!(r-x+1)!}\frac{(N-r)!}{(n-x+1)!(N-r-n+x-1)!}} = \frac{(r-x+1)(n-x+1)}{x(N-r-n+x)}$

$\frac{P(X=x)}{P(X=x-1)} \leq 1$ means that

$(r-x+1)(n-x+1) \leq x(N-r-n+x)$, which, upon simplification, yields

$nr + r - 2x + n + 1 \leq Nx$

Or, $(n+1)(r+1) \leq (N+2)x$

Or, $\frac{(n+1)(r+1)}{N+2} \leq x$

i.e., the mode of the distribution of X is the largest integer less than or equal to $\frac{(n+1)(r+1)}{N+2}$.

2.9.26

Let X be the number of defective batteries. Then,

(a) $P(X=1) = \frac{\binom{4}{1}\binom{8}{2}}{\binom{12}{3}} = \frac{28}{55}$

(b) The probability that the first two batteries work but the third one doesn't is $= (\frac{8}{12}).(\frac{7}{11}).(\frac{4}{10}) = \frac{28}{165}$

2.9.28

Let X and Y be the random variables for the number of patients in the first and second groups whose headache pain is alleviated. Then,

$P(X > Y) = P(X \geq 1, Y = 0) + P(X \geq 2, Y = 1) + P(X \geq 3, Y = 2) + P(X \geq 4, Y = 3) + P(X = 5, Y = 4)$

$= \binom{5}{0}(0.5)^0(0.5)^5 \sum_{x=1}^{5} \binom{5}{x}(0.4)^x(0.6)^{5-x} + \binom{5}{1}(0.5)^1(0.5)^4 \sum_{x=2}^{5} \binom{5}{x}(0.4)^x(0.6)^{5-x} + \cdots +$
$\binom{5}{4}(0.5)^4(0.5)^1 \sum_{x=5}^{5} \binom{5}{x}(0.4)^x(0.6)^{5-x}$

$= 0.26042$

2.9.30

Let W = win \$1 on finite sequence of one or more spins. $P(W) = 1 - P$ (6 straight losses).

For every spin, $P(\text{loss}) = \frac{20}{38} = 0.52632$.

$P(W) = 1 - (0.52632)^6 = 0.97874$.

$P(W_1 \bigcap W_2 \bigcap W_3 \bigcap W_4 \bigcap W_5) = \left(1 - (0.52632)^6\right)^5 = 0.89812$ i.e., about 90% chance of leaving the table \$5 ahead.

Expected winning $(+5)(0.89812) + (-63)(0.10188) = -\$5.97$

2.9.32

(a) $P(\text{2 aces in a random bridge hand}) = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} = 0.2135$.

Let $X \sim B(6, 0.2135)$. Then, $P(\text{exactly 2 aces in 6 hands}) = P(X=2) = \binom{6}{2}(0.2135)^2(0.7865)^4 = 0.2616$

(b) $P(\text{0 aces in a hand}) = \frac{\binom{48}{13}}{\binom{52}{13}} = a$

$P(\text{1 ace in a hand}) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} = b$

$P(\text{at least 2 aces in a hand}) = 1 - a - b$

$P(\text{at least 2 aces in at least 2 hands out of 6}) = 1 - P(X \leq 1)$ where $X \sim B(6, 1-a-b)$, which is ≈ 0.4834.

2.9.34

$E(2^X) = \sum_{x=0}^n 2^x \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^n \binom{n}{x} (2p)^x q^{n-x} = (2p+q)^n = (1+p)^n$

2.9.36

By theorem 2.5.3, $V(X) = np(1-p)$. Let $f(p) = np(1-p)$. Then $f'(p) = n(1-2p)$ gives that $f'(p) = 0$ when $p = \frac{1}{2}$. $f''(p) = -2n$ shows that $f(p)$ attains its maximum at $p = \frac{1}{2}$. Thus, $V(X) \le n(\frac{1}{2})(1-\frac{1}{2}) = \frac{n}{4}$ and $V(X) = \frac{n}{4}$ if and only if $p = \frac{1}{2}$

2.9.38

Let X be the number of trials until r^{th} success. Then $X \sim NB(r,p)$. Here $r = 4$ and $p = \frac{1}{2}$. Hence,

$P(X \ge 7) = 1 - P(X \le 6) = 1 - \sum_{x=4}^6 \binom{x-1}{3} (0.5)^4 (0.5)^{x-4} = 1 - \binom{3}{3}(0.5)^4 - \binom{4}{3}(0.5)^5 - \binom{5}{3}(0.5^6) = 0.65625$

2.9.40

$P(X > k) = \sum_{x=k+1}^{\infty} pq^{x-1} p \sum_{x=k}^{\infty} q^x = p.\frac{q^k}{1-q} = q^k$

2.9.42

Let M_T = {Tony misses the bull's eye}; M_C = {Cleo misses the bull's eye}; H_C ={Cleo hits the bull's eye}

$P(M_T H_C) = (0.9)(0.2)$

$P(M_T M_C M_T H_C) = (0.9)^2(0.8)(0.2)$

$P(M_T M_C M_T M_C M_T H_C) = (0.9)^3(0.8)^2(0.2)$

Thus, the probability that Cleo hits the bull's eye before Tony does is given by:

$= P(M_T H_C) + P(M_T M_C M_T H_C) + P(M_T M_C M_T M_C M_T H_C) + \cdots$

$= (0.9)(0.2) + (0.9)^2(0.8)(0.2) + (0.9)^3(0.8)^2(0.2) + \cdots$

$= \sum_{n=0}^{\infty} (0.9)^{n+1}(0.8)^n(0.2) = (0.9)(0.2)\sum_{n=0}^{\infty}(0.72)^n = 0.18.\frac{1}{1-0.72} = \frac{9}{14}$

2.9.44

First, note that $\sum_{i=0}^n A_i = \frac{1-A^{n+1}}{1-A}$

Now, $\sum_{y=1}^{M-1} ypq^{y-1} = p\sum_{y=1}^{M-1} yq^{y-1} = p\frac{d}{dq}q^y p \sum_{y=1}^{M-1} q^y = p\frac{d}{dq}\left(q.\frac{1-q^{M-1}}{1-q}\right) = p\frac{d}{dq}\left(\frac{q-q^M}{1-q}\right) = \frac{1}{p}(1-q^M - Mpq^{M-1})$

So, $EY = \frac{1}{p}(1-(q+Mp)q^{M-1}) + Mq^{M-1} = \frac{1-(q+Mp)q^{M-1}+Mpq^{M-1}}{p} = \frac{1-q^M - Mpq^{M-1}+Mpq^{M-1}}{p} = \frac{1-q^M}{p} = \frac{1-q^M}{1-q}$

2.9.46

Let N_S = {Somnia not awake}; N_T = {Tarde not awake}; Y_T = {Tarde awake}

$P(N_S Y_T) = (0.8)(0.3)$

$P(N_S N_T N_S Y_T) = (0.8)^2(0.7)(0.3)$

$P(N_S N_T N_S N_T N_S Y_T) = (0.8)^3(0.7)^2(0.3)$

...

Thus, the probability that Tarde wins is given by

$P(N_S Y_T) + P(N_S N_T N_S Y_T) + P(N_S N_T N_S N_T N_S Y_T) + \cdots = \sum_{n=0}^{\infty}(0.8)^{n+1}(0.7)^n(0.3) = \frac{6}{11}$

2.9.48

(a) As mentioned in the hint, this event will happen only on sequences of the type $H, MMH, MMMMH, \cdots$ etc. Therefore, the required probability

$$\begin{aligned} P(H \cup MMH \cup MMMMH \cup \cdots) &= P(H) + P(MMH) + P(MMMMH) + \cdots \\ &= 0.1 + (0.9)(0.8)(0.1) + (0.9)(0.8)(0.9)(0.8)(0.1) + \cdots \\ &= 0.1\left[1 + 0.72 + (0.72)^2 + \cdots\right] = \frac{0.1}{1-0.72} = \frac{5}{14} \end{aligned}$$

(b) This event will happen only on sequences of the type $MH, MMMH, MMMMMH, \cdots$ etc. Therefore, the required probability

$$\begin{aligned} P(MH \cup MMMH \cup MMMMMH \cup \cdots) &= P(MH) + P(MMMH) + P(MMMMMH) + \cdots \\ &= (0.8)(0.1) + (0.8)(0.9)(0.8)(0.1) + (0.8)(0.9)(0.8)(0.9)(0.8)(0.1) + \cdots \\ &= (0.8)(0.1)\left[1 + 0.72 + (0.72)^2 + \cdots\right] = \frac{0.08}{1-0.72} = \frac{2}{7} \end{aligned}$$

2.9.50

The two different ways you can win, as mentioned in the hint, are $WW, WLWW, \cdots$ and $LWW, LWLWW, \cdots$. The probability that you win in the first way is

$$\begin{aligned} P(WW \cup WLWW \cup WLWLWW \cdots) &= P(WW) + P(WLWW) + P(WLWLWW) + \cdots \\ &= (0.6)(0.6) + (0.6)(0.4)(0.6)(0.6) + (0.6)(0.4)(0.6)(0.4)(0.6)(0.6) + \cdots \\ &= (0.6)^2\left[1 + 0.24 + (0.24)^2 + \cdots\right] = \frac{0.36}{1-0.24} = \frac{9}{19} \end{aligned}$$

The probability that you win in the second way is

$$\begin{aligned} P(LWW \cup LWLWW \cup LWLWLWW \cdots) &= P(LWW) + P(LWLWW) + P(LWLWLWW) + \cdots \\ &= (0.4)(0.6)(0.6) + (0.4)(0.6)(0.4)(0.6)(0.6) \\ &\quad + (0.4)(0.6)(0.4)(0.6)(0.4)(0.6)(0.6) + \cdots \\ &= (0.4)(0.6)^2\left[1 + 0.24 + (0.24)^2 + \cdots\right] = \frac{0.144}{1-0.24} = \frac{18}{95} \end{aligned}$$

Therefore, the probability that you win is $= \frac{9}{19} + \frac{18}{95} = \frac{63}{95} \approx 0.6632$

2.9.52

Let X_i be the event that both Eddie and I get the same outcome at trial i. Let W_i be the event that I get H while Eddie gets T at trail i. Then, the probability that I win is

$P(W) = P(W_1) + P(X_1W_2) + P(X_1X_2W_3) + \cdots$

Now, $P(W_i) = (0.6)(0.3) = 0.18$ for any i.

$P(X_i) = P(HH) + P(TT) = (0.6)(0.7) + (0.4)(0.3) = 0.54$ for any i.

Hence, $P(W) = 0.18 + (0.54)(0.18) + (0.54)^2(0.18) + \cdots = 0.18\sum_{n=0}^{\infty}(0.54)^n = \frac{0.18}{1-0.54} = \frac{9}{23}$

2.9.54

By the hint, $P(X_1) = 1$. Consider X_2. There are $n-1$ choices given n coupons. Thus, $X_2 \sim G(\frac{n-1}{n})$. Similarly, we have $X_i \sim G(\frac{n-i+1}{n})$ for $i = 1, \cdots, n$. Hence, $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{n}{n-i+1}$

2.9.56

Let X be the number of emissions per 10 microseconds. Then $X \sim B(10, 0.2) \approx P(2)$. Thus, $P(X=3) \approx \frac{e^{-2}2^3}{3!} = 0.180447$

2.9.58

$E(\frac{1}{X+1}) = \sum_{x=0}^{\infty} \frac{1}{x+1} \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{\lambda} \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} = \frac{1}{\lambda} \sum_{y=1}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} = \frac{1}{\lambda}(1 - P(Y=0)) = \frac{1}{\lambda}(1 - e^{-\lambda})$.

2.9.60

The p.m.f. of $X \sim P(\lambda)$ is $\frac{e^{-\lambda}\lambda^x}{x!}$. Consider the ratio between $P(X=x)$ and $P(X=x+1)$. Then

$\frac{P(X=x)}{P(X=x-1)} = \frac{\frac{e^{-\lambda}\lambda^x}{x!}}{\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}} = \frac{\lambda}{x}$. Now $\frac{\lambda}{x} \geq 1$ if $x \leq \lambda$, and thus $x = \lfloor \lambda \rfloor$, or the greatest integer less than or equal to λ, is the mode of $P(\lambda)$.

2.9.62

$E[Xf(X-1)] = \sum_{x=0}^{\infty} xf(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} xf(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} f(x-1) \frac{e^{-\lambda}\lambda^x}{(x-1)!} = \sum_{y=0}^{\infty} f(y) \frac{e^{-\lambda}\lambda^{y+1}}{y!} = \lambda \sum_{y=0}^{\infty} f(y) \frac{e^{-\lambda}\lambda^y}{y!} = \lambda E[f(X)]$

2.9.64

$p(x) = \frac{\lambda^x e^{-x}}{x!} = \frac{\lambda}{x} \frac{\lambda^{x-1}e^{-x}}{(x-1)!} = \frac{\lambda}{x} p(x-1)$ for all $x \geq 1$.

Let $\lambda = 3$.

$P(0) = e^{-3} = 0.049787$ (Tabled value 0.05)

$P(1) = \frac{3}{1} P(0) = 0.149362$ (Tabled value 0.149)

$P(2) = \frac{3}{2} P(1) = 0.224042$ (Tabled value 0.224)

$P(3) = \frac{3}{3} P(2) = 0.224042$ (Tabled value 0.224)

$P(4) = \frac{3}{4} P(3) = 0.168031$ (Tabled value 0.168)

etc.

2.9.66

$EX = \sum_{x=1}^{\infty} \frac{x}{1-e^{-\lambda}} \frac{\lambda^x e^{-\lambda}}{x!} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=1}^{\infty} \frac{x\lambda^x}{x!} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{y=0}^{\infty} \frac{\lambda^{1+y}}{y!} = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} e^{\lambda} = \frac{\lambda}{1-e^{-\lambda}}$

$E(X(X-1)) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=2}^{\infty} \frac{x(x-1)\lambda^x}{x!} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{y=0}^{\infty} \frac{\lambda^{y+2}}{y!} = \frac{\lambda^2}{1-e^{-\lambda}}$

Hence, $V(X) = E(X(X-1)) + EX - (EX)^2 = \frac{\lambda^2}{1-e^{-\lambda}} + \frac{\lambda}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2} = \frac{\lambda(1-e^{-\lambda}-\lambda e^{-\lambda})}{(1-e^{-\lambda})^2}$

2.9.68

$g_X(t) = \frac{1-t^{n+1}}{(n+1)(1-t)} = \frac{1}{n+1} \sum_{x=0}^{n} t^x$

Since $g_X(t) = \sum_{x=0}^{\infty} t^x p(x) = \sum_{x=0}^{\infty} t^x (\frac{1}{n+1})$, it shows that $p(x) = \frac{1}{n+1}$ for $x = 0, 1, \cdots, n$.

2.9.70

$p(x) = \frac{1}{6}$ for $x = 1, 2, \cdots, 6$. Hence

$g_X(t) = \sum_{x=0}^{\infty} t^x p(x) = \frac{1}{6}\sum_{x=1}^{6} t^x = \frac{t(t^6-1)}{6(t-1)} = \frac{t^7-t}{6(t-1)}$

2.9.72

(a) If $t = 0$, $m(t) = (-0.3)^1 0 \neq 1$. So, this is not a moment generating function.

(b) The p.m.f. is given by $P(X = 0) = 0.2; P(X = 1) = 0.8$. So $X \sim B(1, 0.8)$.

Mean $= EX = 1 \times 0.8 = 0.8$, Variance $= V(X) = 1 \times 0.8 \times 0.2 = 0.16$

(c) $m(t) = \frac{0.5e^t}{1-0.5e^t} = 0.5e^t \sum_{k=0}^{\infty}(0.5e^t)^k = \sum_{k=1}^{\infty}(0.5)^k e^{tk}$

The p.m.f. is given by $P(X = k) = (0.5)^k$; $k = 1, 2, \cdots$ or $X \sim 0.5$. Therefore, Mean $= \frac{1}{0.5} = 2$, Variance $= \frac{0.5}{(0.5)^2} = 2$

(d) $m(t) = \left[\frac{0.6e^t}{1-0.4e^t}\right]^3 = (0.6)^3 e^{3t} \sum_{k=0}^{\infty}\binom{k+2}{k}(0.4e^t)^k = \sum_{k=0}^{\infty}\binom{k+2}{k}(1-0.4)^3(0.4)^k e^{(k+3)t} = \sum_{k=3}^{\infty}\binom{k-1}{k-3}(1-0.4)^3(0.4)^{k-3}e^{kt}$.

Hence the p.m.f. is given by $P(X = k) = \binom{k-1}{k-3}(1-0.4)^3(0.4)^{k-3}$; $k = 3, 4, \cdots$. Hence $X \sim NB(3, 0.4)$

Mean $= \frac{3}{0.4} = 7.5$; Variance $= \frac{3.(0.4)}{(0.6)^2} = \frac{10}{3}$

(e) $m(t) = e^{3(e^t-1)} = e^{3e^t-3} = e^{-3}e^{3e^t} = e^{-3}\sum_{k=0}^{\infty}\frac{(3e^t)^k}{k!} = \sum_{k=0}^{\infty}\frac{e^{-3}3^k}{k!}$.

Hence, the p.m.f. is given by $P(X = k) = \frac{e^{-3}3^k}{k!}$ for $k = 0, 1, 2, \cdots$. So, $X \sim P(3)$.

Mean = 3; Variance = 3.

(f) From the mathematical form of $m(t)$, it is evident that the p.m.f. is $P(X = k) = \frac{k}{12}$ for $k = 2, 4, 6$.

Mean $= 2.\frac{1}{6} + 4.\frac{2}{6} + 6.\frac{3}{6} = \frac{14}{3}$

Variance $= \left[2^2.\frac{1}{6} + 4^2.\frac{2}{6} + 6^2.\frac{3}{6}\right] - (\frac{14}{3})^2 = \frac{20}{9}$

(g) $m(t) = \frac{1}{n}\sum_{j=1}^{n} e^{jt}$. So the p.m.f. is given by $P(X = k) = \frac{1}{n}$ for $k = 1, 2, \cdots, n$. In other words, X follows the discrete uniform distribution on $\{1, 2, \cdots, n\}$.

Mean $= \sum_{j=1}^{n} j.\frac{1}{n} = \frac{n+1}{2}$

Variance $= \sum_{j=1}^{n} j^2\frac{1}{n} - (\frac{n+1}{2})^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)(n-1)}{12}$

2.9.74

(a) $X \sim NB(r, p)$. $P(X = x) = \binom{x-1}{r-1}p^r q^x$; $x = r, r+1, \cdots$.

Let $Y = X - r$. Then $P(Y = y) = \binom{r+y-1}{r-1}p^r q^y$; $y = 0, 1, \cdots$

(b) Now suppose $r \to \infty, p \to 1$ and $rq \to \lambda > 0$. Then

$P(Y = y) = \frac{(r+y-1)\cdots r}{y!}p^r q^y$. And since, in the limit $rq \approx \lambda > 0$ and since y is a fixed finite constant,

$P(Y = y) = \frac{p^r(r+y-1)\cdots r.q^y}{y!} \approx \frac{(1-\frac{\lambda}{r})^r(rq)^y}{y!} \approx \frac{(1-\frac{\lambda}{r})^r\lambda^y}{y!} \to \frac{\lambda^y e^{-y}}{y!}$ for $y = 0, 1, 2, \cdots$