Chapter 2

Logic and Reasoning

Exercises 2.1

1. (ii)

P	Q	$\neg Q$	$P \wedge Q$	$(P \wedge Q) \Leftrightarrow \neg Q$
T	T	F	T	F
T	F	T	F	F
F	T	F	F	T
F	F	T	F	F

(iii)

P	Q	$\neg P$	$Q \Rightarrow \neg P$	$P \veebar (Q \Rightarrow \neg P)$
T	T	F	F	T
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

(iv)

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(vi)

P	Q	R	$\neg Q$	$P \Rightarrow \neg Q$	$(P \Rightarrow \neg Q) \wedge R$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	T	F

(vii)

P	Q	R	$R \Leftrightarrow Q$	$\neg(R \Leftrightarrow Q)$	$P \vee \neg(R \Leftrightarrow Q)$
T	T	T	T	F	T
T	T	F	F	T	T
T	F	T	F	T	T
T	F	F	T	F	T
F	T	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T
F	F	F	T	F	F

(viii)

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg R$	$P \wedge \neg R$	$\neg(P \wedge Q) \Rightarrow (P \wedge \neg R)$
T	T	T	T	F	F	F	T
T	T	F	T	F	T	T	T
T	F	T	F	T	F	F	F
T	F	F	F	T	T	T	T
F	T	T	F	T	F	F	F
F	T	F	F	T	T	F	F
F	F	T	F	T	F	F	F
F	F	F	F	T	T	F	F

(ix)

P	Q	R	$Q \Leftrightarrow R$	$P \vee (Q \Leftrightarrow R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	T	T

(x)

P	Q	R	$\neg P$	$\neg Q$	$P \wedge R$	$\neg P \wedge \neg Q$	$(P \wedge R) \vee (\neg P \wedge \neg Q)$
T	T	T	F	F	T	F	T
T	T	F	F	F	F	F	F
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

2. (iii) *The wind blows or the rain falls and, in addition, the temperature does not rise.*

   (iv) *If it is not the case that the sun shines and the wind blows then the rain falls or the temperature doesn't rise.*

3. (i) $W \wedge \neg R$

   (iii) $T \Leftrightarrow (S \wedge \neg R)$

   (v) $((\neg S \vee (W \wedge R)) \Rightarrow \neg T$

4. (i)

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Since $P \wedge \neg P$ is always false, it is a contradiction.

(ii)

P	$\neg P$	$\neg P \vee P$
T	F	T
F	T	T

Since $\neg P \vee P$ is always true, it is a tautology.

(iii)

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Since all its truth values are T, it follows that $P \Rightarrow (P \vee Q)$ is a tautology.

(vi)

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Since its truth values are neither all T nor all F, it follows that $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ is neither a tautology nor a contradiction.

(vii) $P \wedge \neg P$ is a contradiction. Hence, for any proposition A, $(P \wedge \neg P) \Rightarrow A$ is always true. Therefore $(P \wedge \neg P) \Rightarrow (Q \vee R)$ is a tautology.

(viii)

P	Q	R	$\neg Q$	$\neg R$	$P \Rightarrow \neg Q$	$\neg R \Rightarrow P$	$(P \Rightarrow \neg Q) \wedge (\neg R \Rightarrow P)$
T	T	T	F	F	F	T	F
T	T	F	F	T	F	T	F
T	F	T	T	F	T	T	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	T	T
F	T	F	F	T	T	F	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

Since its truth values are neither all T nor all F, it follows that $(P \Rightarrow \neg Q) \wedge (\neg R \Rightarrow P)$ is neither a tautology nor a contradiction.

5. (ii) A compound proposition that is a tautology gives truth value T for all combinations of truth values of its 'input' simple propositions. Thus, replacing all occurrences of one of its simple propositions with another proposition cannot give rise to a truth value F. Hence any substitution instance of a tautology is also a tautology.

A similar argument works for contradictions.

Exercises 2.2

1. (i)

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the truth values in the fourth and seventh columns are equal, $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

(ii)

P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

Since the truth values in the first and fourth columns are equal, $P \vee (P \wedge Q) \equiv P$.

(v)

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the truth values in the fifth and eighth columns are equal,
$(P \vee Q) \Rightarrow R \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$.

(vii)

						A	B	
P	Q	R	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow R$	$\neg P$	$\neg P \Rightarrow R$	$Q \Rightarrow R$	$A \wedge B$
T	T	T	T	T	F	T	T	T
T	T	F	T	F	F	T	F	F
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F	F
F	F	T	T	T	T	T	T	T
F	F	F	T	F	T	F	T	F

Since the truth values in the fifth and ninth columns are equal,
$(P \Rightarrow Q) \Rightarrow R \equiv (\neg P \Rightarrow R) \wedge (Q \Rightarrow R)$.

(viii) For space reasons, we have omitted the columns of the truth table for $\neg P$ and $\neg R$.

						A	B		
P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg(P \wedge Q) \wedge R$	$\neg P \wedge R$	$Q \vee \neg R$	$\neg B$	$A \vee \neg B$
T	T	T	T	F	F	F	T	F	F
T	T	F	T	F	F	F	T	F	F
T	F	T	F	T	T	F	F	T	T
T	F	F	F	T	F	F	T	F	F
F	T	T	F	T	T	T	T	F	T
F	T	F	F	T	F	F	T	F	F
F	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	F	T	F	F

Since the truth values in the sixth and tenth columns are equal,
$\neg(P \wedge Q) \wedge R \equiv (\neg P \wedge R) \vee \neg(Q \vee \neg R)$.

(ix) There is a typographical error in the question: $(P \Rightarrow R) \wedge (Q \Rightarrow \neg P)$ should be replaced with $(P \Rightarrow R) \vee (Q \Rightarrow \neg P)$.

						A	B	
P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\neg P$	$P \Rightarrow R$	$Q \Rightarrow \neg P$	$A \vee B$
T	T	T	T	T	F	T	F	T
T	T	F	T	F	F	F	F	F
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	F	T	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Since the truth values in the fifth and ninth columns are equal,

$(P \wedge Q) \Rightarrow R \equiv (P \Rightarrow R) \vee (Q \Rightarrow \neg P)$.

(x)

					A	C	B	D	
P	Q	R	$P \Leftrightarrow Q$	$(P \Leftrightarrow Q) \vee R$	$Q \vee R$	$P \Rightarrow A$	$P \vee R$	$Q \Rightarrow B$	$C \wedge D$
T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T	T
T	F	F	F	F	F	F	T	T	F
F	T	T	F	T	T	T	T	T	T
F	T	F	F	F	T	T	F	F	F
F	F	T	T	T	T	T	T	T	T
F	F	F	T	T	F	T	F	T	T

Since the truth values in the fifth and tenth columns are equal,

$(P \Leftrightarrow Q) \vee R \equiv (P \Rightarrow (Q \vee R)) \wedge (Q \Rightarrow (P \vee R))$.

2. (i)

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

The only row where $P \wedge Q$ is true is row 1 and $P \vee Q$ is also true in row 1.

Hence $(P \wedge Q) \vDash (P \vee Q)$.

(iii) There is a typographical error in the question: $\neg(P \vee Q)$ should be replaced with $\neg(P \veebar Q)$.

P	Q	$P \Rightarrow Q$	$\neg Q$	$P \vee \neg Q$	$P \veebar Q$	$\neg(P \veebar Q)$
T	T	T	F	T	F	T
T	F	F	T	T	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	T

The only rows where $P \Rightarrow Q$ and $P \vee \neg Q$ are both true are rows 1 and 4. In teach of these rows, $\neg(P \veebar Q)$ is also true.

Hence $\{P \wedge Q, P \vee \neg Q\} \vDash \neg(P \veebar Q)$.

(v) There is a typographical error in the question: $R \Rightarrow P$ should be replaced with $P \Rightarrow R$.

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \Rightarrow R$	$\neg Q$	$\neg Q \Rightarrow R$
T	T	T	T	T	T	F	T
T	T	F	F	T	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	F	T
F	T	F	F	F	T	F	T
F	F	T	F	F	T	T	T
F	F	F	F	F	T	T	F

The propositions $P \vee (Q \wedge R)$ and $P \Rightarrow R$ are both true only in rows 1,3 and 5. In each of these rows, $\neg Q \Rightarrow R$ is also true.

Therefore $\{P \vee (Q \wedge R), P \Rightarrow R\} \vDash \neg Q \Rightarrow R$.

(vi)

P	Q	R	$P \Rightarrow Q$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$P \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	F	F	T	T
T	F	F	F	F	T	F
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

The propositions $P \Rightarrow Q$ and $(P \wedge Q) \Rightarrow R$ are both true in rows 1,5,6,7 and 8. In each of these rows, $P \Rightarrow R$ is also true.

Therefore $\{P \Rightarrow Q, (P \wedge Q) \Rightarrow R\} \vDash \neg P \Rightarrow R$.

3. (i)
$$\begin{aligned}
\neg P \vee (P \wedge Q) &\equiv (\neg P \vee P) \wedge (\neg P \vee Q) && \text{Distributive law}\\
&\equiv (P \vee \neg P) \wedge (\neg P \vee Q) && \text{Commutative law}\\
&\equiv \text{true} \wedge (\neg P \vee Q) && \text{Complement law}\\
&\equiv (\neg P \vee Q) \wedge \text{true} && \text{Commutative law}\\
&\equiv \neg P \vee Q && \text{Identity law}
\end{aligned}$$

(iii)
$$\begin{aligned}
\neg(\neg P \wedge Q) \wedge (P \vee \neg Q) &\equiv (\neg(\neg P) \vee \neg Q) \wedge (P \vee \neg Q) && \text{De Morgan's law}\\
&\equiv (P \vee \neg Q) \wedge (P \vee \neg Q) && \text{Involution law}\\
&\equiv P \vee \neg Q && \text{Idempotent law}
\end{aligned}$$

(v)
$$\begin{aligned}
P \wedge [(P \wedge Q) \vee \neg P] &\equiv [P \wedge (P \wedge Q)] \vee [P \wedge \neg P] && \text{Distributive law}\\
&\equiv [(P \wedge P) \wedge Q)] \vee \text{false} && \text{Associative \& complement laws}\\
&\equiv P \wedge Q && \text{Idempotent \& identity laws}
\end{aligned}$$

(vii)
$$\begin{aligned}
\neg(P \wedge \neg(Q \wedge R)) &\equiv \neg P \vee \neg(\neg(Q \wedge R)) && \text{De Morgan's law} \\
&\equiv \neg P \vee (Q \wedge R) && \text{Involution law} \\
&\equiv (\neg P \vee Q) \wedge (\neg P \vee R) && \text{Distributive law} \\
&\equiv (\neg P \vee Q) \wedge (\neg P \vee \neg(\neg R)) && \text{Involution law} \\
&\equiv (\neg P \vee Q) \wedge \neg(P \wedge \neg R) && \text{De Morgan's law}
\end{aligned}$$

4. (i) There is a typographical error in the question: $\neg(P \wedge Q)$ should be replaced with $\neg(P \vee Q)$.

$$\begin{aligned}
\neg Q \wedge (P \Rightarrow Q) &\equiv \neg Q \wedge (\neg P \vee Q) && \text{Material implication law} \\
&\equiv (\neg Q \wedge \neg P) \vee (\neg Q \wedge Q) && \text{Distributive law} \\
&\equiv (\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) && \text{Commutative law; twice} \\
&\equiv \neg(P \vee Q) \vee \text{false} && \text{De Morgan's \& complement laws} \\
&\equiv \neg(P \vee Q) && \text{Identity law}
\end{aligned}$$

(iii)
$$\begin{aligned}
(P \wedge Q) \Rightarrow R &\equiv \neg(P \wedge Q) \vee R && \text{Material implication law} \\
&\equiv (\neg P \vee \neg Q) \vee R && \text{De Morgan's law} \\
&\equiv \neg P \vee (\neg Q \vee R) && \text{Associative law} \\
&\equiv \neg P \vee (Q \Rightarrow R) && \text{Material implication law} \\
&\equiv P \Rightarrow (Q \Rightarrow R) && \text{Material implication law}
\end{aligned}$$

(v)
$$\begin{aligned}
P \Rightarrow (Q \vee R) &\equiv \neg P \vee (Q \vee R) && \text{Material implication law} \\
&\equiv (\neg P \vee \neg P) \vee (Q \vee R) && \text{Idempotent law} \\
&\equiv \neg P \vee (\neg P \vee (Q \vee R)) && \text{Associative law} \\
&\equiv \neg P \vee ((\neg P \vee Q) \vee R) && \text{Associative law} \\
&\equiv \neg P \vee ((Q \vee \neg P) \vee R) && \text{Commutative law} \\
&\equiv \neg P \vee (Q \vee (\neg P \vee R)) && \text{Associative law} \\
&\equiv (\neg P \vee Q) \vee (\neg P \vee R) && \text{Associative law} \\
&\equiv (P \Rightarrow Q) \vee (P \Rightarrow R) && \text{Material implication law, twice}
\end{aligned}$$

Exercises 2.3

1. (i) Universe: Students
   Predicates: P : ... *likes to party.*
   Then symbolise as: $\forall x \bullet P(x)$.

   (iii) Universe: People
   Predicates: A : ... *went to the auction*
   B : ... *bought something.*
   Then symbolise as: $\forall x \bullet A(x) \Rightarrow B(x)$.

(v) Universe: Celebrities
Predicates: G : … *gives interviews*
P : … *participates in reality TV*.

Then symbolise as: $\forall x \bullet G(x) \wedge \neg\forall x \bullet P(x)$.

(vi) Universe: Athletes
Predicates: R : … *can run fast*.

Then symbolise as: $\neg\forall x \bullet R(x)$.

(viii) Universe: Graham's friends
Predicates: P : … *came to the party*
S : … *would have to stand*.

Then symbolise as: $(\forall x \bullet P(x)) \Rightarrow (\exists x \bullet S(x))$.

(ix) Universe: My friends
Predicates: B : … *believes in global warming*
C : … *drives a large car*.

Then symbolise as: $(\forall x \bullet B(x)) \wedge (\exists x \bullet C(x))$.

(x) Universe: People
Predicates: $A(x, y)$: … x *applauds* y
$C(x)$: … x *is courageous*.

Then symbolise as: $\forall x\, \forall y \bullet C(y) \Rightarrow A(x, y)$.

Note that, although '*everyone applauds someone who is courageous*' appears to indicate an existential quantifier, the meaning is really '*everyone applauds anyone who is courageous*' which explains the two universal quantifiers.

(xi) Universe: People who went to the races
Predicates: C : … *was cold*
L : … *lost money*.

Then symbolise as: $(\neg\exists x \bullet C(x)) \wedge (\exists x \bullet L(x))$.

2. (i) *Julie smiles a lot but she's prone to depression.*

(iii) *Everyone who is likeable smiles a lot.*

(v) *Everyone who is not likeable or doesn't smile a lot is prone to depression.*

(vii) *Not everyone who is prone to depression smiles a lot.*

3. (i) $\forall y \bullet T(\text{Carl}, y) \Rightarrow P(\text{Carl}, y)$

(ii) $\forall x \bullet T(x, \text{Statistics}) \Rightarrow E(x, \text{Statistics})$

(iv) $\exists x\, \exists y \bullet T(x, y) \wedge \neg E(x, y)$

(vii) $\exists y\, \forall x \bullet T(x, y) \Rightarrow P(x, y)$

(viii) Note that the phrase '*all students take courses that they don't enjoy*' may be interpreted either as '*all students take some courses that they don't enjoy*' or as '*all students take courses all of which they don't enjoy*'.

The first interpretation is symbolised as

$$(\forall x\, \exists y \bullet T(x,y) \wedge \neg E(x,y)) \Rightarrow (\neg \exists x\, \forall y \bullet T(x,y) \Rightarrow P(x,y))$$

and the second interpretation is symbolised as

$$(\forall x\, \forall y \bullet T(x,y) \Rightarrow \neg E(x,y)) \Rightarrow (\neg \exists x\, \forall y \bullet T(x,y) \Rightarrow P(x,y)).$$

This is a situation where the natural language sentence is ambiguous and the two interpretations are equally 'valid'.

4. (i) (a) *All students enjoy logic.*
   (b) $\neg \forall x \bullet E(x, \text{Logic}) \equiv \exists x \bullet \neg E(x, \text{Logic})$
   (c) *Some students don't enjoy logic.*

   (ii) (a) *Some students do not pass logic.*
   (b) $\neg \exists x \bullet \neg P(x, \text{Logic}) \equiv \forall x \bullet P(x, \text{Logic})$
   (c) *All students pass logic.*

   (iv) (a) *Poppy takes a course that she does not enjoy.*
   (b) $\neg \exists y \bullet T(\text{Poppy}, y) \wedge \neg E(\text{Poppy}, y) \equiv \forall y \bullet T(\text{Poppy}, y) \Rightarrow E(\text{Poppy}, y)$
   Note that this uses the equivalences $\neg(P \wedge \neg Q) \equiv \neg P \vee Q \equiv P \Rightarrow Q$.
   (c) *Poppy enjoys every course she takes.*

   (vi) (a) *No student passes every course.*
   (b) $\neg(\neg \exists x\, \forall y \bullet P(x,y)) \equiv \exists x\, \forall y \bullet P(x,y)$
   (c) *Some student passes every course.*

   (viii) (a) *If some students do not take every course then not all students pass some course.*
   (b) $\neg[(\exists x\, \forall y \bullet \neg T(x,y)) \Rightarrow (\neg \forall x\, \exists y \bullet P(x,y))]$
   $\equiv (\exists x\, \forall y \bullet \neg T(x,y)) \wedge \neg(\neg \forall x\, \exists y \bullet P(x,y))$ (using $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$)
   $\equiv (\exists x\, \forall y \bullet \neg T(x,y)) \wedge (\forall x\, \exists y \bullet P(x,y))$
   (c) *Some students do not take every course and every student passes some course.*

Exercises 2.4

1. (i)

1.	$P \Rightarrow Q$	premise
2.	$P \wedge R$	premise
3.	P	2. Simplification
4.	Q	1, 3. Modus ponens

(iii)

1.	$P \Leftrightarrow Q$	premise
2.	$\neg(P \wedge Q)$	premise
3.	$\neg P \vee \neg Q$	2. Equivalence: De Morgan's law
4.	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	1. Equivalence: biconditional law
5.	$P \Rightarrow Q$	4. Simplification
6.	$\neg P \vee Q$	5. Equivalence: material implication law
7.	$Q \vee \neg P$	6. Equivalence: commutative law
8.	$\neg Q \vee \neg P$	3. Equivalence: commutative law
9.	$\neg P \vee \neg P$	7, 8. Resolution
10.	$\neg P$	9. Equivalence: idempotent law
11.	$(Q \Rightarrow P) \wedge (P \Rightarrow Q)$	4. Equivalence: commutative law
12.	$Q \Rightarrow P$	11. Simplification
13.	$\neg Q \vee P$	12. Equivalence: material implication law
14.	$P \vee \neg Q$	13. Equivalence: commutative law
15.	$\neg Q \vee \neg Q$	3, 14. Resolution
16.	$\neg Q$	15. Equivalence: idempotent law
17.	$\neg P \wedge \neg Q$	10, 16. Conjunction

(v)

1.	$(Q \vee S) \Rightarrow R$	premise
2.	$Q \vee P$	premise
3.	$\neg R$	premise
4.	$\neg(Q \vee S)$	1, 3. Modus tollens
5.	$\neg Q \wedge \neg S$	4. Equivalence: De Morgan's law
6.	$\neg Q$	5. Simplification
7.	P	2, 6. Disjunctive syllogism

(vii)

1.	$(P \vee Q) \Rightarrow (R \wedge S)$	premise
2.	P	premise
3.	$P \vee Q$	2. Addition
4.	$R \wedge S$	1, 3. Modus ponens
5.	R	4. Simplification

(ix)

1.	$P \Rightarrow \neg Q$	premise
2.	$Q \vee (R \wedge S)$	premise
3.	$\neg(\neg Q) \vee (R \wedge S)$	3. Equivalence: involution law
4.	$\neg Q \Rightarrow (R \wedge S)$	4. Equivalence: material implication law
5.	$P \Rightarrow (R \wedge S)$	1, 4. Hypothetical syllogism

2. (ii)

1.	$P \Leftrightarrow Q$	premise
2.	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	1. Equivalence: biconditional law
3.	$P \Rightarrow Q$	2. Simplification

(iv)

1.	$P \Rightarrow R$	premise
2.	$Q \Rightarrow R$	premise
3.	$\neg P \vee R$	1. Equivalence: material implication law
4.	$\neg Q \vee R$	2. Equivalence: material implication law
5.	$R \vee \neg P$	3. Equivalence: commutative law
6.	$R \vee \neg Q$	4. Equivalence: commutative law
7.	$(R \vee \neg P) \wedge (R \vee \neg Q)$	5, 6. Conjunction
8.	$R \vee (\neg P \wedge \neg Q)$	7. Equivalence: distributive law
9.	$R \vee \neg(P \vee Q)$	7. Equivalence: De Morgan's law
10.	$\neg(P \vee Q) \vee R$	9. Equivalence: commutative law
11.	$(P \vee Q) \Rightarrow R$	4. Equivalence: material implication law

3. (i) Let W: *Mary drinks wine*
C: *Mary eats cheese*
L: *Mary eats chocolate*
H: *Mary gets a headache.*

The argument has premises: $(W \vee C) \Rightarrow H$ and $W \wedge L$.

The conclusion is H.

1.	$(W \vee C) \Rightarrow H$	premise
2.	$W \wedge L$	premise
3.	W	2. Simplification
4.	$W \vee C$	3. Addition
5.	H	1, 4. Modus ponens

(ii) Let D: *You get a degree*
J: *You get a good job*
S: *You will be successful*
H: *You will be happy.*

The argument has premises: $(D \vee J) \Rightarrow (S \wedge H)$ and J.

The conclusion is H.

1.	$(D \vee J) \Rightarrow (S \wedge H)$	premise
2.	J	premise
3.	$J \vee D$	2. Addition
4.	$D \vee J$	3. Equivalence: commutative law
5.	$S \wedge H$	1, 4. Modus ponens
6.	$H \wedge S$	5. Equivalence: commutative law
7.	H	6. Simplification

(iv) Let P: *The project was a success*
I: *Sally invested her inheritance*
S: *Sally was sensible*
B: *Sally is broke.*

The argument has premises: $\neg P \vee \neg I$, $S \Rightarrow I$, P and $(\neg S \wedge \neg I) \Rightarrow B$.

The conclusion is B.

1.	$\neg P \vee \neg I$	premise
2.	$S \Rightarrow I$	premise
3.	P	premise
4.	$(\neg S \wedge \neg I) \Rightarrow B$	premise
5.	$\neg(\neg P)$	3. Equivalence: involution law
6.	$\neg I$	1, 5. Disjunctive syllogism
7.	$\neg S$	2, 6. Modus tollens
8.	$\neg S \wedge \neg I$	6, 7. Conjunction
9.	B	4, 8. Modus ponens

(v) Let A: *The murder was committed by A*
B: *The murder was committed by B*
C: *The murder was committed by C*
P: *The victim was poisoned.*

The argument has premises: $A \vee (B \wedge C)$ and $A \Rightarrow P$.

The conclusion is $C \vee P$.

1.	$A \vee (B \wedge C)$	premise
2.	$A \Rightarrow P$	premise
3.	$\neg A \vee P$	2. Equivalence: material implication law
4.	$(B \wedge C) \vee P$	1, 3. Resolution
5.	$P \vee (B \wedge C)$	4. Equivalence: commutative law
6.	$(P \vee B) \wedge (P \vee C)$	5. Equivalence: distributive law
7.	$(P \vee C) \wedge (P \vee B)$	6. Equivalence: commutative law
8.	$P \vee C$	7. Simplification
9.	$C \vee P$	8. Equivalence: commutative law

(vii) Let U: *It is useful*
K: *I will keep it*
V: *It is valuable*
B: *It belonged to Ben.*

The argument has premises: $(U \Rightarrow K) \wedge (V \Rightarrow K)$, $B \Rightarrow (U \vee V)$ and B.

The conclusion is K.

1.	$(U \Rightarrow K) \wedge (V \Rightarrow K)$	premise
2.	$B \Rightarrow (U \vee V)$	premise
3.	B	premise
4.	$U \Rightarrow K$	1. Simplification
5.	$(V \Rightarrow K) \wedge (U \Rightarrow K)$	1. Equivalence: commutative law
6.	$V \Rightarrow K$	5. Simplification
7.	$\neg U \vee K$	4. Equivalence: material implication law
8.	$\neg V \vee K$	6. Equivalence: material implication law
9.	$K \vee \neg U$	7. Equivalence: commutative law
10.	$K \vee \neg V$	8. Equivalence: commutative law
11.	$(K \vee \neg U) \wedge (K \vee \neg V)$	9, 10. Conjunction
12.	$K \vee (\neg U \wedge \neg V)$	11. Equivalence: distributive law
13.	$K \vee \neg(U \vee V)$	12. Equivalence: De Morgan's law
14.	$\neg(U \vee V) \vee K$	13. Equivalence: commutative law
15.	$U \vee V$	2, 3. Modus ponens
16.	$\neg(\neg(U \vee V))$	15. Equivalence: involution law
17.	K	14, 16. Disjunctive syllogism

Note that, with one additional application of the material implication law after line 14, lines 4 – 14 are essentially establishing the logical equivalence

$$(U \Rightarrow K) \wedge (V \Rightarrow K) \equiv (U \vee V) \Rightarrow K.$$

Unfortunately, this is not one of our named logical equivalences. Were this a named equivalence, we would have the following much simpler deduction available.

1.	$(U \Rightarrow K) \wedge (V \Rightarrow K)$	premise
2.	$B \Rightarrow (U \vee V)$	premise
3.	B	premise
4.	$(U \vee V) \Rightarrow K$	1. Equivalence
5.	$B \Rightarrow K$	2, 4. Hypothetical syllogism
6.	K	3, 5. Modus ponens

(viii) Let R: *It will rain*
S: *I will will go shopping*
U: *I take an umbrella*
C: *I go by car.*

The argument has premises: $\neg R \Rightarrow S$, $S \Rightarrow (\neg U \Rightarrow R)$ and $C \Rightarrow \neg U$.

The conclusion is $R \vee \neg C$.

1.	$\neg R \Rightarrow S$	premise
2.	$S \Rightarrow (\neg U \Rightarrow R)$	premise
3.	$C \Rightarrow \neg U$	premise
4.	$\neg R \Rightarrow (\neg U \Rightarrow R)$	1, 2. Hypothetical syllogism
5.	$\neg R \Rightarrow (\neg\neg U \vee R)$	4. Equivalence: material implication law
6.	$\neg R \Rightarrow (U \vee R)$	5. Equivalence: involution law
7.	$\neg\neg R \vee (U \vee R)$	6. Equivalence: material implication law
8.	$R \vee (U \vee R)$	7. Equivalence: involution law
9.	$R \vee (R \vee U)$	8. Equivalence: commutative law
10.	$(R \vee R) \vee U$	9. Equivalence: associative law
11.	$R \vee U$	10. Equivalence: idempotent law
12.	$U \vee R$	11. Equivalence: commutative law
13.	$\neg C \vee \neg U$	3. Equivalence: material implication law
14.	$\neg U \vee \neg C$	13. Equivalence: commutative law
15.	$R \vee \neg C$	12, 14. Resolution

(x) Let C: *Tim committed the crime*
F: *Tim will flee the country*
S: *We will see Tim again*
T: *Tim is Tom's friend.*

The argument has premises: $C \Rightarrow (F \wedge \neg S)$ and $S \Rightarrow \neg T$.
The conclusion is $(C \vee T) \Rightarrow \neg S$.

1.	$C \Rightarrow (F \wedge \neg S)$	premise
2.	$S \Rightarrow \neg T$	premise
3.	$\neg S \vee \neg T$	2. Equivalence: material implication law
4.	$\neg C \vee (F \wedge \neg S)$	1. Equivalence: material implication law
5.	$(\neg C \vee F) \wedge (\neg C \vee \neg S)$	4. Equivalence: distributive law
6.	$(\neg C \vee \neg S) \wedge (\neg C \vee F)$	5. Equivalence: commutative law
7.	$\neg C \vee \neg S$	6. Simplification
8.	$\neg S \vee \neg C$	7. Equivalence: commutative law
9.	$(\neg S \vee \neg T) \wedge (\neg S \vee \neg C)$	3, 8. Conjunction
10.	$\neg S \vee (\neg T \wedge \neg C)$	9. Equivalence: distributive law
11.	$\neg S \vee \neg(T \vee C)$	10. Equivalence: De Morgan's law
12.	$\neg S \vee \neg(C \vee T)$	11. Equivalence: commutative law
13.	$\neg(C \vee T) \vee \neg S$	12. Equivalence: commutative law
14.	$(C \vee T) \Rightarrow \neg S$	13. Equivalence: material implication law

4. (i) Let W: *Mary drinks wine*
C: *Mary eats cheese*
L: *Mary eats chocolate*
H: *Mary gets a headache.*

The argument has premises: $(W \wedge C) \Rightarrow H$ and $W \vee L$.

The conclusion is H.

Suppose W is true, C is false and H is false. (The truth value of L may be either true or false.)

Then the premises are both true and the conclusion is false. Therefore the argument is not valid.

(iii) Let R: *It will rain*
S: *I will will go shopping*
U: *I take an umbrella*
C: *I go by car.*

The argument has premises: $\neg R \Rightarrow S$, $S \Rightarrow (\neg U \Rightarrow R)$ and $C \Rightarrow \neg U$.

The conclusion is $R \wedge \neg C$.

Suppose R is false, S is true, U is true and C is false.

Then the premises are all true and the conclusion is false. Therefore the argument is not valid.

(v) Unfortunately the argument as stated is valid.

Let C: *Tim committed the crime*
F: *Tim will flee the country*
S: *We will see Tim again*
T: *Tim is Tom's friend.*

The argument has premises: $C \Rightarrow (F \wedge \neg S)$ and $S \Rightarrow \neg T$.

The conclusion is $T \Rightarrow \neg S$.

Since the second premise is logically equivalent to the conclusion, whenever the premises are all true, so too is the conclusion.

Suppose we change the conclusion to $\neg T \Rightarrow \neg S$: *if he's not Tom's friend the we will never see him again.*

Now suppose C is false, S is true and T is false. (The truth value of F may be either true or false.)

Now the premises are both true and the conclusion is false. Therefore the revised argument is not valid.

5.

J	L	G	$\neg J$	$\neg L$	$\neg G$	$\neg J \Rightarrow \neg L$	$\neg J \wedge G$	$L \vee \neg G$
T	T	T	F	F	F	T	F	T
T	T	F	F	F	T	T	F	T
T	F	T	F	T	F	T	F	F
T	F	F	F	T	T	T	F	T
F	T	T	T	F	F	F	T	T
F	T	F	T	F	T	F	F	T
F	F	T	T	T	F	T	T	F
F	F	F	T	T	T	T	F	T

There is no row of the table where $\neg J \Rightarrow \neg L$, $\neg J \wedge G$ and $L \vee \neg G$ are all true.

Hence the premises in example 2.14.4 are inconsistent.

6. (i) Define the universe to be 'people' and symbolise the predicates as follows.

Let G: ... *is good-looking*
R: ... *is rich*
D: ... *is dishonest.*

Then the argument has premises: $\exists x \bullet G(x) \wedge R(x)$ and $\forall x \bullet R(x) \Rightarrow D(x)$.

The conclusion is $\exists x \bullet G(x) \wedge D(x)$.

1.	$\exists x \bullet G(x) \wedge R(x)$	premise
2.	$\forall x \bullet R(x) \Rightarrow D(x)$	premise
3.	$G(a) \wedge R(a)$	1. $\exists$-elimination
4.	$R(a) \Rightarrow D(a)$	2. $\forall$-elimination
5.	$G(a)$	3. Simplification
6.	$R(a) \wedge G(a)$	3. Equivalence: commutative law
7.	$R(a)$	6. Simplification
8.	$D(a)$	4, 7. Modus ponens
9.	$G(a) \wedge D(a)$	5, 8. Conjunction
10.	$\exists x \bullet G(x) \wedge D(x)$	9. $\exists$-introduction

(ii) Using the same universe and predicates as in part (i), the argument has premises: $\exists x \bullet G(x) \wedge R(x)$ and $\forall x \bullet R(x) \Rightarrow D(x)$.

The conclusion is $\neg \forall x \bullet G(x) \Rightarrow \neg D(x)$.

The first ten lines of the deduction are the same as in part (i) and the remaining deduction is as follows.

11.	$\exists x \bullet \neg\neg(G(x) \wedge D(x))$	10. Equivalence: involution law
12.	$\exists x \bullet \neg(\neg G(x) \vee \neg D(x)$	11. Equivalence: De Morgan's law
13.	$\exists x \bullet \neg(G(x) \Rightarrow \neg D(x))$	12. Equivalence: material implication law
14.	$\neg \forall x \bullet G(x) \Rightarrow \neg D(x)$	13. Equivalence: negating quantified propositions

(iv) Define the universe to be 'numbers' and symbolise the predicates as follows.

Let I: ... *is an integer*
E: ... *is even*
O: ... *is odd*
N: ... *is non-zero.*

Then the argument has premises: $\forall x \bullet I(x) \Rightarrow (E(x) \vee O(x))$, $\forall x \bullet I(x) \Rightarrow (E(x) \vee N(x))$, and $\exists x \bullet I(x)$.

The conclusion is $\exists x \bullet E(x) \vee (O(x) \wedge N(x))$.

1.	$\forall x \bullet I(x) \Rightarrow (E(x) \vee O(x))$	premise
2.	$\forall x \bullet I(x) \Rightarrow (E(x) \vee N(x))$	premise
3.	$\exists x \bullet I(x)$	premise
4.	$I(a)$	3. $\exists$-elimination
5.	$I(a) \Rightarrow (E(a) \vee O(a))$	1. $\forall$-elimination
6.	$I(a) \Rightarrow (E(a) \vee N(a))$	2. $\forall$-elimination
7.	$E(a) \vee O(a)$	4, 5. Modus ponens
8.	$E(a) \vee N(a)$	4, 6. Modus ponens
9.	$(E(a) \vee O(a)) \wedge (E(a) \vee N(a))$	7, 8. Conjunction
10.	$E(a) \vee (O(a) \wedge N(a))$	9. Equivalence: distributive law
11.	$\exists x \bullet E(x) \vee (O(x) \wedge N(x))$	10. $\exists$-introduction

(v) Define the universe to be 'animals' and symbolise the predicates as follows.

Let F: . . . *has feathers*
A: . . . *is aquatic*
S: . . . *lives in the sea.*

Then the argument has premises: $\forall x \bullet F(x) \Rightarrow \neg A(x)$ and $\exists x \bullet A(x) \wedge S(x)$.

The conclusion is $\exists x \bullet S(x) \wedge \neg F(x)$.

1.	$\forall x \bullet F(x) \Rightarrow \neg A(x)$	premise
2.	$\exists x \bullet A(x) \wedge S(x)$	premise
3.	$A(a) \wedge S(a)$	2. $\exists$-elimination
4.	$F(a) \Rightarrow \neg A(a)$	1. $\forall$-elimination
5.	$A(a)$	3. Simplification
6.	$\neg\neg A(a)$	5. Equivalence: involution law
7.	$\neg F(a)$	4, 6. Modus tollens
8.	$S(a) \wedge A(a)$	3. Equivalence: commutative law
9.	$S(a)$	8. Simplification
10.	$S(a) \wedge \neg F(a)$	7, 9. Conjunction
11.	$\exists x \bullet S(x) \wedge \neg F(x)$	10. $\exists$-introduction

(vi) Define the universe to be 'functions' and symbolise the predicates as follows.

Let C: . . . *is continuous*
D: . . . *is differentiable*
A: . . . *is defined for all values of x.*

Then the argument has premises: $\exists x \bullet C(x) \wedge D(x)$ and $\forall x \bullet C(x) \Rightarrow A(x)$.

The conclusion is $\exists x \bullet A(x) \wedge D(x)$.

1.	$\exists x \bullet C(x) \wedge D(x)$	premise
2.	$\forall x \bullet C(x) \Rightarrow A(x)$	premise
3.	$C(a) \wedge D(a)$	1. $\exists$-elimination
4.	$C(a) \Rightarrow A(a)$	2. $\forall$-elimination
5.	$C(a)$	3. Simplification
6.	$A(a)$	4, 5. Modus ponens
7.	$D(a) \wedge C(a)$	3. Equivalence: commutative law
8.	$D(a)$	7. Simplification
9.	$A(a) \wedge D(a)$	6, 8. Conjunction
10.	$\exists x \bullet A(x) \wedge D(x)$	9. $\exists$-introduction

(vii) Define the universe to be 'things' and symbolise the predicates as follows.

Let E: *. . . is enjoyable*
C: *. . . is cheap*
B: *. . . is harmful to one's health*
H: *. . . is a holiday*.

Then the argument has premises: $\forall x \bullet (E(x) \wedge C(x) \Rightarrow B(x)$, $\forall x \bullet H(x) \Rightarrow E(x)$ and $\exists x \bullet H(x) \wedge \neg B(x)$.

The conclusion is $\exists x \bullet \neg C(x)$.

1.	$\forall x \bullet (E(x) \wedge C(x)) \Rightarrow B(x)$	premise
2.	$\forall x \bullet H(x) \Rightarrow E(x)$	premise
3.	$\exists x \bullet H(x) \wedge \neg B(x)$	premise
4.	$H(a) \wedge \neg B(a)$	3. $\exists$-elimination
5.	$(E(a) \wedge C(a)) \Rightarrow B(a)$	1. $\forall$-elimination
6.	$H(a) \Rightarrow E(a)$	2. $\forall$-elimination
7.	$H(a)$	4. Simplification
8.	$E(a)$	6, 7. Modus ponens
9.	$\neg B(a) \wedge H(a)$	4. Equivalence: commutative law
10.	$\neg B(a)$	9. Simplification
11.	$\neg(E(a) \wedge C(a))$	5, 10. Modus tollens
12.	$\neg E(a) \vee \neg C(a)$	11. Equivalence: De Morgan's law
13.	$\neg(\neg E(a))$	8. Equivalence: involution law
14.	$\neg C(a)$	12, 13. Disjunctive syllogism
15.	$\exists x \bullet \neg C(x)$	14. $\exists$-introduction