Chapter 2: Second-order Differential Equations

2.3 *Second-order differential equations*

Problem 1. Solve equation

$$(x-3)y''+y'=0.$$

<u>*Solution.*</u>

Let us $y'=z(x)$. Then $z'=\dfrac{dz}{dx}=y''$. Equation now reads

$$(x-3)z'+z=0.$$

This is the linear nonhomogeneous first-order equation. Its general solution is:

$$\int\frac{dz}{z}=-\int\frac{dx}{x-3} \quad\Rightarrow\quad \ln|z|=-\ln|x-3|+\ln|C| \quad\Rightarrow\quad z=\frac{C_1}{(x-3)}.$$

Next, integrate

$$y'=\frac{C_1}{(x-3)} \quad\Rightarrow\quad \int dy=C_1\int\frac{dx}{x-3} \quad\Rightarrow y=C_1\ln|x-3|+C_2.$$

Thus, *the general solution is*

$$y=C_1\ln|x-3|+C_2.$$

Problem 2. Solve equation

$$y^3y''=1.$$

<u>*Solution.*</u>

Let us $y'=z(y)$. Differentiation of $z(y)$ gives $y''=\dfrac{dz}{dy}\dfrac{dy}{dx}$, and replacing here y' by $z(y)$ gives $y''=z(y)\dfrac{dz}{dy}$.

Equation now reads

$$y^3z\frac{dz}{dy}=1.$$

This is first-order equation. Its general solution is:

$$\int zdz=\int\frac{dy}{y^3} \quad\Rightarrow\quad z^2=-\frac{1}{y^2}+C_1 \quad\Rightarrow\quad z=\frac{\sqrt{C_1y^2-1}}{y}.$$

Next, integrate

$$y'=\frac{\sqrt{C_1y^2-1}}{y} \quad\Rightarrow\quad \frac{ydy}{\sqrt{C_1y^2-1}}=dx \quad\Rightarrow$$

$$\Rightarrow\quad \frac{1}{2C_1}\int\frac{d\left(C_1y^2-1\right)}{\sqrt{C_1y^2-1}}=\int dx \quad\Rightarrow\quad \sqrt{C_1y^2-1}=C_1x+C_2$$

Thus, *the general solution is*

$$C_1y^2-1=(C_1x+C_2)^2.$$

Problem 3. Solve equation

$$y'^2 + 2yy'' = 0\,.$$

<u>*Solution.*</u>

Let us $y' = z(y)$. Differentiation of $z(y)$ gives $y'' = \dfrac{dz}{dy}\dfrac{dy}{dx}$, and replacing here y' by $z(y)$ gives $y'' = z(y)\dfrac{dz}{dy}$.

Equation now reads

$$z^2 + 2yzz' = 0\,.$$

This is first-order equation. Its general solution is:

$$\int \frac{2dz}{z} = -\int \frac{dy}{y} \quad \Rightarrow \quad 2\ln|z| = -\ln|y| - \ln|C| \quad \Rightarrow \quad z^2 = \frac{1}{C_1 y}\,.$$

Next, integrate

$$y' = \frac{1}{C_1 y} \quad \Rightarrow \quad C_1\sqrt{y}dy = dx \quad \Rightarrow$$

$$\Rightarrow \quad C_1\int\sqrt{y}dy = \int dx \quad \Rightarrow \quad \frac{3}{2}C_1\sqrt{y^3} = x + C_2$$

Thus, *the general solution is*

$$y^3 = C_1(x + C_2)^2\,.$$

Problem 4. Solve equation

$$y''(e^x + 1) + y' = 0\,.$$

<u>*Solution.*</u>

Let us $y' = z(x)$. Then $z' = \dfrac{dz}{dx} = y''$. Equation now reads

$$z'(e^x + 1) + z = 0\,.$$

This is first-order equation. Its general solution is:

$$\int \frac{dz}{z} = -\int \frac{dx}{e^x + 1} \quad \Rightarrow \quad \int \frac{dz}{z} = -\int \frac{e^{-x}dx}{1 + e^{-x}} \quad \Rightarrow \quad \int \frac{dz}{z} = \int \frac{d\left(1 + e^{-x}\right)}{1 + e^{-x}} \quad \Rightarrow.$$

$$\Rightarrow \quad \ln|z| = \ln\left(1 + e^{-x}\right) - \ln|C| \quad \Rightarrow \quad z = C_1\left(1 + e^{-x}\right)$$

Next, integrate

$$y' = C_1\left(1 + e^{-x}\right) \quad \Rightarrow \quad \int dy = C_1\int\left(1 + e^{-x}\right)dx \quad \Rightarrow y = C_1\left(x - e^{-x}\right) + C_2\,.$$

Thus, *the general solution is*

$$y = C_1(x - e^{-x}) + C_2\,.$$

Problem 5. Solve equation

$$yy'' = y'^2 - y'^3\,.$$

<u>*Solution.*</u>

Let us $y' = z(y)$. Differentiation of $z(y)$ gives $y'' = \frac{dz}{dy}\frac{dy}{dx}$, and replacing here y' by $z(y)$ gives $y'' = z(y)\frac{dz}{dy}$.

Equation now reads

$$yzz' = z^2 - z^3.$$

This is first-order equation.

$$y\frac{z'}{z^2} = \frac{1}{z} - 1 \quad \Rightarrow \quad -y\frac{d}{dy}\left(\frac{1}{z} - 1\right) = \frac{1}{z} - 1 \quad \Rightarrow$$

$$\int \frac{d\left(\frac{1}{z} - 1\right)}{\frac{1}{z} - 1} = -\int \frac{dy}{y} \quad \Rightarrow \quad \ln\left(\frac{1}{z} - 1\right) = -\ln|y| - \ln|C_1| \quad \Rightarrow \quad \frac{1}{z} - 1 = \frac{1}{C_1 y}.$$

Next, integrate

$$y' = \frac{C_1 y}{C_1 y + 1} \quad \Rightarrow \quad \left(1 + \frac{1}{C_1 y}\right)dy = dx \quad \Rightarrow$$

$$\Rightarrow \quad \int dy + \frac{1}{C_1}\int \frac{dy}{y} = \int dx \quad \Rightarrow \quad y + C_1 \ln|y| = x + C_2$$

Thus, *the general solution is*

$$y + C_1 \ln|y| = x + C_2.$$

Problem 6. Solve equation

$$x(y'' + 1) + y' = 0.$$

<u>*Solution.*</u>

Let us $y' = z(x)$. Then $z' = \frac{dz}{dx} = y''$. Equation now reads

$$z' + 1 + \frac{z}{x} = 0.$$

This is the linear nonhomogeneous first-order equation.

The general solution of the homogeneous equation $z' + \frac{z}{x} = 0$ is:

$$\int \frac{dz}{z} = -\int \frac{dx}{x} \quad \Rightarrow \quad \ln|z| = -\ln|x| - \ln|C| \quad \Rightarrow \quad z = \frac{C}{x}.$$

Next, using variation of the parameters we look for the particular solution of the inhomogeneous equation in the form $\overline{z}(x) = \frac{C(x)}{x}$.

$$\overline{z}'(x) = \frac{xC'(x) - C(x)}{x^2} \quad \Rightarrow \quad \frac{xC'(x) - C(x)}{x^2} + \frac{C(x)}{x^2} + 1 = 0 \quad \Rightarrow$$

$$\Rightarrow \quad C'(x) = -x \quad \Rightarrow \quad dC'(x) = -xdx \quad \Rightarrow \quad C(x) = -\frac{x^2}{2} + C_1$$

This gives $\bar{z}(x) = -\frac{x}{2} + \frac{C_1}{x}$. Next, integrate

$$y' = -\frac{x}{2} + \frac{C_1}{x} \quad \Rightarrow \quad \int dy = -\frac{1}{2}\int x dx + C_1 \int \frac{dx}{x} \quad \Rightarrow y = -\frac{x^2}{4} + C_1 \ln|x| + C_2 .$$

Thus, *the general solution is*

$$y = C_1 \ln|x| - \frac{x^2}{4} + C_2 .$$

Problem 7. Solve equation

$$y'' = \sqrt{1-(y')^2} .$$

<u>*Solution.*</u>

Let us $y' = z(x)$. Then $z' = \frac{dz}{dx} = y''$. Equation now reads

$$z' = \sqrt{1-z^2} .$$

This is first-order equation. Its general solution is:

$$\int \frac{dz}{\sqrt{1-z^2}} = \int dx \quad \Rightarrow \quad \arcsin z = x + C_1 \quad \Rightarrow \quad z = \sin(x + C_1).$$

Next, integrate

$$y' = \sin(x + C_1) \quad \Rightarrow \quad y = -\cos(x + C_1) + C_2 .$$

Thus, *the general solution is*

$$y = C_2 - \cos(x + C_1).$$

Problem 8. Solve equation

$$y''' + 2xy'' = 0 .$$

<u>*Solution.*</u>

Let us $y'' = z(x)$. Then $z' = \frac{dz}{dx} = y'''$. Equation now reads

$$z' + 2xz = 0 .$$

This is first-order equation. Its general solution is:

$$\int \frac{dz}{z} = -2\int x dx \quad \Rightarrow \quad \ln|z| = -x^2 + \ln|C_1| \quad \Rightarrow \quad z = C_1 e^{-x^2} .$$

Next, integrate

$$y'' = C_1 e^{-x^2} \quad \Rightarrow \quad y' = C_1 \int e^{-x^2} dx + C_2 .$$

Then,

$$y = \int \left[C_1 \int e^{-x^2} dx + C_2 \right] dx + C_3 = C_1 \int \left[\int e^{-x^2} dx \right] dx + C_2 x + C_3 .$$

Integrating $I = \int \left[\int e^{-x^2} dx \right] dx$ by parts we obtain:

$$u = \int e^{-x^2} dx, \quad \mathrm{d}v = dx \quad \Rightarrow \quad du = e^{-x^2} dx, \quad v = x,$$

$$I = x \int e^{-x^2} dx - \int x e^{-x^2} dx = x \int e^{-x^2} dx + \frac{1}{2} e^{-x^2} .$$

Thus, *the general solution is*

$$y = C_1\left(\frac{1}{2}e^{-x^2} + x\int e^{-x^2}dx\right) + C_2x + C_3.$$

Problem 9. Solve equation

$$y'' + 2y' = e^x y'^2.$$

<u>*Solution.*</u>

Let us $y' = z(x)$. Then $z' = \frac{dz}{dx} = y''$. Equation now reads

$$z' + 2x = e^x z^2.$$

This is the Bernoulli equation. Dividing by z^2 we get

$$z^{-2}z' + 2z^{-1} = e^x.$$

Let us $u = z^{-1}$. Then $u' = -z^{-2}z'$ and the equation becomes

$$-u' + 2u = e^x.$$

The general solution of the homogeneous equation $-u' + 2u = 0$ is:

$$\int\frac{du}{u} = 2\int dx \quad\Rightarrow\quad \ln|u| = 2x + C \quad\Rightarrow\quad u = Ce^{2x}.$$

Next, using variation of parameter method we look for the particular solution of the inhomogeneous equation in the form $\overline{u}(x) = C(x)e^{2x}$.

$$\overline{u}'(x) = C'(x)e^{2x} + C(x)2e^{2x} \quad\Rightarrow\quad -C'(x)e^{2x} - C(x)2e^{2x} + 2C(x)e^{2x} = e^x \quad\Rightarrow$$

$$\Rightarrow\quad -C'(x) = e^{-x} \quad\Rightarrow\quad C(x) = e^{-x} \quad\Rightarrow \overline{u}(x) = e^x$$

This gives $u(x) = C_1e^{2x} + e^x$ and $z(x) = \dfrac{1}{e^x\left(1 + C_1e^x\right)} = y'$.

Next, integrate

$$y = \int\frac{dx}{e^x\left(1 + C_1e^x\right)} = -\int\frac{e^{-x}d\left(e^{-x}\right)}{\left(e^{-x} + C_1\right)} =$$

$$= -\int\left(1 - \frac{C_1}{e^{-x} + C_1}\right)d\left(e^{-x}\right) = -e^{-x} + C_1\ln\left|e^{-x} + C_1\right| + C_2.$$

Thus, *the general solution is*

$$y = -e^{-x} + C_1\ln\left|e^{-x} + C_1\right| + C_2.$$

When divided by z, the solution $y = C$ was lost. Thus, also $y = C$.

Problem 10. Solve equation

$$yy'' = (y')^3.$$

<u>*Solution.*</u>

Let us $y' = z(y)$. Differentiation of $z(y)$ gives $y'' = \frac{dz}{dy}\frac{dy}{dx}$, and replacing here y' by $z(y)$ gives $y'' = z(y)\frac{dz}{dy}$.

Equation now reads

$$yzz' = z^3.$$

This is first-order equation.

$$y\frac{z'}{z^2} = 1 \quad \Rightarrow \quad -y\frac{d}{dy}\left(\frac{1}{z}\right) = 1 \quad \Rightarrow \quad \int d\left(\frac{1}{z}\right) = -\int \frac{dy}{y} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{1}{z} = -\ln|y| + C_1 \quad \Rightarrow \quad z = \frac{1}{C_1 - \ln|y|}.$$

Next, integrate

$$y' = \frac{1}{C_1 - \ln|y|} \quad \Rightarrow \quad \left(C_1 - \ln|y|\right)dy = dx \quad \Rightarrow$$

$$\Rightarrow \quad C_1\int dy - \int \ln|y|dy = dx \quad \Rightarrow \quad C_1 y + y\ln|y| = x + C_2$$

Thus, *the general solution is*

$$y\ln|y| + x + C_1 y + C_2 = 0.$$

When divided by z, the solution $y' = 0$ was lost. Thus, also $y = C$.

Problem 11. Solve equation

$$\left(1 - x^2\right)y'' - \frac{y'}{2} + \frac{1}{2} = 0.$$

<u>*Solution.*</u>

Let us $y' = z(x)$. Then $z' = \frac{dz}{dx} = y''$. Equation now reads

Equation now reads

$$\left(1 - x^2\right)z' - \frac{1}{2}(z - 1) = 0.$$

This is first-order equation. Its general solution is:

$$\frac{1}{2}\int \frac{dz}{z-1} = \int \frac{dx}{1 - x^2} \quad \Rightarrow \quad \frac{1}{2}\ln|z - 1| = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C_1 \quad \Rightarrow \quad z = 1 + C_1\frac{x+1}{x-1}.$$

Next, integrate

$$y' = 1 + C_1\frac{x+1}{x-1} \quad \Rightarrow \quad y = \int dx + C_1\int\left[1 + \frac{2}{x-1}\right]dx = x + C_1\left[x + 2\ln|x - 1|\right] + C_2.$$

Thus, *the general solution is*

$$y = x + C_1\left[x + 2\ln|x - 1|\right] + C_2.$$

Problem 12. Solve equation

$$xy''' - y'' = 0.$$

Solution.

Let us $y'' = z(x)$. Then $z' = \frac{dz}{dx} = y'''$. Equation now reads

Equation now reads

$$xz' - z = 0.$$

This is first-order equation. Its general solution is:

$$\int \frac{dz}{z} = \int \frac{dx}{x} \quad \Rightarrow \quad \ln|z| = \ln|x| + \ln|C| \quad \Rightarrow \quad z = Cx.$$

Next, integrate

$$y'' = Cx \quad \Rightarrow \quad y' = C\int x dx + C_2 = C\frac{x^2}{2} + C_2.$$

$$y' = \int \left[C\frac{x^2}{2} + C_2 \right] dx + C_3 = C\frac{x^3}{6} + C_2 x + C_3.$$

Thus*, the general solution is*

$$y = C_1 x^3 + C_2 x + C_3.$$

2.5 ***Linear n-th order equations***

Check if the following functions are linearly independent

Problem 1. $e^x,\ e^{x-1}$.

Solution.

These functions are linearly dependent: $e^x = e \cdot e^{x-1}$.

Problem 2. $\cos x,\ \sin x$.

Solution.

These functions are linearly independent, since the Wronskian

$$W\left(y_1(x), y_2(x)\right) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0.$$

Problem 3. $1,\ x,\ x^2$.

Solution.

These functions are linearly independent, since the Wronskian

$$W\left(y_1(x), y_2(x), y_3(x)\right) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2x^2 \neq 0$$

on any interval.

Problem 4. $4-x,\ 2x+3,\ 6x+8$.

<u>*Solution.*</u>

These functions are linearly dependent, since the Wronskian equals zero.

Problem 5. $e^x,\ e^{2x},\ e^{3x}$.

<u>*Solution.*</u>

These functions are linearly independent, since the Wronskian is not zero.

Problem 6. $x,\ e^x,\ xe^x$.

<u>*Solution.*</u>

Since

$$W\left(y_1(x), y_2(x), y_3(x)\right) = \begin{vmatrix} x & e^x & xe^x \\ 1 & e^x & (1+x)e^x \\ 0 & e^x & (2+x)e^x \end{vmatrix} = (x-2)e^{2x} \neq 0$$

on any interval, these functions are linearly independent.

Problem 7. $2^x,\ 3^x,\ 6^x$.

<u>*Solution.*</u>

Since

$$W\left(y_1(x), y_2(x), y_3(x)\right) = \begin{vmatrix} 2^x & 3^x & 6^x \\ 2^x \ln 2 & 3^x \ln 3 & 6^x \ln 6 \\ 2^x (\ln 2)^2 & 3^x (\ln 3)^2 & 6^x (\ln 6)^2 \end{vmatrix} =$$

$$= 36^x \cdot \ln 2 \cdot \ln 3 \cdot \ln 6 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \ln 2 & \ln 3 & \ln 6 \end{vmatrix} = 0,$$

these functions are linearly dependent.

Problem 8. $\cos x,\ \sin x,\ \sin 2x$.

<u>*Solution.*</u>

Since

$$W\left(y_1(x), y_2(x), y_3(x)\right) = -3\sin 2x \neq 0,$$

these functions are linearly independent.

2.6 *Linear second-order equations with constant coefficients*

Problem 1. Solve equation

$$y'' - 5y' - 6y = 0.$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 5k - 6 = 0$ has two real and distinct roots:

$$k_{1,2} = \frac{5 \pm \sqrt{25+24}}{2} \quad \Rightarrow \quad k_1 = -1,\ k_2 = 6.$$

Thus, *the general solution* is

$$y(x) = C_1 e^{6x} + C_2 e^{-x}.$$

Problem 2. Solve equation

$$y''' - 6y'' + 13y' = 0.$$

<u>*Solution.*</u>

The characteristic equation $k^3 - 6k^2 + 13k = 0$ has one real and two complex conjugated roots:

$$k\left(k^2 - 6k + 13\right) = 0 \quad \Rightarrow \quad k_1 = 0,\ k_{2,3} = \frac{6 \pm \sqrt{36-52}}{2} = 3 \pm 2i.$$

Thus, *the general solution* is

$$y(x) = C_1 + e^{3x}\left(C_2 \cos 2x + C_3 \sin 2x\right).$$

Problem 3. Solve equation

$$y^{(4)} - y = 0.$$

<u>*Solution.*</u>

The characteristic equation $k^4 - 1 = 0$ has two real and two complex conjugated roots:

$$k^2 = 1 \quad \Rightarrow \quad k_{1,2} = \pm 1; \qquad k^2 = -1 \quad \Rightarrow \quad k_{3,4} = \pm i.$$

Thus, *the general solution* is

$$y = C_1 e^{-x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x.$$

Problem 4. Solve equation

$$y^{(4)} + 13y^{(2)} + 36y = 0.$$

<u>*Solution.*</u>

The characteristic equation $k^4 + 13k^2 + 36 = 0$ has two pairs of complex conjugated roots:

$$k_{1,2}^2 = \frac{-13 + \sqrt{169-144}}{2} = -4 \quad \Rightarrow \quad k_{1,2} = \pm 2i,$$

$$k_{3,4}^2 = \frac{-13 - \sqrt{169-144}}{2} = -9 \quad \Rightarrow \quad k_{3,4} = \pm 3i.$$

Thus, *the general solution* is

$$y = C_1 \cos 2x + C_2 \sin 2x + C_3 \cos 3x + C_4 \sin 3x.$$

Problem 5. Solve equation

$$y'' - 5y' + 6y = 0.$$

Solution.

The characteristic equation $k^2 - 5k + 6 = 0$ has two real and distinct roots:

$$k_{1,2} = \frac{5 \pm \sqrt{25-24}}{2} \quad \Rightarrow \quad k_1 = 2,\ k_2 = 3.$$

Thus, *the general solution* is

$$y(x) = C_1 e^{2x} + C_2 e^{3x}.$$

Problem 6. Solve equation

$$y''' - 4y'' + 3y' = 0.$$

Solution.

The characteristic equation $k^3 - 4k^2 + 3k = 0$ has three real and distinct roots:

$$k\left(k^2 - 4k + 3\right) = 0 \quad \Rightarrow \quad k_1 = 0,\ \ k_{2,3} = \frac{4 \pm \sqrt{16-12}}{2} \Rightarrow \begin{cases} k_2 = 1, \\ k_3 = 3. \end{cases}$$

Thus, *the general solution* is

$$y = C_1 + C_2 e^{x} + C_3 e^{3x}.$$

Problem 7. Solve equation

$$y''' + 6y'' + 25y' = 0.$$

Solution.

The characteristic equation $k^3 + 6k^2 + 25k = 0$ has one real and two complex conjugated roots:

$$k\left(k^2 + 6k + 25\right) = 0 \quad \Rightarrow \quad k_1 = 0,\ \ k_{2,3} = \frac{-6 \pm \sqrt{36-100}}{2} = -3 \pm 4i.$$

Thus, *the general solution* is

$$y = C_1 + e^{-3x}(C_2 \cos 4x + C_3 \sin 4x).$$

Problem 8. Solve equation

$$y''' + 5y'' = 0.$$

Solution.

The characteristic equation $k^3 + 5k^2 = 0$ has three real roots:

$$k^2(k+5) = 0 \quad \Rightarrow \quad k_1 = -5,\ \ k_2 = k_3 = 0.$$

Thus, *the general solution* is

$$y = C_1 + C_2 x + C_3 e^{-5x}.$$

Problem 9. Solve equation

$$y''' - 3y'' + 3y' - y = 0.$$

Solution.

The characteristic equation $k^3 - 3k^2 + 3k - 1 = 0$ has repeated real root:

$$k^3 - 3k^2 + 3k - 1 = 0 \quad \Rightarrow \quad (k-1)^3 = 0 \quad \Rightarrow \quad k_1 = k_2 = k_3 = 1.$$

Thus, *the general solution* is

$$y = e^x (C_1 + C_2 x + C_3 x^2).$$

Problem 10. Solve equation

$$y^{(4)} - 8y^{(2)} - 9y = 0.$$

Solution.

The characteristic equation $k^4 - 8k^2 - 9 = 0$ has two real and two complex conjugated roots:

$$k_{1,2}^2 = \frac{8 + \sqrt{64+36}}{2} = 9 \quad \Rightarrow \quad k_{1,2} = \pm 3,$$

$$k_{3,4}^2 = \frac{8 - \sqrt{64+36}}{2} = -1 \quad \Rightarrow \quad k_{3,4} = \pm i.$$

Thus, *the general solution* is

$$y = C_1 e^{-3x} + C_2 e^{3x} + C_3 \cos x + C_4 \sin x.$$

Problem 11. Solve equation

$$y'' + 4y' + 3y = 0.$$

Solution.

The characteristic equation $k^2 + 4k + 3 = 0$ has two real and distinct roots:

$$k_{1,2} = \frac{-4 \pm \sqrt{16-12}}{2} \quad \Rightarrow \quad k_1 = -1,\ k_2 = -3.$$

Thus, *the general solution* is

$$y = C_1 e^{-x} + C_2 e^{-3x}.$$

Problem 12. Solve equation

$$y'' - 2y' = 0.$$

Solution.

The characteristic equation $k^2 - 2k = 0$ has two real and distinct roots:

$$k^2 - 2k = k(k-2) = 0 \quad \Rightarrow \quad k_1 = 0,\ k_2 = 2.$$

Thus, *the general solution* is

$$y = C_1 + C_2 e^{2x}.$$

Problem 13. Solve equation

$$2y'' - 5y' + 2y = 0.$$

Solution.

The characteristic equation $2k^2 - 5k + 2 = 0$ has two real and distinct roots:

$$k_{1,2} = \frac{5 \pm \sqrt{25-16}}{4} \quad \Rightarrow \quad k_1 = 2,\ k_2 = \frac{1}{2}.$$

Thus, *the general solution* is

$$y = C_1 e^{2x} + C_2 e^{x/2}.$$

Problem 14. Solve equation

$$y'' - 4y' + 5y = 0.$$

Solution.

The characteristic equation $k^2 - 4k + 5 = 0$ has complex conjugated roots:

$$k_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i.$$

Thus, *the general solution* is

$$y = e^{2x}(C_1 \cos x + C_2 \sin x).$$

Problem 15. Solve equation

$$y''' - 8y = 0.$$

Solution.

The characteristic equation $k^3 - 8 = 0$ has one real and two complex conjugated roots:

$$(k-2)\left(k^2 + 2k + 4\right) = 0 \quad \Rightarrow \quad k_1 = 2,\ \ k_{2,3} = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm i\sqrt{3}.$$

Thus, *the general solution* is

$$y = C_1 e^{2x} + e^{-x}(C_2 \cos x\sqrt{3} + C_3 \sin x\sqrt{3}).$$

Problem 16. Solve equation

$$y^{(4)} + 4y = 0.$$

Solution.

The characteristic equation $k^4 + 4 = 0$ has two pairs of complex conjugated roots:

$$k_{1,2} = 1 \pm i, \quad k_{3,4} = -1 \pm i.$$

Thus, *the general solution* is

$$y = e^{x}(C_1 \cos x + C_2 \sin x) + e^{-x}(C_3 \cos x + C_4 \sin x).$$

Problem 17. Solve equation

$$y'' - 2y' + y = 0.$$

Solution.

The characteristic equation $k^2 - 2k + 1 = 0$ has repeated real root:

$$k^2 - 2k + 1 = (k-1)^2 \quad \Rightarrow \quad k_1 = k_2 = 1.$$

Thus, *the general solution* is

$$y = e^{x}(C_1 + C_2 x).$$

Problem 18. Solve equation

$$4y''+4y'+y=0.$$

<u>*Solution.*</u>

The characteristic equation $4k^2+4k+1=0$ has repeated real root:

$$4k^2+4k+1=4\left(k+\frac{1}{2}\right)^2 \quad \Rightarrow \quad k_1=k_2=-\frac{1}{2}.$$

Thus, *the general solution* is

$$y=e^{-x/2}(C_1+C_2x).$$

Problem 19. Solve equation

$$y^{(5)}-6y^{(4)}+9y'''=0.$$

<u>*Solution.*</u>

The characteristic equation $k^5-6k+9=0$ has repeated real roots:

$$k^5-6k+9=k^3(k-3)^2 \quad \Rightarrow \quad k_1=k_2=k_3=0,\ \ k_4=k_5=3.$$

Thus, *the general solution* is

$$y=C_1+C_2x+C_3x^2+e^{3x}(C_4+C_5x).$$

Problem 20. Solve equation

$$y'''-5y''+6y'=0.$$

<u>*Solution.*</u>

The characteristic equation $k^3-5k^2+6k=0$ has three real and distinct roots:

$$k\left(k^2-5k+6\right)=0 \quad \Rightarrow \quad k_1=0,\ \ k_{2,3}=\frac{5\pm\sqrt{25-24}}{2} \Rightarrow \begin{cases} k_2=2, \\ k_3=3. \end{cases}$$

Thus, *the general solution* is

$$y=C_1+C_2e^{2x}+C_3e^{3x}.$$

Problem 21. Solve IVP

$$y''+2y=0, \quad y(0)=0,\ \ y'(0)=1.$$

<u>*Solution.*</u>

The characteristic equation $k^2+2=0$ has complex conjugated roots $k_{1,2}=\pm\sqrt{2}i$. The *general solution* in real form is

$$y(x)=C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x.$$

Substitution of $y(x)$ and $y'(x)$ in the initial conditions gives

$$\begin{cases} C_1=0, \\ \sqrt{2}C_2=1, \end{cases}$$

which has the solution $C_1=0,\ \ C_2=1/\sqrt{2}$.

Thus, *the solution of IVP* is

$$y(x) = \frac{1}{\sqrt{2}} \sin \sqrt{2} x .$$

Problem 22. Solve IVP

$$y''' - y' = 0, \quad y(2) = 1, \; y'(2) = 0, \; y''(2) = 0 .$$

<u>*Solution.*</u>

The characteristic equation $k^3 - k = 0$ has three real and distinct roots:

$$k^3 - k = k(k+1)(k-1) = 0 \quad \Rightarrow \quad k_1 = 0, \; k_2 = 1, \; k_3 = -1 .$$

Thus, *the general solution* is

$$y(x) = C_1 + C_2 e^x + C_3 e^{-x} .$$

Substitution of $y(x)$, $y'(x)$ and $y''(x)$ in the initial conditions gives

$$\begin{cases} y(2) = C_1 + C_2 e^2 + C_3 e^{-2} = 1, \\ y'(2) = C_2 e^2 - C_3 e^{-2} = 0, \\ y''(2) = C_2 e^2 + C_3 e^{-2} = 0, \end{cases}$$

which has the solution $C_1 = 1, \; C_2 = 0, \; C_2 = 0$.

Thus, *the solution of IVP* is

$$y(x) = 1 .$$

Problem 23. Solve IVP

$$y'' + 4y' + 5y = 0, \quad y(0) = -3, \; y'(0) = 0 .$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 4k + 5 = 0$ has complex conjugated roots:

$$k_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i .$$

Thus, *the general solution* is

$$y = e^{-2x}(C_1 \cos x + C_2 \sin x) .$$

Substitution of $y(x)$ and $y'(x)$ in the initial conditions gives

$$\begin{cases} y(0) = C_1 = -3, \\ y'(0) = -2C_1 + C_2 = 0, \end{cases}$$

which has the solution $C_1 = -3, \; C_2 = -6$.

Thus, *the solution of IVP* is

$$y = -3e^{-2x}(\cos x + 2 \sin x) .$$

***Inhomogeneous equations**: Method of undetermined coefficients*

Problem 1. Solve the equation

$$y'' + 6y' + 5y = 25x^2 - 2 .$$

Solution.

The characteristic equation $k^2+6k+5=0$ has the roots $k_1=-1,\ k_2=-5$. These roots are simple, thus *the general solution of the homogeneous equation* is

$$Y(x)=C_1e^{-5x}+C_2e^{-x}.$$

The right side of the equation is a polynomial $P_n(x)$. So, function $\overline{y}(x)$ has to be taken in the form

$$\overline{y}(x)=x^rQ_n(x)$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of zero roots of the characteristic equation. As we see, $P_n(x)=P_2(x)=25x^2-2$, thus, we search the particular solution as

$$\overline{y}(x)=ax^2+bx+c$$

(with $r=0$ – there is no roots $k=0$ of the characteristic equation) and *a*, *b*, *c* will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$2a+12ax+6b+5ax^2+5bx+5c=25x^2-2.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} x^0: & 2a+6b+5c=-2, \\ x^1: & 12a+5b=0, \\ x^2: & 5a=25. \end{cases}$$

From here $a=5$, $b=-12$, $c=12$, thus,

$$\overline{y}(x)=5x^2-12x+12.$$

Finally, *the general solution of the given equation* is

$$y(x)=Y(x)+\overline{y}(x)=C_1e^{-5x}+C_2e^{-x}+5x^2-12x+12.$$

Problem 2. Solve the equation

$$y^{(4)}+3y''=9x^2.$$

Solution.

The characteristic equation $k^4+3k^2=k^2\left(k^2+3\right)=0$ has double real root $k_{1,2}=0$ and complex conjugated roots $k_{3,4}=\pm\sqrt{3}i$. Thus *the general solution of the homogeneous equation* is

$$Y(x)=C_1+C_2x+C_3\cos\sqrt{3}x+C_4\sin\sqrt{3}x.$$

The right side of the equation is a polynomial $P_n(x)$. So, function $\overline{y}(x)$ has to be taken in the form

$$\overline{y}(x)=x^rQ_n(x)$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of zero roots of the characteristic equation. As we see, $P_n(x)=P_2(x)=9x^2$, thus, we search the particular solution as

$$\bar{y}(x) = x^2\left(ax^2 + bx + c\right)$$

(with $r = 2$ – there are two zero roots, $k = 0$, of the characteristic equation) and a, b, c will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$24a + 36ax^2 + 18bx + 6c = 9x^2.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} x^0: \quad 24a + 6c = 0, \\ x^1: \quad 18b = 0, \\ x^2: \quad 36a = 9. \end{cases}$$

From here $a = 1/4$, $b = 0$, $c = -1$, thus,

$$\bar{y}(x) = \frac{x^4}{4} - x^2.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 + C_2 x + C_3 \cos\sqrt{3}x + C_4 \sin\sqrt{3}x + \frac{x^4}{4} - x^2.$$

Problem 3. Solve the equation

$$y'' + 6y = 5e^x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 6 = 0$ has complex conjugated roots $k_{1,2} = \pm\sqrt{6}i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x.$$

The right side of the equation is a product of a polynomial $P_n(x) = P_0(x) = 5$ and $e^{\gamma x} = e^x$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x)$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_0(x) = 5$, $\gamma = 1$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\bar{y}(x) = ae^x$$

Coefficient a will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$ae^x + 6e^x = 5e^x.$$

From here $a = 5/7$, thus,

$$\bar{y}(x) = \frac{5}{7}e^x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x + \frac{5}{7}e^x.$$

Problem 4. Solve the equation

$$y'' + 6y' + 9y = 10\sin x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 6k + 9 = (k+3)^2 = 0$ has repeated real root $k_{1,2} = -3$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = e^{-3x}(C_1 + C_2 x).$$

The particular solution $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r (A\cos\delta x + A\sin\delta x),$$

where *A* and *B* are coefficients to be determined, and the value of *r* equals the number of roots of the characteristic equation equal to *iδ*.

Since $\delta = 1$ and there is no root equal to $i\delta = i$, thus $r = 0$ and we search $\bar{y}(x)$ as

$$\bar{y}(x) = A\cos x + B\sin x.$$

Coefficient *A* and *B* will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$-A\cos x - B\sin x - 6A\sin x + 6B\cos x + 9A\cos x + 9B\sin x = 10\sin x.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} \cos x: & -A + 6B + 9A = 0, \\ \sin x: & -B - 6A + 9B = 10. \end{cases}$$

From here $A = -0.6$, $B = 0.8$, thus,

$$\bar{y}(x) = -0.6\cos x + 0.8\sin x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = e^{-3x}(C_1 + C_2 x) - 0.6\cos x + 0.8\sin x.$$

Problem 5. Solve the equation

$$\frac{d^2x}{dt^2} - 2x = te^{-t}.$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 2 = 0$ has real and distinct roots $k_{1,2} = \pm\sqrt{2}$. Thus *the general solution of the homogeneous equation* is

$$X(t) = C_1 e^{-t\sqrt{2}} + C_2 e^{t\sqrt{2}}.$$

The right side of the equation is a product of a polynomial $P_n(t)$ and $e^{\gamma t}$. So, function $\bar{x}(t)$ has to be taken in the form

$$\bar{x}(t) = t^r e^{\gamma t} Q_n(t),$$

where $Q_n(t)$ is a polynomial having the same order as $P_n(t)$ (with coefficients to be determined), and the value of *r* equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(t) = P_1(t) = t$, $\gamma = -1$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\overline{x}(t) = e^{-t}(at + b).$$

Coefficient a and b will be determined by substitution this $\overline{x}(t)$ and its derivatives into equation. It gives:

$$e^{-t}\left(at + b - 2a - 2at - 2b\right) = te^{-t}.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} t^0: \quad b - 2a - 2b = 0, \\ t^1: \quad a - 2a = 1. \end{cases}$$

From here $a = -1$, $b = 2$, thus,

$$\overline{x}(t) = e^{-t}(2 - t).$$

Finally, *the general solution of the given equation* is

$$x(t) = X(t) + \overline{x}(t) = C_1 e^{-t\sqrt{2}} + C_2 e^{t\sqrt{2}} + e^{-t}(2 - t).$$

Problem 6. Solve the equation

$$y'' - 2y' = 4(x + 1).$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 2k = k(k - 2) = 0$ has two real roots $k_1 = 0$ and $k_2 = 2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 + C_2 e^{2x}.$$

The right side of the equation is a polynomial $P_n(x)$. So, function $\overline{y}(x)$ has to be taken in the form

$$\overline{y}(x) = x^r Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of zero roots of the characteristic equation. As we see, $P_n(x) = P_1(x) = 4x + 4$, thus, we search the particular solution as

$$\overline{y}(x) = x\left(ax + b\right)$$

with $r = 1$ (one zero root, $k_1 = 0$, of the characteristic equation) and a, b will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$2a - 4ax - 2b = 4x + 4.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} x^0: \quad 2a - 2b = 4, \\ x^1: \quad -4a = 4. \end{cases}$$

From here $a = -1$, $b = -3$, thus,

$$\overline{y}(x) = -x^2 - 3x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \overline{y}(x) = C_1 + C_2 e^{2x} - x^2 - 3x.$$

Problem 7. Solve the equation

$$y'' - 2y' - 3y = e^{4x}.$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 2k - 3 = 0$ has two real and distinct roots $k_1 = -1$ and $k_2 = 3$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 e^{-x} + C_2 e^{3x}.$$

The right side of the equation is a product of a polynomial $P_n(x) = P_0(x) = 1$ and $e^{\gamma x} = e^{4x}$. So, the particular solution function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $\gamma = 4$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\bar{y}(x) = ae^{4x}.$$

Coefficient a will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$e^{4x}(16a - 8a - 3a) = e^{4x}.$$

From here $a = 1/5$, thus,

$$\bar{y}(x) = \frac{1}{5} e^{4x}.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 e^{-x} + C_2 e^{3x} + \frac{1}{5} e^{4x}.$$

Problem 8. Solve the equation

$$y'' + y = 4xe^x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 1 = 0$ has complex conjugated roots $k_{1,2} = \pm i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 \cos x + C_2 \sin x.$$

The right side of the equation is a product of a polynomial $P_n(x)$ and $e^{\gamma x}$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_1(x) = 4x$, $\gamma = 1$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\bar{y}(x) = e^x(ax + b).$$

Coefficients a and b will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$e^x\left(ax+b+2a+ax+b\right)=4xe^x .$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} x^0: \quad 2a+2b=0, \\ x^1: \quad 2a=4. \end{cases}$$

From here $a=2$, $b=-2$, thus,

$$\overline{y}(x)=2e^x(x-1).$$

Finally, *the general solution of the given equation* is

$$y(x)=Y(x)+\overline{y}(x)=C_1\cos x+C_2\sin x+2e^x(x-1).$$

Problem 9. Solve the equation

$$y''-y=2e^x-x^2.$$

<u>*Solution.*</u>

The characteristic equation $k^2-1=0$ has two real and distinct roots $k_1=1$, $k_2=-1$. Thus *the general solution of the homogeneous equation* is

$$Y(x)=C_1e^x+C_2e^{-x}.$$

The right side of the equation is a sum of two functions

$$f(x)=f_1(x)+f_2(x)=2e^x-x^2,$$

so we'll use the principle of superposition to obtain the particular solution to the given equation.

First find the particular solution $\overline{y}_1(x)$ of the equation

$$y''-y=2e^x.$$

The function $\overline{y}_1(x)$ has to be taken in the form

$$\overline{y}_1(x)=x^re^{\gamma x}Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)=P_0(x)=2$ and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $\gamma=1$ and since $\gamma=k_1$ we must take $r=1$, therefore, we search the particular solution as

$$\overline{y}_1(x)=axe^x.$$

Coefficient a will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$e^x\left(ax+2a-ax\right)=2e^x.$$

From here $a=1$, thus,

$$\overline{y}_1(x)=xe^x.$$

It remains to find the particular solution $\overline{y}_2(x)$ of equation

$$y''-y=-x^2.$$

The right side of this equation is the second-degree polynomial and there is no roots $k = 0$ of the characteristic equation, thus we search the particular solution in the form

$$\bar{y}_1(x) = ax^2 + bx + c.$$

Substitution of $\bar{y}_2(x)$ and its derivatives into equation gives:

$$2a - ax^2 - bx - c = -x^2.$$

From here $a = 1$, $b = 0$, $c = 2$, thus,

$$\bar{y}_2(x) = x^2 + 2.$$

Finally, *the general solution of the given equation* is the sum

$$y(x) = Y(x) + \bar{y}_1(x) + \bar{y}_2(x) = C_1e^x + C_2e^{-x} + xe^x + x^2 + 2.$$

Problem 10. Solve the equation

$$y'' + y' - 2y = 3xe^x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + k - 2 = 0$ has two real and distinct roots $k_1 = 1$ and $k_2 = -2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1e^x + C_2e^{-2x}.$$

The right side of the equation is a product of a polynomial $P_n(x)$ and $e^{\gamma x}$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_1(x) = 3x$, $\gamma = 1$ and since $\gamma = k_1$ we must take $r = 1$, therefore, we search the particular solution as

$$\bar{y}(x) = xe^x(ax + b).$$

Coefficients a and b will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$e^x\left[x^2(a + a - 2a) + x(b + 4a + b + 2a - 2b) + 2b + 2a + b\right] = 3xe^x.$$

From here $a = 1/2$, $b = -1/3$, thus,

$$\bar{y}(x) = xe^x\left(\frac{x}{2} - \frac{1}{3}\right).$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1e^x + C_2e^{-2x} + xe^x\left(\frac{x}{2} - \frac{1}{3}\right).$$

Problem 11. Solve the equation

$$y'' - 3y' + 2y = \sin x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 3k + 2 = 0$ has two real and distinct roots $k_1 = 1$ and $k_2 = 2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 e^x + C_2 e^{2x}.$$

The particular solution $\overline{y}(x)$ has to be taken in the form

$$\overline{y}(x) = x^r (A\cos\delta x + A\sin\delta x),$$

where *A* and *B* are coefficients to be determined, and the value of *r* equals the number of roots of the characteristic equation equal to *iδ*. Since $\delta = 1$ and there is no root equal to $i\delta = i$, thus $r = 0$ and we search $\overline{y}(x)$ as

$$\overline{y}(x) = A\cos x + B\sin x.$$

Coefficient *A* and *B* will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$-A\cos x - B\sin x + 3A\sin x - 3B\cos x + 2A\cos x + 2B\sin x = \sin x.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} \cos x: & -A - 3B + 2A = 0, \\ \sin x: & -B + 3A + 2B = 1. \end{cases}$$

From here $A = 0.3$, $B = 0.1$, thus,

$$\overline{y}(x) = 0.3\cos x + 0.1\sin x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \overline{y}(x) = C_1 e^x + C_2 e^{2x} + 0.3\cos x + 0.1\sin x.$$

Problem 12. Solve the equation

$$y'' + y = 4\sin x.$$

Solution.

The characteristic equation $k^2 + 1 = 0$ has complex conjugated roots $k_{1,2} = \pm i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1\cos x + C_2\sin x.$$

The particular solution $\overline{y}(x)$ has to be taken in the form

$$\overline{y}(x) = x^r (A\cos\delta x + A\sin\delta x),$$

where *A* and *B* are coefficients to be determined, and the value of *r* equals the number of roots of the characteristic equation equal to *iδ*. Since $\delta = 1$ and there is a root equal to $i\delta = i$, thus $r = 1$ and we search $\overline{y}(x)$ as

$$\overline{y}(x) = x(A\cos x + B\sin x).$$

Coefficient *A* and *B* will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$-x(A\cos x + B\sin x) - 2A\sin x + 2B\cos x + x(A\cos x + B\sin x) = 4\sin x.$$

From here $A = -2$, $B = 0$, thus,

$$\overline{y}(x) = -2x\cos x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \overline{y}(x) = C_1\cos x + C_2\sin x - 2x\cos x.$$

Problem 13. Solve the equation

$$y'' - 3y' + 2y = x\cos x.$$

<u>Solution.</u>

The characteristic equation $k^2 - 3k + 2 = 0$ has two real and distinct roots $k_1 = 1$ and $k_2 = 2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 e^x + C_2 e^{2x}.$$

The right side of the equation has a form $P_1(x)\cos\delta x$ where $P_1(x)$ is polynomial of the first order. So, the particular solution $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r\left[M_1(x)\cos\delta x + N_1(x)\sin\delta x\right],$$

where $M_1(x)$ and $N_1(x)$ are polynomials of the first order and the value of *r* equals the number of roots of the characteristic equation equal to *iδ*. . Since $\delta = 1$ and there is no root equal to $i\delta = i$, thus $r = 0$ and we search $\bar{y}(x)$ as

$$\bar{y}(x) = (Ax + B)\cos x + (Cx + D)\sin x.$$

Coefficient *A*, *B*, *C* and *D* will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. Differentiation of $\bar{y}(x)$ gives:

$$\bar{y}'(x) = -(Ax + B)\sin x + A\cos x + (Cx + D)\cos x + C\sin x,$$

$$\bar{y}''(x) = -(Ax + B)\cos x - 2A\sin x - (Cx + D)\sin x + 2C\cos x.$$

Substituting this in the given equation and equating the coefficients of the like terms in both sides gives:

$$\begin{cases} \cos x: \quad -B + 2C - 3A - 3D + 2B = 0, \\ \sin x: \quad -2A - D + 3B - 3C + 2D = 0, \\ x\cos x: \quad -A - 3C + 2A = 1, \\ x\sin x: \quad -C + 3A + 2C = 0. \end{cases}$$

From here $A = 0.1$, $B = 0.12$, $C = -0.3$, $D = -0.34$, thus,

$$\bar{y}(x) = (0.1x - 0.12)\cos x - (0.3x + 0.34)\sin x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 e^x + C_2 e^{2x} + \\ + (0.1x - 0.12)\cos x - (0.3x + 0.34)\sin x.$$

Problem 14. Solve the equation

$$y'' + 3y' - 4y = e^{-4x} + xe^{-x}.$$

<u>Solution.</u>

The characteristic equation $k^2 + 3k - 4 = 0$ has two real and distinct roots $k_1 = 1$, $k_2 = -4$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 e^x + C_2 e^{-4x}.$$

The right side of the equation is a sum of two functions

$$f(x) = f_1(x) + f_2(x) = e^{-4x} + xe^{-x},$$

so we'll use the principle of superposition to obtain the particular solution to the given equation.

First find the particular solution $\overline{y}_1(x)$ of the equation

$$y'' + 3y' - 4y = e^{-4x}.$$

The function $\overline{y}_1(x)$ has to be taken in the form

$$\overline{y}_1(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x) = P_0(x) = 1$ and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $\gamma = -4$ and since $\gamma = k_2$ we must take $r = 1$, therefore, we search the particular solution as

$$\overline{y}_1(x) = axe^{-4x}.$$

Coefficient a will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$e^{-4x}\left(-4a + 16ax - 4a + 3a - 12ax - 4ax\right) = e^{-4x}.$$

From here $a = -1/5$, thus,

$$\overline{y}_1(x) = -\frac{x}{5}e^{-4x}.$$

It remains to find the particular solution $\overline{y}_2(x)$ of equation

$$y'' + 3y' - 4y = xe^{-x}.$$

The right side of the equation is a product of a polynomial $P_n(x)$ and $e^{\gamma x}$. So, function $\overline{y}_2(x)$ has to be taken in the form

$$\overline{y}_2(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_1(x) = 3$, $\gamma = -1$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution $\overline{y}_2(x)$ as

$$\overline{y}_2(x) = e^{-x}(ax + b).$$

Coefficients a and b will be determined by substitution this $\overline{y}_2(x)$ and its derivatives into equation. It gives:

$$e^{-x}\left(ax + b - 2a - 3ax - 3b + 3a - 4ax - 4b\right) = xe^{-x}.$$

Substitution of $\overline{y}_2(x)$ and its derivatives into equation gives:

$$2a - ax^2 - bx - c = -x^2.$$

From here $a = -1/6$, $b = -1/36$, thus,

$$\overline{y}_2(x) = -e^{-x}\left(\frac{x}{6} + \frac{1}{36}\right).$$

Finally, *the general solution of the given equation* is the sum

$$y(x) = Y(x) + \overline{y}_1(x) + \overline{y}_2(x) = C_1 e^{x} + C_2 e^{-4x} - \frac{x}{5}e^{-4x} - \frac{e^{-x}}{6}\left(x + \frac{1}{6}\right).$$

Problem 15. Solve the equation

$$y'' - 4y' + 8y = e^{2x} + \sin 2x.$$

<u>Solution.</u>

The characteristic equation $k^2 - 4k + 8 = 0$ has complex conjugated roots $k_{1,2} = 2 \pm 2i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = e^{2x}(C_1 \cos 2x + C_2 \sin 2x).$$

The right side of the equation is a sum of two functions

$$f(x) = f_1(x) + f_2(x) = e^{2x} + \sin 2x,$$

so we'll use the principle of superposition to obtain the particular solution to the given equation.

First find the particular solution $\overline{y}_1(x)$ of the equation

$$y'' - 4y' + 8y = e^{2x}.$$

The function $\overline{y}_1(x)$ has to be taken in the form

$$\overline{y}_1(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x) = P_0(x) = 1$ and the value of *r* equals the number of roots of the characteristic equation equal to γ. As we see, $\gamma = 2$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\overline{y}_1(x) = ae^{2x}.$$

Coefficient *a* will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$e^{2x}\left(4a - 8a + 8a\right) = e^{2x}.$$

From here $a = 1/4 = 0.25$, thus,

$$\overline{y}_1(x) = 0.25e^{2x}.$$

It remains to find the particular solution $\overline{y}_2(x)$ of equation

$$y'' - 4y' + 8y = \sin 2x.$$

The particular solution $\overline{y}_2(x)$ has to be taken in the form

$$\overline{y}(x) = x^r (A\cos \delta x + A\sin \delta x)$$

where *A* and *B* are coefficients to be determined, and the value of *r* equals the number of roots of the characteristic equation equal to *iδ*. Since $\delta = 2$ and there is no root equal to $i\delta = 2i$, thus $r = 0$ and we search $\overline{y}(x)$ as

$$\overline{y}_2(x) = A\cos 2x + B\sin 2x.$$

Coefficient *A* and *B* will be determined by substitution this $\overline{y}_2(x)$ and its derivatives into equation. It gives:

$$-4A\cos 2x - 4B\sin 2x + 8A\sin 2x - 8B\cos 2x + 8A\cos 2x + 8B\sin 2x = \sin 2x.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} \cos 2x: & -4A - 8B + 8A = 0, \\ \sin 2x: & -4B + 8A + 8B = 1. \end{cases}$$

From here $A = 0.1$, $B = 0.05$, thus,

$$\bar{y}_2(x) = 0.1\cos 2x + 0.05\sin 2x.$$

Finally, *the general solution of the given equation* is the sum

$$y(x) = Y(x) + \bar{y}_1(x) + \bar{y}_2(x) = e^{2x}(C_1\cos 2x + C_2\sin 2x) +$$
$$+ 0.25e^{2x} + 0.1\cos 2x + 0.05\sin 2x.$$

Problem 16. Solve the equation

$$y'' + y = x\sin x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 1 = 0$ has complex conjugated roots $k_{1,2} = \pm i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1\cos x + C_2\sin x.$$

The right side of the equation has a form $P_1(x)\sin\delta x$ where $P_1(x)$ is polynomial of the first order. So, the particular solution $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r\left[M_1(x)\cos\delta x + N_1(x)\sin\delta x\right],$$

where $M_1(x)$ and $N_1(x)$ are polynomials of the first order and the value of r equals the number of roots of the characteristic equation equal to $i\delta$. . Since $\delta = 1$ and the root $k_1 = i\delta = i$, thus $r = 1$ and we search $\bar{y}(x)$ as

$$\bar{y}(x) = x\left[(Ax + B)\cos x + (Cx + D)\sin x\right].$$

Coefficient *A*, *B*, *C* and *D* will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. Differentiation of $\bar{y}(x)$ gives:

$$\bar{y}'(x) = \left(2Ax + B + Cx^2 + Dx\right)\cos x + \left(-Ax^2 - Bx + 2Cx + D\right)\sin x,$$

$$\bar{y}''(x) = \left[-Ax^2 + (4C - B)x + 2A + 2D\right]\cos x +$$
$$+\left[-Cx^2 - (4A + D)x - 2B + 2C\right]\sin x.$$

Substituting this in the given equation and equating the coefficients of the like terms in both sides gives:

$$\begin{cases} \cos x: & 2A + 2D = 0, \\ \sin x: & -2B + 2C = 0, \\ x\cos x: & B + 4C - B = 0, \\ x\sin x: & D - 4A - D = 1. \end{cases}$$

From here $A = -\dfrac{1}{4}$, $B = 0$, $C = 0$, $D = \dfrac{1}{4}$, thus,

$$\bar{y}(x) = -\frac{x^2}{4}\cos x + \frac{x}{4}\sin x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = \left(C_1 - \frac{x^2}{4}\right)\cos x + \left(C_2 + \frac{x}{4}\right)\sin x.$$

Problem 17. Solve the equation

$$y'' + 4y' + 4y = xe^{2x}.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 4k + 4 = 0$ has repeated real root $k_{1,2} = -2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = e^{-2x}\left(C_1 + C_2 x\right).$$

The right side of the equation is a product of a polynomial $P_n(x)$ and $e^{\gamma x}$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_1(x) = x$, $\gamma = 2$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\bar{y}(x) = e^{2x}(ax + b).$$

Coefficient a and b will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$e^{2x}\left(4ax + 4b + 4a + 8ax + 8b + 4a + 4ax + 4b\right) = xe^{2x}.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} x^0: \ \ 8a + 16b = 0, \\ x^1: \ \ 16a = 1. \end{cases}$$

From here $a = 1/16$, $b = -1/32$, thus,

$$\bar{y}(x) = e^{2x}\left(\frac{x}{16} - \frac{1}{32}\right).$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = e^{-2x}\left(C_1 + C_2 x\right) + e^{2x}\frac{(2x-1)}{32}.$$

Problem 18. Solve the equation

$$y'' - 5y' = 3x^2 + \sin 5x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 5k = k(k-5) = 0$ has real and distinct roots $k_1 = 0$ and $k_2 = 5$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 + C_2 e^{5x}.$$

The right side of the equation is a sum of two functions

$$f(x) = f_1(x) + f_2(x) = 3x^2 + \sin 5x,$$
so we'll use the principle of superposition to obtain the particular solution to the given equation.

First find the particular solution $\overline{y}_1(x)$ of the equation
$$y'' - 5y' = 3x^2.$$

The right side of the equation is a polynomial $P_n(x)$. So, function $\overline{y}_1(x)$ has to be taken in the form
$$\overline{y}_1(x) = x^r Q_n(x),$$
where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of *r* equals the number of zero roots of the characteristic equation. As we see, $P_n(x) = P_2(x) = 3x^2$, thus, we search the particular solution as
$$\overline{y}_1(x) = x\left(ax^2 + bx + c\right)$$
(with $r = 1$ – there is zero root, $k_1 = 0$, of the characteristic equation) and *a*, *b*, *c* will be determined by substitution this $\overline{y}_1(x)$ and its derivatives into equation. It gives:
$$6ax + 2b - 15ax^2 - 10bx - 5c = 3x^2.$$
Next, equating the coefficients of the like terms at both sides of this equation gives
$$\begin{cases} x^0: & 2b - 5c = 0, \\ x^1: & 6a - 10b = 0, \\ x^2: & -15a = 3. \end{cases}$$
From here $a = -0.2$, $b = -0.12$, $c = -0.048$, thus,
$$\overline{y}(x) = -0.2x^3 - 0.12x^2 - 0.048x.$$

It remains to find the particular solution $\overline{y}_2(x)$ of equation
$$y'' - 5y' = \sin 5x.$$

The particular solution $\overline{y}_2(x)$ has to be taken in the form
$$\overline{y}(x) = x^r(A\cos\delta x + A\sin\delta x),$$
where *A* and *B* are coefficients to be determined, and the value of *r* equals the number of roots of the characteristic equation equal to *iδ*. Since $\delta = 5$ and there is no root equal to $i\delta = 5i$, thus $r = 0$ and we search $\overline{y}(x)$ as
$$\overline{y}_2(x) = A\cos 5x + B\sin 5x.$$
Coefficient *A* and *B* will be determined by substitution this $\overline{y}_2(x)$ and its derivatives into equation. It gives:
$$-25A\cos 5x - 25B\sin 5x + 25A\sin 5x - 25B\cos 5x = \sin 5x.$$
Next, equating the coefficients of the like terms at both sides of this equation gives
$$\begin{cases} \cos 5x: & -25A - 25B = 0, \\ \sin 2x: & -25B + 25A = 1. \end{cases}$$
From here $A = 0.02$, $B = -0.02$, thus,
$$\overline{y}_2(x) = 0.02\cos 5x - 0.02\sin 5x.$$

Finally, *the general solution of the given equation* is the sum

$$y(x) = Y(x) + \bar{y}_1(x) + \bar{y}_2(x) = C_1 + C_2 e^{5x} -$$
$$- 0.2x^3 - 0.12x^2 - 0.048x + 0.02(\cos 5x - \sin 5x).$$

Problem 19. Solve the IVP

$$y'' + y = 4e^x, \quad y(0) = 0, \ y'(0) = -3.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 1 = 0$ has complex conjugated roots $k_{1,2} = \pm i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 \cos x + C_2 \sin x.$$

The right side of the equation is a product of a polynomial $P_n(x) = P_0(x) = 4$ and $e^{\gamma x} = e^x$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_0(x) = 4$, $\gamma = 1$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\bar{y}(x) = ae^x.$$

Coefficient a will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$ae^x + ae^x = 4e^x.$$

From here $a = 2$, thus,

$$\bar{y}(x) = 2e^x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 \cos x + C_2 \sin x + 2e^x.$$

Substituting $y(x)$ and $y'(x) = -C_1 \sin x + C_2 \cos x + 2e^x$ in initial conditions

$$\begin{cases} C_1 + 2 = 0, \\ C_2 + 2 = -3, \end{cases}$$

gives $C_1 = -2, \ C_2 = -5$.

Thus *the solution of the IVP* is

$$y(x) = -2\cos x - 5\sin x + 2e^x.$$

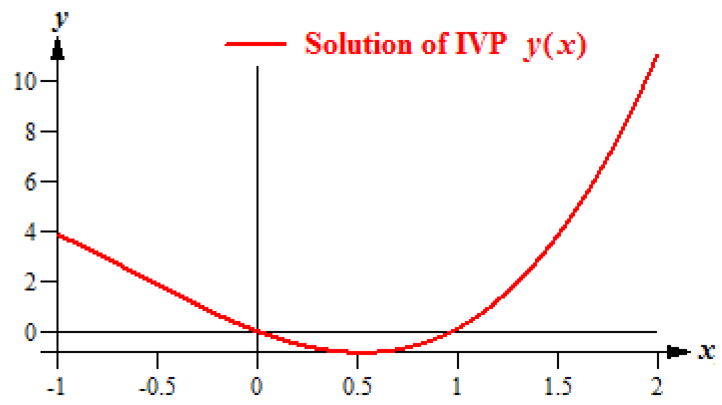

Fig. 1. The solution of the IVP for Problem 19.

Problem 20. Solve the IVP

$$y'' - 2y' = 2e^x, \qquad y(1) = -1, \;\; y'(1) = 0.$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 2k = k(k-2) = 0$ has real and distinct roots $k_1 = 0$ and $k_2 = 2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 + C_2 e^{2x}.$$

The right side of the equation is a product of a polynomial $P_n(x) = P_0(x) = 2$ and $e^{\gamma x} = e^x$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_0(x) = 4$, $\gamma = 1$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\bar{y}(x) = ae^x.$$

Coefficient a will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$ae^x - 2ae^x = 2e^x.$$

From here $a = -2$, thus,

$$\bar{y}(x) = -2e^x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 + C_2 e^{2x} - 2e^x.$$

Substituting $y(x)$ and $y'(x) = 2C_2 e^{2x} - 2e^x$ in initial conditions

$$\begin{cases} C_1 + C_2 e^2 - 2e = -1, \\ 2C_2 e^2 - 2e = 0, \end{cases}$$

gives $C_1 = e - 1,\; C_2 = 1/e$.

Thus *the solution of the IVP* is

$$y(x) = e^{2x-1} - 2e^x + e - 1.$$

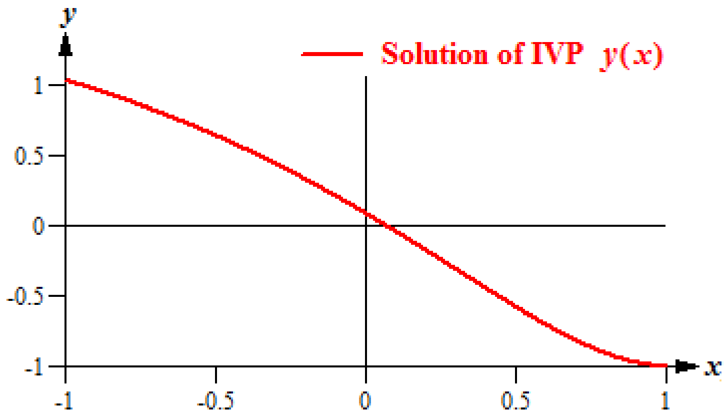

Fig. 2. The solution of the IVP for Problem 20.

Problem 21. Solve the IVP

$$y'' + 2y' + 2y = xe^{-x}, \quad y(0) = y'(0) = 0.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 2k + 2 = 0$ has complex conjugated roots $k_{1,2} = -1 \pm i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = e^{-x}\left(C_1 \cos x + C_2 \sin x\right).$$

The right side of the equation is a product of a polynomial $P_n(x)$ and $e^{\gamma x}$. So, function $\overline{y}(x)$ has to be taken in the form

$$\overline{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_1(x) = x$, $\gamma = -1$ and since $\gamma \neq k_{1,2}$ we must take $r = 0$, therefore, we search the particular solution as

$$\overline{y}(x) = e^{-x}(ax + b).$$

Coefficients a and b will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$e^{-x}\left[x(a - 2a + 2a) + b - 2a - 2b + 2a + 2b\right] = xe^{-x}.$$

From here $a = 1$, $b = 0$, thus,

$$\overline{y}(x) = xe^{-x}.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \overline{y}(x) = e^{-x}\left(C_1 \cos x + C_2 \sin x\right) + xe^{-x}.$$

Substituting $y(x)$ and $y'(x) = e^{-x}\left[\cos x\left(C_2 - C_1\right) - \sin x\left(C_1 + C_1\right) + 1 - x\right]$ in initial conditions

$$\begin{cases} C_1 = 0, \\ C_2 + 1 = 0, \end{cases}$$

gives $C_1 = 0,\ C_2 = -1$.

Thus *the solution of the IVP* is

$$y = e^{-x}(x - \sin x).$$

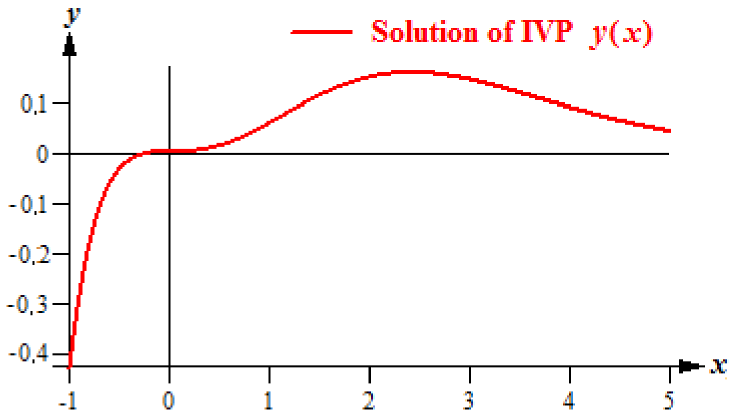

Fig. 3. The solution of the IVP for Problem 21.

Problem 22. Solve the equation

$$y'' - y = 2\sin x - 4\cos x.$$

<u>*Solution.*</u>

The characteristic equation $k^2 - 1 = (k+1)(k-1) = 0$ has real and distinct roots $k_1 = 1$ and $k_2 = -1$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 e^x + C_2 e^{-x}.$$

The particular solution $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r (A\cos\delta x + A\sin\delta x),$$

where A and B are coefficients to be determined, and the value of r equals the number of roots of the characteristic equation equal to $i\delta$. Since $\delta = 1$ and there is no root equal to $i\delta = i$, thus $r = 0$ and we search $\bar{y}(x)$ as

$$\bar{y}(x) = A\cos x + B\sin x.$$

Coefficient A and B will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$-A\cos x - B\sin x - A\cos x - B\sin x = 2\sin x - 4\cos x.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} \cos x: & -A - A = -4, \\ \sin x: & -B - B = 2. \end{cases}$$

From here $A = 2$, $B = -1$, thus,

$$\bar{y}(x) = 2\cos x - \sin x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 e^x + C_2 e^{-x} + 2\cos x - \sin x.$$

Problem 23. Solve the equation

$$y'' + y = e^x + \cos x.$$

<u>*Solution.*</u>

The characteristic equation $k^2+1=0$ has complex conjugated roots $k_{1,2}=\pm i$. Thus *the general solution of the homogeneous equation* is

$$Y(x)=C_1\cos x+C_2\sin x.$$

The right side of the equation is a sum of two functions

$$f(x)=f_1(x)+f_2(x)=e^x+\cos x,$$

so we'll use the principle of superposition to obtain the particular solution to the given equation.

First find the particular solution $\overline{y}_1(x)$ of the equation

$$y''+y=e^x.$$

The function $\overline{y}_1(x)$ has to be taken in the form

$$\overline{y}_1(x)=x^r e^{\gamma x}Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)=P_0(x)=1$ and the value of *r* equals the number of roots of the characteristic equation equal to γ. As we see, $\gamma=1$ and since $\gamma\neq k_{1,2}$ we must take $r=0$, therefore, we search the particular solution as

$$\overline{y}_1(x)=ae^x.$$

Coefficient *a* will be determined by substitution this $\overline{y}(x)$ and its derivatives into equation. It gives:

$$e^x\left(a+a\right)=e^x.$$

From here $a=1/2$, thus,

$$\overline{y}_1(x)=\frac{1}{2}e^x.$$

It remains to find the particular solution $\overline{y}_2(x)$ of equation

$$y''+y=\cos x.$$

The particular solution $\overline{y}_2(x)$ has to be taken in the form

$$\overline{y}_2(x)=x^r(A\cos\delta x+A\sin\delta x),$$

where *A* and *B* are coefficients to be determined, and the value of *r* equals the number of roots of the characteristic equation equal to *iδ*. Since $\delta=1$ and there is a root equal to $i\delta=i$, thus $r=1$ and we search $\overline{y}(x)$ as

$$\overline{y}(x)=x(A\cos x+B\sin x).$$

Coefficient *A* and *B* will be determined by substitution this $\overline{y}_2(x)$ and its derivatives into equation. It gives:

$$\cos x(Ax+2B-Ax)+\sin x(-Bx+Bx)=\cos x.$$

From here $A=0$, $B=1/2$, thus,

$$\overline{y}_2(x)=\frac{x}{2}\sin x.$$

Finally, *the general solution of the given equation* is the sum

$$y(x)=Y(x)+\overline{y}_1(x)+\overline{y}_2(x)=C_1\cos x+C_2\sin x+\frac{1}{2}\left(e^x+x\sin x\right).$$

Problem 24. Solve the equation

$$y'' + y = \cos x + \cos 2x .$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 1 = 0$ has complex conjugated roots $k_{1,2} = \pm i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 \cos x + C_2 \sin x .$$

The right side of the equation is a sum of two functions

$$f(x) = f_1(x) + f_2(x) = \cos x + \cos 2x ,$$

so we'll use the principle of superposition to obtain the particular solution to the given equation.

First find the particular solution $\overline{y}_1(x)$ of the equation

$$y'' + y = \cos x .$$

The function $\overline{y}_1(x)$ has to be taken in the form

$$\overline{y}_1(x) = x^r (A\cos \delta x + A\sin \delta x),$$

where A and B are coefficients to be determined, and the value of r equals the number of roots of the characteristic equation equal to $i\delta$. Since $\delta = 1$ and there is a root equal to $i\delta = i$, thus $r = 1$ and we search $\overline{y}_1(x)$ as

$$\overline{y}_1(x) = x\left(A\cos x + B\sin x\right).$$

Coefficient A and B will be determined by substitution this $\overline{y}_1(x)$ and its derivatives into equation. It gives:

$$\cos x(Ax + 2B - Ax) + \sin x(-Bx + Bx) = \cos x .$$

From here $A = 0$, $B = 1/2$, thus,

$$\overline{y}_1(x) = \frac{x}{2}\sin x .$$

It remains to find the particular solution $\overline{y}_2(x)$ of equation

$$y'' + y = \cos 2x .$$

The particular solution $\overline{y}_2(x)$ has to be taken in the form

$$\overline{y}_2(x) = x^r (A\cos \delta x + A\sin \delta x),$$

where A and B are coefficients to be determined, and the value of r equals the number of roots of the characteristic equation equal to $i\delta$. Since $\delta = 2$ and there is no root equal to $i\delta = 2i$, thus $r = 0$ and we search $\overline{y}(x)$ as

$$\overline{y}_2(x) = A\cos 2x + B\sin 2x .$$

Coefficient A and B will be determined by substitution this $\overline{y}_2(x)$ and its derivatives into equation. It gives:

$$-4A\cos 2x - 4B\sin 2x + A\cos 2x + B\sin 2x = \cos 2x .$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} \cos 2x: & -4A + A = 1, \\ \sin 2x: & -4B + B = 0. \end{cases}$$

From here $A = -1/3$, $B = 0$, thus,

$$\bar{y}(x) = -\frac{1}{3}\cos 2x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2} x \sin x - \frac{1}{3}\cos 2x.$$

Problem 25. Solve the equation

$$y^{(4)} + 2y'' + y = \cos x.$$

<u>*Solution.*</u>

The characteristic equation $k^4 + 2k^2 + 1 = \left(k^2 + 1\right)^2 = 0$ has two pairs of complex conjugated roots $k_{1,2} = i$, $k_{3,4} = -i$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x.$$

The particular solution $\bar{y}(x)$ has to be taken in the form

$$\bar{y}_1(x) = x^r (A\cos\delta x + A\sin\delta x)$$

where A and B are coefficients to be determined, and the value of r equals the number of roots of the characteristic equation equal to $i\delta$. Since $\delta = 1$ and there are two roots equal to $i\delta = i$, thus $r = 2$ and we search $\bar{y}(x)$ as

$$\bar{y}(x) = x^2 \left(A\cos x + B\sin x\right).$$

Coefficient A and B will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. Differentiation of $\bar{y}(x)$ gives:

$$\bar{y}'(x) = \cos x\left(2Ax + Bx^2\right) + \sin x\left(-Ax^2 + 2Bx\right),$$

$$\bar{y}''(x) = \cos x\left(2A + 4Bx - Ax^2\right) + \sin x\left(-4Ax + 2B - Bx^2\right),$$

$$\bar{y}'''(x) = \cos x\left(6B - 6Ax - Bx^2\right) + \sin x\left(-6A - 6Bx + Ax^2\right),$$

$$\bar{y}^{(4)}(x) = \cos x\left(-12A - 8Bx + Ax^2\right) + \sin x\left(-12B + 8Ax + Bx^2\right),$$

Substituting this in the given equation and equating the coefficients of the like terms in both sides gives $A = -\frac{1}{8}$, $B = 0$. Thus,

$$\bar{y}(x) = -\frac{x^2}{8}\cos x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x - \frac{x^2}{8}\cos x.$$

Problem 26. Solve the IVP

$$y'' - 4y = \sin 2x, \quad y(0) = 0, \;\; y'(0) = 0.$$

Solution.

The characteristic equation $k^2 - 4 = (k-2)(k+2) = 0$ has real and distinct roots $k_1 = 2$, $k_2 = -2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 e^{2x} + C_2 e^{-2x}.$$

The particular solution $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r (A\cos\delta x + A\sin\delta x),$$

where *A* and *B* are coefficients to be determined, and the value of *r* equals the number of roots of the characteristic equation equal to $i\delta$. Since $\delta = 2$ and there is no root equal to $i\delta = 2i$, thus $r = 0$ and we search $\bar{y}(x)$ as

$$\bar{y}(x) = A\cos 2x + B\sin 2x.$$

Coefficient *A* and *B* will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$-4A\cos 2x - 4B\sin 2x - 4A\cos 2x - 4B\sin 2x = \sin 2x.$$

Next, equating the coefficients of the like terms at both sides of this equation gives

$$\begin{cases} \cos 2x: & -4A + 4A = 0, \\ \sin 2x: & -4B - 4B = 1. \end{cases}$$

From here $A = 0$, $B = -1/8$, thus,

$$\bar{y}(x) = -\frac{1}{8}\sin 2x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{8}\sin 2x.$$

Substituting $y(x)$ and $y'(x) = 2C_1 e^{2x} - 2C_2 e^{-2x} - \frac{1}{4}\cos 2x$ in initial conditions

$$\begin{cases} C_1 + C_2 = 0, \\ 2C_1 - 2C_2 - \frac{1}{4} = 0, \end{cases}$$

gives $C_1 = 1/16$, $C_2 = -1/16$.

Thus *the solution of the IVP* is

$$y(x) = \frac{1}{16}\left(e^{2x} - e^{-2x} - 2\sin 2x\right).$$

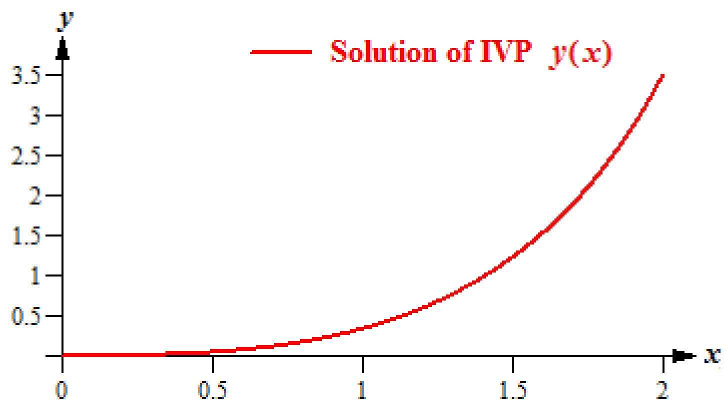

Fig. 4. The solution of the IVP for Problem 26.

Problem 27. Solve the IVP

$$y'' - y = x, \quad y(0) = 1, \; y'(0) = -1.$$

<u>Solution.</u>

The characteristic equation $k^2 - 1 = (k-1)(k+1) = 0$ has real and distinct roots $k_1 = 1,\; k_2 = -1$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = C_1 e^x + C_2 e^{-x}.$$

The right side of the equation is a polynomial $P_n(x)$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of zero roots of the characteristic equation. As we see, $P_n(x) = P_1(x) = x$, thus, we search the particular solution as

$$\bar{y}(x) = ax + b$$

(with $r = 0$ – no zero roots the characteristic equation) and a, b will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$-ax - b = x.$$

From here $a = -1,\; b = 0$, thus,

$$\bar{y}(x) = -x.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = C_1 e^x + C_2 e^{-x} - x.$$

Substituting $y(x)$ and $y'(x) = C_1 e^x - C_2 e^{-x} - 1$ in initial conditions

$$\begin{cases} C_1 + C_2 = 1, \\ C_1 - C_2 - 1 = -1, \end{cases}$$

gives $C_1 = 1/2,\; C_2 = 1/2$.

Thus *the solution of the IVP* is

$$y(x) = \frac{e^x + e^{-x}}{2} - x = \cosh x - x.$$

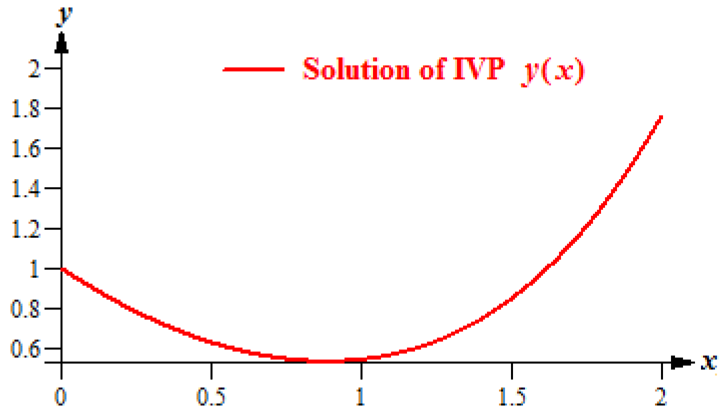

Fig. 5. The solution of the IVP for Problem 27.

Problem 28. Solve the IVP

$$y'' + 4y' + 4y = 3e^{-2x}, \quad y(0) = 0, \; y'(0) = 0.$$

<u>*Solution.*</u>

The characteristic equation $k^2 + 4k + 4 = (k+2)^2 = 0$ has repeated root $k_{1,2} = -2$. Thus *the general solution of the homogeneous equation* is

$$Y(x) = e^{-2x}\left(C_1 + xC_2\right).$$

The right side of the equation is a product of a polynomial $P_n(x)$ and $e^{\gamma x}$. So, function $\bar{y}(x)$ has to be taken in the form

$$\bar{y}(x) = x^r e^{\gamma x} Q_n(x),$$

where $Q_n(x)$ is a polynomial having the same order as $P_n(x)$ (with coefficients to be determined), and the value of r equals the number of roots of the characteristic equation equal to γ. As we see, $P_n(x) = P_0(x) = 3$, $\gamma = -2$ and since $\gamma = k_{1,2}$ we must take $r = 2$, therefore, we search the particular solution as

$$\bar{y}(x) = ax^2 e^{-2x}.$$

Coefficients a and b will be determined by substitution this $\bar{y}(x)$ and its derivatives into equation. It gives:

$$e^{-2x}\left[4ax^2 - 8ax + 2a - 8ax^2 + 8ax + 4ax^2\right] = 3e^{-2x}.$$

From here $a = 3/2$, thus,

$$\bar{y}(x) = \frac{3}{2}x^2 e^{-2x}.$$

Finally, *the general solution of the given equation* is

$$y(x) = Y(x) + \bar{y}(x) = e^{-2x}\left(C_1 + xC_2 + \frac{3}{2}x^2\right).$$

Substituting $y(x)$ and $y'(x) = e^{-2x}\left(-2C_1 + C_2 - 2xC_2 - 3x^2 + 3x\right)$ in initial conditions

$$\begin{cases} C_1 = 0, \\ -2C_1 + C_2 = 0, \end{cases}$$

gives $C_1 = 0,\ C_2 = 0$.

Thus *the solution of the IVP* is

$$y = \frac{3}{2}x^2 e^{-2x}.$$

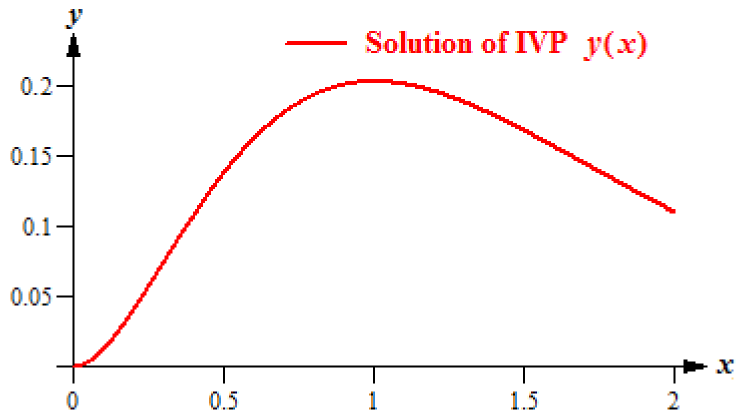

Fig. 6. The solution of the IVP for Problem 28.

Problem 29. Solve the equation

$$x^2 y'' - xy' + y = 0.$$

<u>*Solution.*</u>

The given equation is the *Euler equation*. Search solution in the form $y = x^k$. Substituting in the given equation and canceling x^k, we obtain the equation for k

$$k(k-1) - k + 1 = (k-1)^2 = 0$$

which has repeated root: $k_{1,2} = 1$.

Thus, $y_1 = x$, $y_2 = x\ln|x|$ and *the general solution* is

$$y = x(C_1 + C_2 \ln|x|).$$

Problem 30. Solve the equation

$$x^2 y'' - 4xy' + 6y = 0.$$

<u>*Solution.*</u>

The given equation is the *Euler equation*. Search solution in the form $y = x^k$. Substituting in the given equation and canceling x^k, we obtain the equation for k

$$k(k-1) - 4k + 6 = 0$$

which has two roots: $k_1 = 2$, $k_2 = 3$.

Thus, $y_1 = x^2$, $y_2 = x^3$ and *the general solution* is

$$y = C_1 x^2 + C_2 x^3.$$

Problem 31. Solve the equation

$$x^2 y'' - xy' - 3y = 0.$$

Solution.

The given equation is the *Euler equation*. Search solution in the form $y = x^k$. Substituting in the given equation and canceling x^k, we obtain the equation for k

$$k(k-1) - 2k - 3 = 0$$

which has two roots: $k_1 = -1$, $k_2 = 3$.

Thus, $y_1 = x^{-1}$, $y_2 = x^3$ and *the general solution* is

$$y = C_1 x^{-1} + C_2 x^3.$$

Problem 32. Solve the equation

$$x^3 y''' + xy' - y = 0.$$

Solution.

The given equation is the *Euler equation*. Search solution in the form $y = x^k$. Substituting in the given equation and canceling x^k, we obtain the equation for k

$$k(k-1)(k-2) + k - 1 = (k-1)^3 = 0$$

which has repeated root: $k_{1,2,3} = 1$.

Thus, $y_1 = x$, $y_2 = x \ln|x|$, $y_3 = x \ln^2|x|$ and *the general solution* is

$$y = x\left(C_1 + C_2 \ln|x| + C_3 \ln^2|x|\right).$$