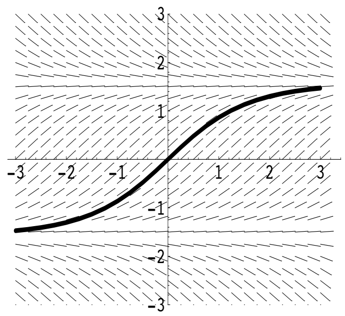
Chapter 2

Geometrical and Numerical Methods for First-Order Equations

2.1 Direction Fields—the Geometry of Differential Equations

1. Looking at the point $(2, 1)$, $y' = \frac{9}{2}$, which matches graph b.
2. Looking at the point $(2, 1)$, $y' = 18$, which matches graph c.
3. Looking at the point $(2, 1)$, $y' = \frac{2}{9}$, which matches graph a.
4. Looking at the point $(2, 1)$, $y' = \frac{1}{18}$, which matches graph d.
5. Graph B
6. Graph C
7. Graph D
8. Graph A
9. $y' = \cos y$

#9



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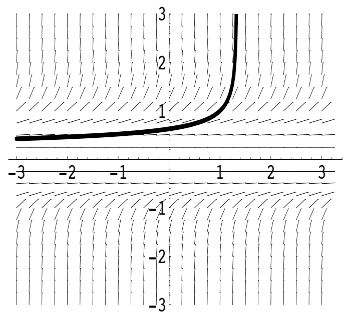
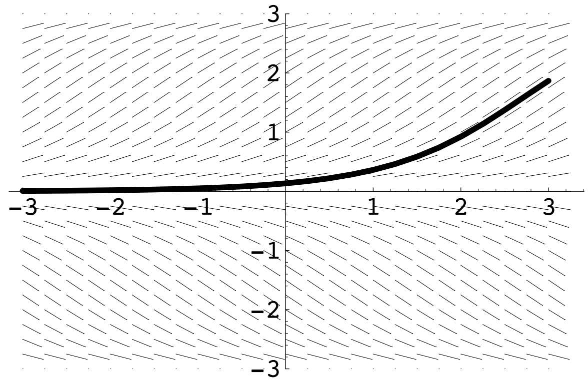
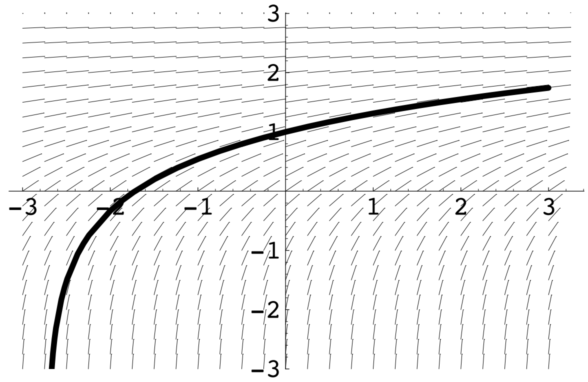


10. $y' = y^4$

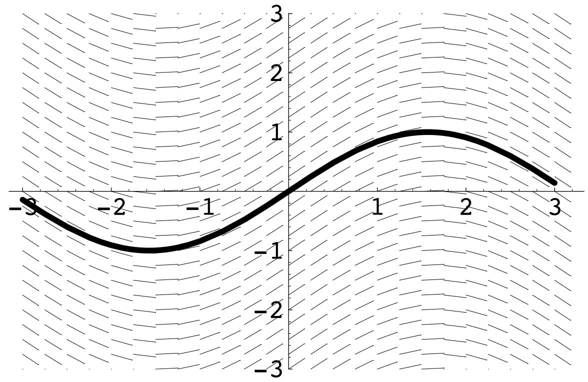
11. $y' = \sin y$



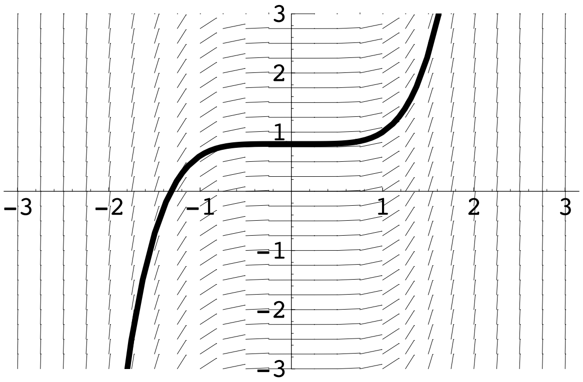
12. $y' = e^{-y}$



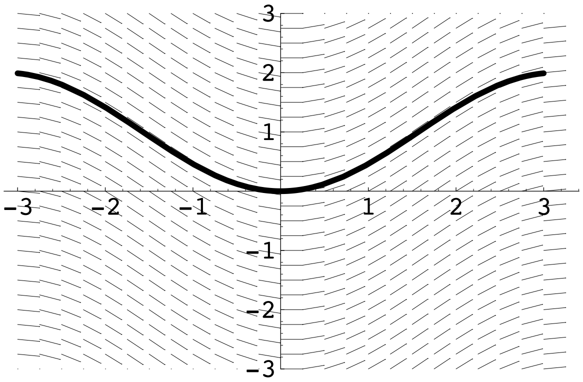
13. $y' = \cos x$



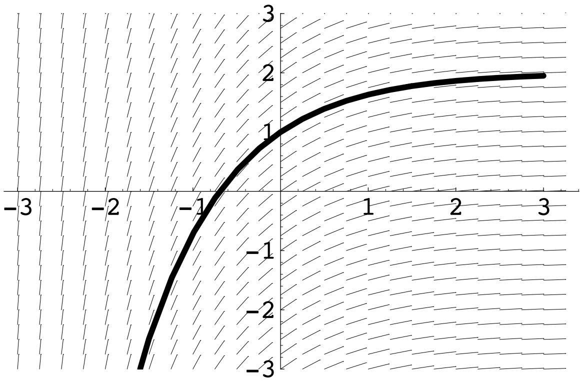
14. $y' = x^4$



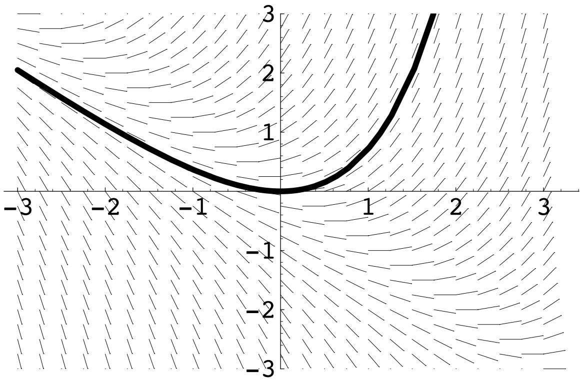
15. $y' = \sin x$



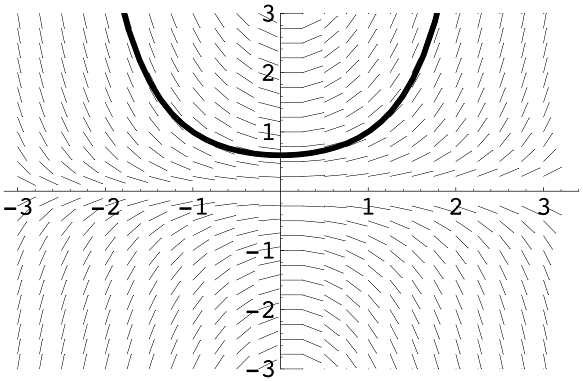
16. $y' = e^{-x}$



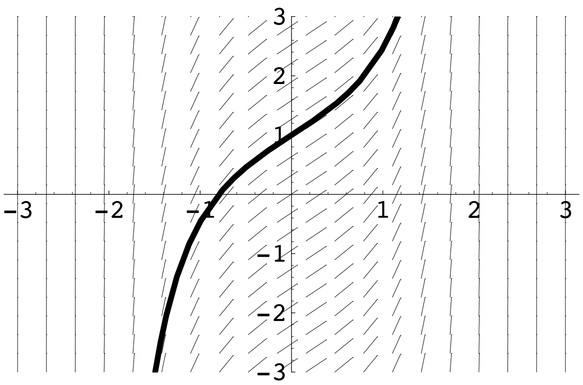
17. $y' = x + y$



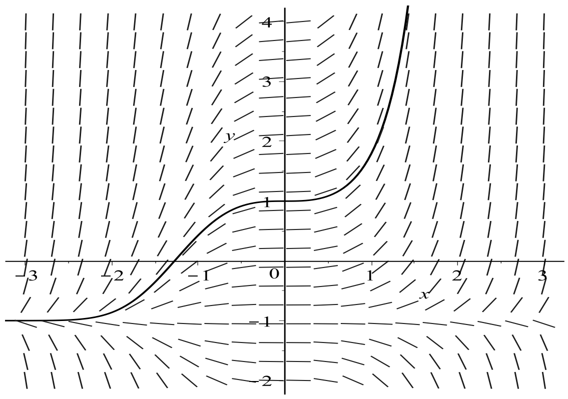
18. $y' = xy$



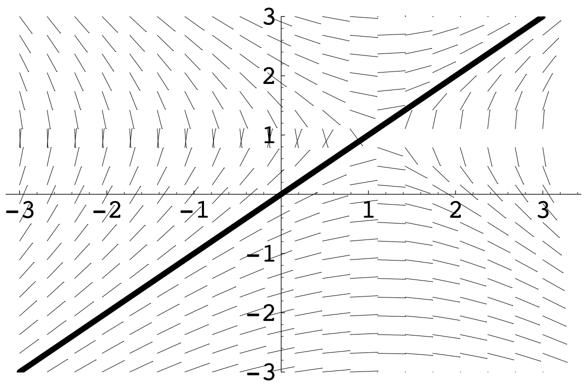
19. $y' = e^{x^2}$



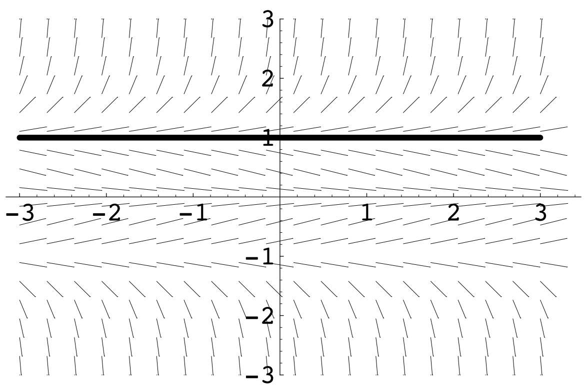
20. $y' = x^2(y+1)$



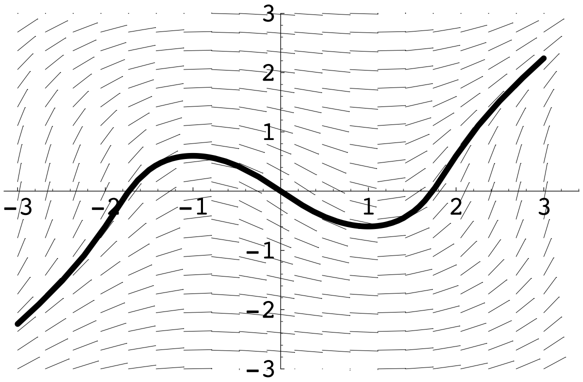
21. $y' = \dfrac{x-1}{y-1}$



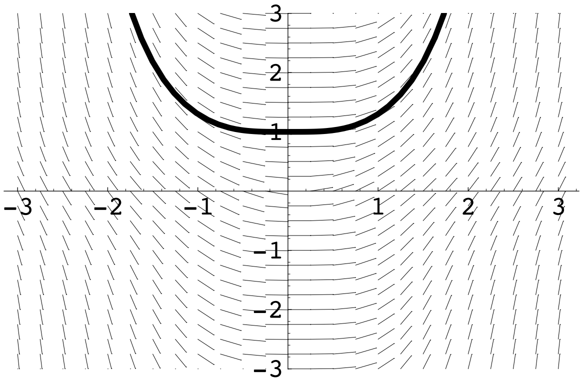
22. $y' = y(y^2 - 2)$



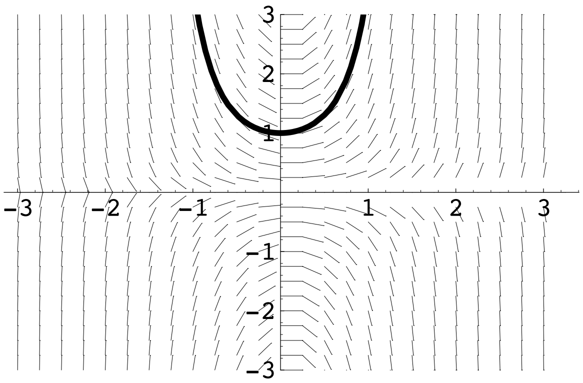
23. $y' = \dfrac{x^2 - 1}{y^2 + 1}$



24. $y' = \dfrac{x^3(y^2 + 1)}{y^2 + x^2}$



25. $xy(x^2 + 2)$



2.2 Existence and Uniqueness for First-Order Equations

1. $y' = y^2 - x^2$, $y(0) = 0$, $f = y^2 - x^2$, $\dfrac{\partial f}{\partial y} = 2y$. The theorem guarantees that a solution exists and is unique on some interval.

2. $y' = x^2 - y^2$, $y(0) = 1$, $f = x^2 - y^2$, $\dfrac{\partial f}{\partial y} = -2y$. The theorem guarantees that a solution exists and is unique on some interval.

3. $y' = y^{-2} - x^2$, $y(0) = 0$, $f = \dfrac{1}{y^2} - x^2$, $\dfrac{\partial f}{\partial y} = -y^{-3}$. The theorem does *not* guarantee that a solution exists or is unique on some interval.

4. $y' = x \ln y$, $y(1) = 1$, $f = x \ln y$, $\dfrac{\partial f}{\partial y} = \frac{x}{y}$. The theorem guarantees that a solution exists and is unique on some interval ($y \neq 0$).

5. $y' = y + \dfrac{1}{1-x}$, $y(1) = 0$, $f = y + \dfrac{1}{1-x}$, $\dfrac{\partial f}{\partial y} = 1$. The theorem does *not* guarantee that a solution exists or is unique on some interval.

6. $y' = e^y + \csc x$, $y(0) = 0$, $f = e^y + \csc x$, $\dfrac{\partial f}{\partial y} = e^y$. The theorem does *not* guarantee that a solution exists or is unique on some interval.

7. $f(x, y) = 3x(y + 2)^{2/3}$, $\dfrac{\partial f}{\partial y} = \dfrac{2x}{(y+2)^{1/3}}$, which is discontinuous when $y = -2$, so a solution exists and is unique everywhere except possibly along $y = -2$.

8. $\dfrac{\partial f}{\partial y} = \dfrac{4}{3} x y^{(-1/3)}$ which is not continuous at $y = 0$. Hence a solution exists and is unique everywhere except possibly along $y = 0$.
   Actual solution: $y = \dfrac{x^6}{27}$. This solution crosses $y = 0$ when $x = 0$.

9. $f(x, y) = (x - y)^{1/5}$, $\dfrac{\partial f}{\partial y} = \dfrac{-1}{5(x-y)^{4/5}}$, which is discontinuous when $y = x$, so solutions exist and are unique everywhere except possibly along $y = x$. Alternatively, $f(x, y) = \dfrac{(x-y)'}{5}$, $\dfrac{\partial f}{\partial y} = -\dfrac{1}{5}$, so solutions exist and are unique everywhere.

10. $\dfrac{\partial f}{\partial y} = 2x^{(2/3)}$ which, along with $f(x, y)$, are continuous everywhere. Hence solution exists and is unique.
    Actual solution: $y = 0$.

11. $f(x,y) = x^2y^{-1}$, $\dfrac{\partial f}{\partial y} = -x^2y^{-2}$, which are both discontinuous when $y = 0$, so solutions exist and are unique everywhere except possibly along $y = 0$.

12. $\dfrac{\partial f}{\partial y} = -\dfrac{2}{3}xy^{(-5/3)}$ which is not continuous at $y = 0$. Hence a solution exists and is unique everywhere except possibly along $y = 0$.
Actual solution: $y = \left(\dfrac{5}{6}\right)^{(3/5)} x^{(6/5)}$ This solution crosses $y = 0$ when $x = 0$.

13. $f(x,y) = (x+y)^{-2}$, $\dfrac{\partial f}{\partial y} = \dfrac{-2}{(x+y)^3}$, which are both discontinuous when $y = -x$, so solutions exist and are unique everywhere except possibly along $y = -x$.

14. $\dfrac{\partial f}{\partial y} = \dfrac{2}{3}xy^{(-1/3)}$ which is not continuous at $y = 0$. Hence a solution exists and is unique everywhere except possibly along $y = 0$.
Actual solution: $y = \dfrac{x^6}{216}$ which crosses $y = 0$ when $x = 0$.

15. $f(x,y) = 5(y-2)^{3/5}$ and $y(0) = 2$. $\dfrac{\partial y}{\partial y} = \dfrac{3}{(y-2)^{2/5}}$ is discontinuous when $y = 2$, so solutions exist and are unique everywhere except possibly along $y = 2$. Solving for the initial value,

$$\begin{aligned} y' = 5(y-2)^{3/5} &\Rightarrow \int (y-2)^{-3/5}\,dy = \int 5\,dx \\ &\Rightarrow \frac{5}{2}(y-2)^{2/5} = 5x + C \end{aligned}$$

Applying initial conditions,

$$\begin{aligned} \frac{5}{2}(2-2)^{2/5} = 5(0) + C \Rightarrow C = 0 \\ \Rightarrow (y-2)^{2/5} = 2x \Rightarrow y - 2 = \pm(2x)^{5/2} \\ \Rightarrow y = 2 \pm (2x)^{5/2} \end{aligned}$$

Solutions passing through $(0,2)$ are NOT unique.

16. $\dfrac{\partial f}{\partial y} = -2xy^{-3}$ which is continuous everywhere except $y = 0$, so solutions exist and are unique everywhere except possibly along $y = 0$.
Actual solution: $\left(\dfrac{-3}{2} + \dfrac{x^2}{2}\right)^{1/3}$

17. $f(x,y) = 2\sqrt{y}$ is continuous for $y \geq 0$ and $\dfrac{\partial f}{\partial y} = \dfrac{1}{\sqrt{y}}$ is continuous for $y > 0$. Hence, by the theorem, a solution exists and is unique for $y(1) = 3$.
    Actual solution is given by: $y = (x + \sqrt{3} - 1)^2$.

18. $\dfrac{\partial f}{\partial y} = x^{(1/3)}(-2y)$ which is continuous everywhere (as is f), therefore, unique solution will exist everywhere.

19. $\dfrac{\partial f}{\partial y} = (y-1)^{(-2/3)}$ which is continuous everywhere except $y = 1$, so solutions exist and are unique everywhere except possibly along $y = 1$.
    Actual solution: $1 \pm 2\sqrt{2}x^{(3/2)}$

20. $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are not continuous at $y = 1$, so solutions exist and are unique everywhere except possibly along $y = 1$.
    Actual solution: $y = 1 + \sqrt[3]{3 - 3\cos x}$ which IS unique.

21. $\dfrac{\partial f}{\partial y}$ DNE at $y = 0$; if $y > 0$, then $\frac{\partial f}{\partial y} = 1$ and if $y < 0$, then $\frac{\partial f}{\partial y} = -1$.
    Hence,
    $\dfrac{\partial f}{\partial y} = \lim\limits_{h \to 0} \dfrac{f(x, y+h) - f(x,y)}{h}$ DNE. Therefore, the solutions will exist and be unique except possibly at $y = 0$.
    If $y \geq 0$, $y = e^C e^x$ which will never be 0. If $y < 0$, $y = -e^C e^{-x}$ which will never be 0. Thus, the only solution to this DE is $y = 0$ which IS unique.

2.3 First-Order Autonomous Equations—Geometrical Insight

1. $y' = 2y + 3$

Root	$y = -\frac{3}{2}$
Multiplicity	1

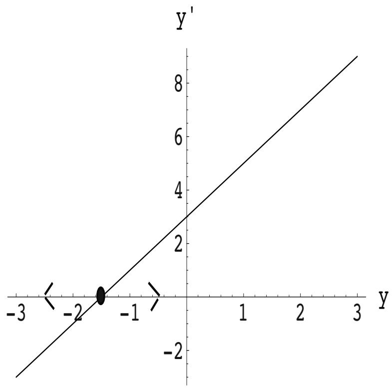
   (i) $y - y'$-plane; see graph

   (ii) By the phase line diagram, $y = -\frac{3}{2}$ is an unstable equilibrium point.

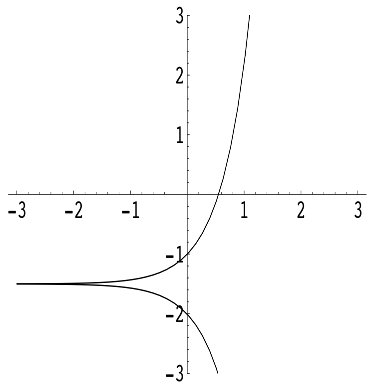
   (iii) $y > -\frac{3}{2}$, $y \to \infty$ as $x \to \infty$
   $y < -\frac{3}{2}$, $y \to -\infty$ as $x \to \infty$

(iv) xy-plane; see graph

2.3.1(i)



2.3.1(iv)



2. $y' = -3y + 2$

Root	$y = \frac{2}{3}$
Multiplicity	1

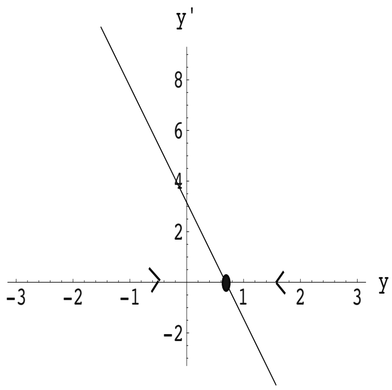
(i) $y - y'$-plane; see graph

(ii) By the phase line diagram, $y = \frac{2}{3}$ is a stable equilibrium point.

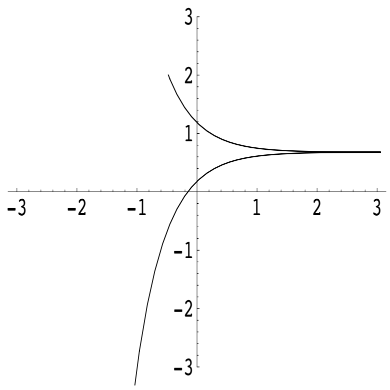
(iii) $y < \frac{2}{3}$, $y \to \frac{2}{3}$ as $x \to \infty$
$y > \frac{2}{3}$, $y \to \frac{2}{3}$ as $x \to \infty$

(iv) xy-plane; see graph

2.3.2(i)



2.3.2(iv)



3. $x' = x^2 - x - 6 = (x - 3)(x + 2)$

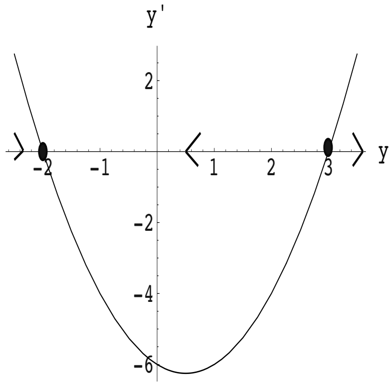
(i) $y - y'$-plane; see graph

(ii) $x = -2$ is a stable equilibrium
$x = 3$ is an unstable equilibrium

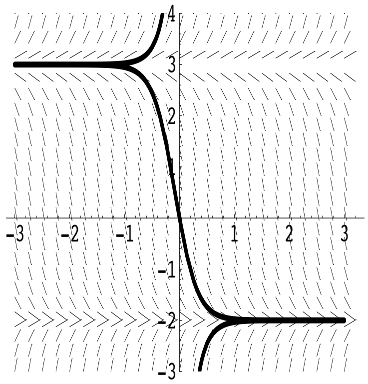
(iii) If $x_0 > 3$, then $x(t) \to \infty$ as $t \to \infty$.
If $-2 < x_0 < 3$, then $x(t) \to -3$ as $t \to \infty$.
If $x_0 < -2$, then $x(t) \to -3$ as $t \to \infty$.

(iv) xy-plane; see graph

2.3.3(i)



2.3.3(iv)



4. $x' = x(x+2)(x-3)$

Equilibrium	Multiplicity
$x = 0$	1
$x = -2$	1
$x = 3$	1

Highest power and coefficient: $x(+x)(+x) = +x^3$

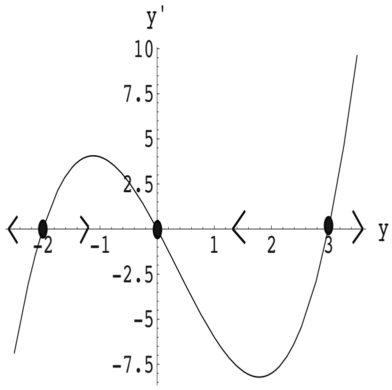
(i) $y - y'$-plane; see graph

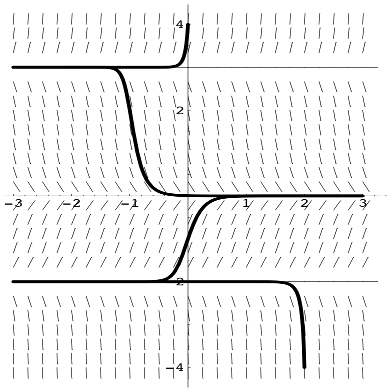
(ii) $x = -2, 3$ unstable; $x = 0$ stable

(iii) For $x_0 \in (-\infty, -2)$, $x \to -\infty$ as $t \to \infty$
For $x_0 \in (-2, 3)$, $x \to 0$ as $t \to \infty$
For $x_0 \in (3, \infty)$, $x \to \infty$ as $t \to \infty$

(iv) xy-plane; see graph

2.3.4(i) 2.3.4(iv)





5. $y' = y^2 + 4y + 4$, $y^2 + 4y + 4 = 0$, $(y+2)^2 = 0$

Root	$y = -2$
Multiplicity	2

End behavior: y^2

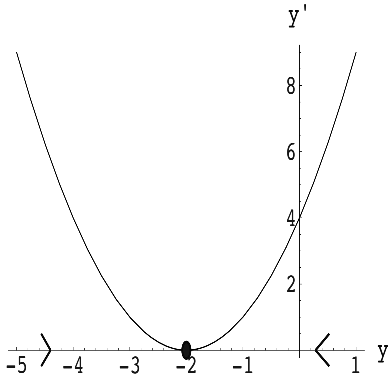
(i) $y - y'$-plane; see graph

(ii) By the phase line diagram, $y = -2$ is a half-stable equilibrium point.

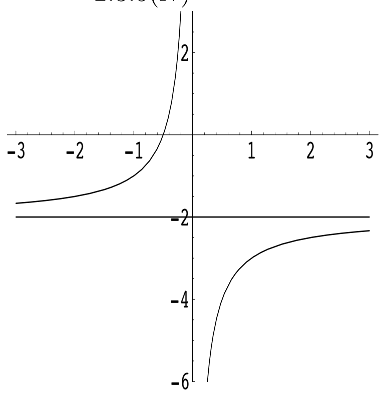
(iii) $y > -2$, $y \to \infty$ as $x \to \infty$
$y < -2$, $y \to -2$ as $x \to \infty$

(iv) xy-plane; see graph

2.3.5(i)



2.3.5(iv)



6. $y' = -y^2$, $-y^2 = 0$

Root	$y = 0$
Multiplicity	2

End behavior: $-y^2$

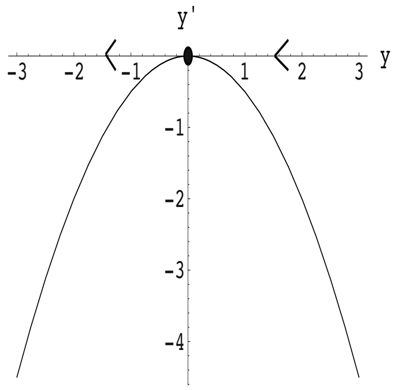
(i) $y - y'$-plane; see graph

(ii) By the phase line diagram, $y = 0$ is a half-stable equilibrium point.

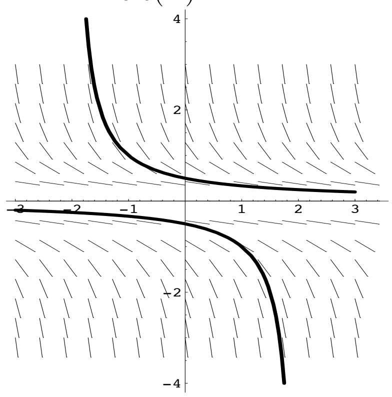
(iii) For $y > 0$, $y \to 0$ as $x \to \infty$.
For $y < 0$, $y \to -\infty$ as $x \to \infty$.

(iv) xy-plane; see graph

2.3.6(i)



2.3.6(iv)



7. $y' = y^2(2 - y)$

Equilibrium	Multiplicity
$y = 0$	2
$y = 2$	1

Highest power and coefficient: $y^2(-y) = -y^3$

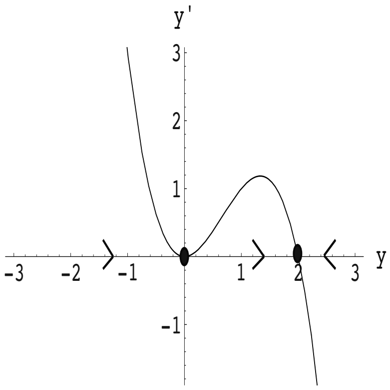
(i) $y - y'$-plane; see graph

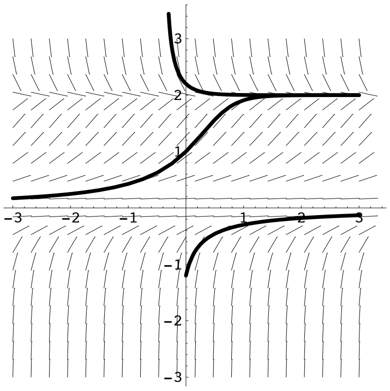
(ii) $y = 0$ is a half-stable point; $y = 2$ is a stable point.

(iii) For $y \in (-\infty, 0)$, $y \to 0$ as $x \to \infty$.
For $y \in (0, \infty)$, $y \to 2$ as $x \to \infty$

(iv) xy-plane; see graph

2.3.5(i) 2.3.5(iv)





8. $y' = (y-1)^2(y-2)^3(1+y)$

Equilibrium	1	2	-1
Multiplicity	2	3	1

Highest power and coefficient: $(y)^2(y)^3(+y) = +y^6$

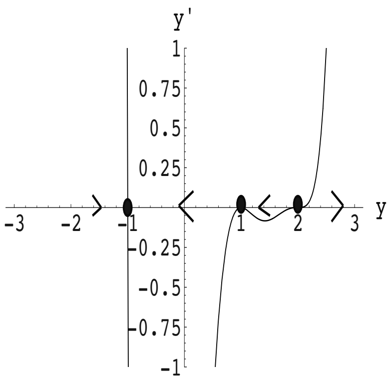
(i) $y - y'$-plane; see graph

(ii) $y = -1$ is stable; $y = 1$ is half-stable; $y = 2$ is unstable.

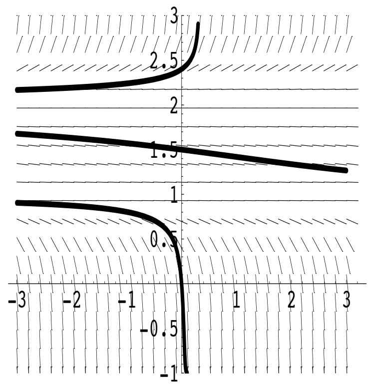
(iii) If $y_0 > 2$, then $y \to \infty$ as $x \to \infty$.
If $1 < y_0 < 2$, then $y \to 1$ as $x \to \infty$.
If $-1 < y_0 < 1$, then $y \to -1$ as $x \to \infty$.
If $y_0 < -1$, then $y \to -1$ as $x \to \infty$.

(iv) xy-plane; see graph

2.3.8(i)



2.3.8(iv)



9. $y' = (y-2)^3(y^2-9) = (y-2)^3(y+3)(y-3)$

Equilibrium	2	-3	3
Multiplicity	3	1	1

Highest power coefficient: $(y)^3(y^2) = +y^5$

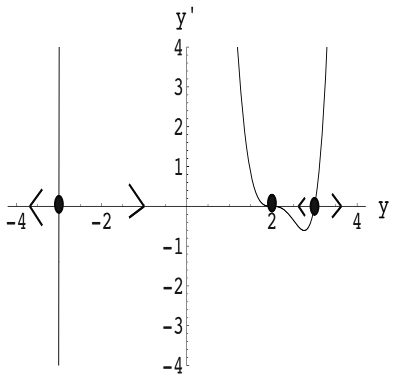
(i) $y - y'$-plane; see graph

(ii) $y = -3, 3$ unstable, $y = 2$ stable

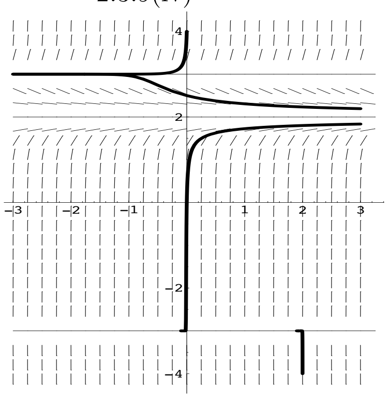
(iii) For $y_0 \in (-\infty, -3)$, $y \to -\infty$ as $x \to \infty$
For $y_0 \in (-3, 3)$, $y \to 2$ as $x \to \infty$
For $y_0 \in (3, \infty)$, $y \to \infty$ as $x \to \infty$

(iv) xy-plane; see graph

2.3.9(i)



2.3.9(iv)



10. $y' = \cos y + 1$, $-2\pi < y < 2\pi$. $\cos y + 1 = 0$, $\cos y = -1$.

Roots	$y = -\pi$	$y = \pi$
Multiplicity	1	1

End behavior: cosine wave

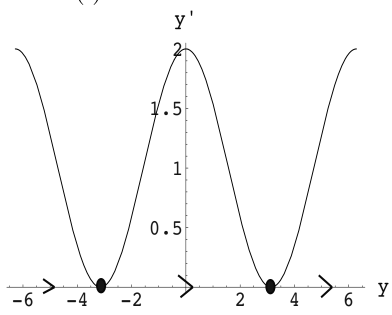
(i) $y - y'$-plane; see graph

(ii) By the phase line diagram, $y = -\pi$ is a half-stable equilibrium point. $y = \pi$ is a half-stable equilibrium point.

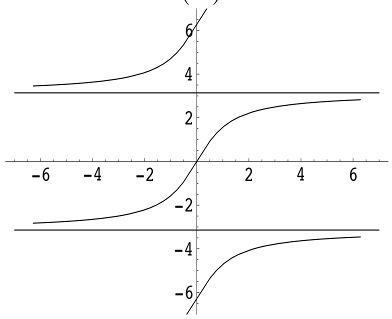
(iii) For $y > \pi$, $y \to \infty$ as $x \to \infty$.
For $-\pi < y < \pi$, $y \to \pi$ as $x \to \infty$.
For $y < -\pi$, $y \to -\pi$ as $x \to \infty$.

(iv) xy-plane;see graph

2.3.10(i)



2.3.10(iv)



11. $y' = \sin y$, $-2\pi < y < 2\pi$
$\sin y = 0$, $-2\pi < y < 2\pi$

Roots	$y = -\pi$	$y = 0$	$y = \pi$
Multiplicity	1	1	1

End behavior: trigonometric sine wave

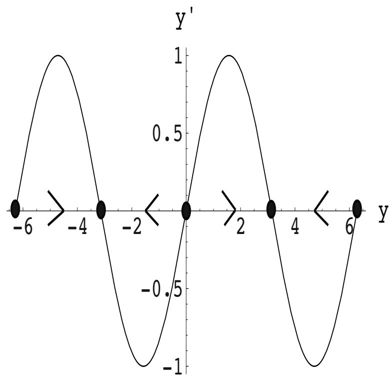
(i) $y - y'$-plane; see graph

(ii) By the phase line diagram, $y = -\pi$ is a stable equilibrium point.
$y = 0$ is an unstable equilibrium point.
$y = \pi$ is a stable equilibrium point.

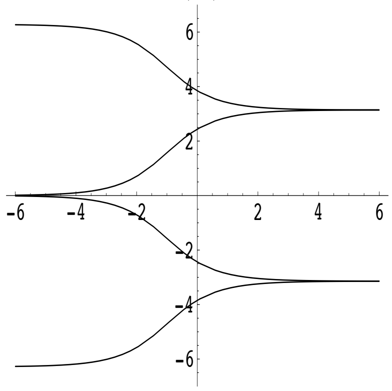
(iii) For $y > \pi$, $y \to \pi$ as $x \to \infty$
For $0 < y < \pi$, $y \to \pi$ as $x \to \infty$
For $-\pi < y < 0$, $y \to -\pi$ as $x \to \infty$
For $y < -\pi$, $y \to -\pi$ as $x \to \infty$

(iv) xy-plane; see graph

2.3.11(i)



2.3.11(iv)



12. $v' = g - \frac{k}{m}v$. Equilibrium satisfies $g - (k/m)v = 0$, so $v = \dfrac{gm}{k}$

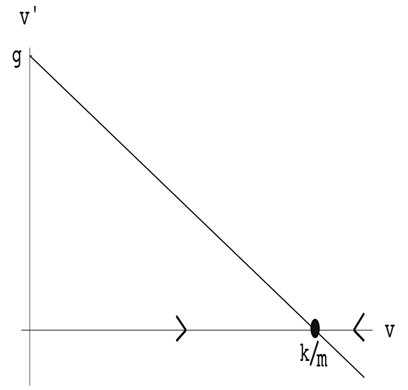
 (i) $v - v'$-plane; see graph

 (ii) $v = \dfrac{gm}{k}$ is a stable equilibrium point.

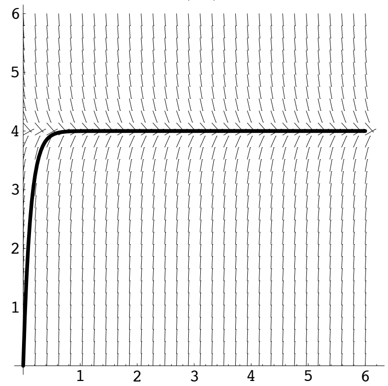
 (iii) For $v \in [0, \infty)$, $v \to \frac{gm}{k}$ as $t \to \infty$

 (iv) Figures will vary. See graph with $g = 32$, $m = .25$, $k = 2$ as in example 2 in 1.3.

2.3.12(i)



2.3.12(iv)



13. $v' = g - \frac{k}{m}v^2$. Equilibrium satisfies $g - (k/m)v^2 = 0$, so $v^2 = gm/k$ yields $v = \pm\sqrt{gm/k}$. Note that this is a free-fall problem where $v > 0$ in the downward direction. We thus ignore $v < 0$.

 (i) Graphs will vary

 (ii) $\sqrt{\dfrac{gm}{k}}$ is a stable equilibrium point

 (iii) For $v \in [0, \infty)$, $v \to +\sqrt{gm/k}$ as $t \to \infty$

 (iv) Graphs will vary

14. $x' = (2 - x)^3(x^2 + 4)$, $(2 - x)^3(x^2 + 4) = 0$

Roots	2	$\pm 2i$
Multiplicity	3	ignore

End behavior: $(-x)^3(x^2) = -x^5$

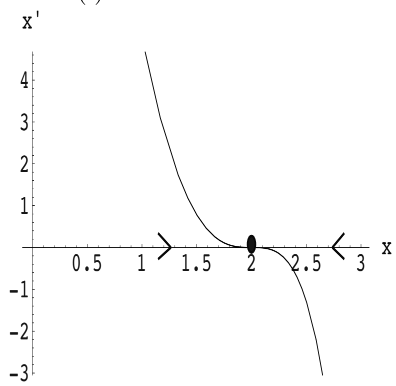
 (i) $x - x'$-plane; see graph

 (ii) By the phase line diagram, $x = 2$ is a stable equilibrium point.

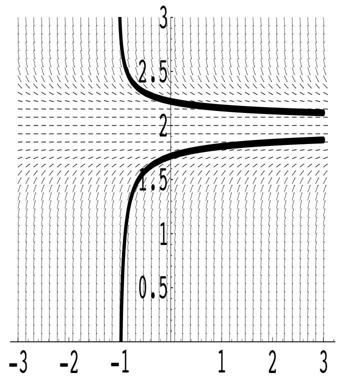
 (iii) For $x > 2$, $\to 2$ as $t \to \infty$.
 For $x < -2$, $\to 2$ as $t \to \infty$.

(iv) tx-plane; see graph

2.3.14(i)



2.3.14(iv)



15. $y' = (2-y)^3(y^2+4)^2$

Root	2
Multiplicity	3

Highest power and coefficient: $(-y)^3(y^2)^2 = -y^7$

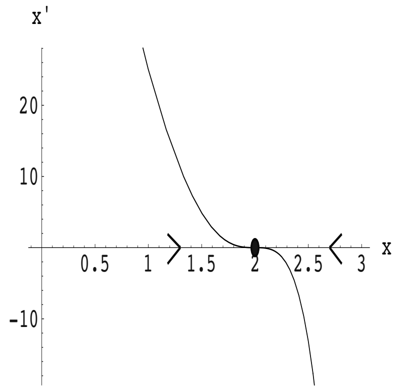
(i) $x - x'$-plane; see graph

(ii) By the phase line diagram, $y = 2$ is a stable equilibrium point.

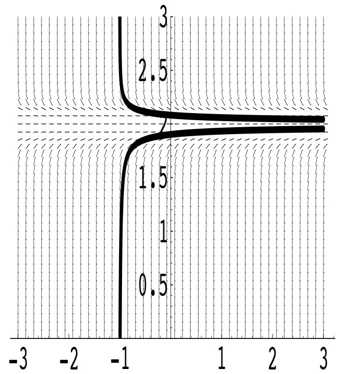
(iii) If $y_0 < 2$, then $y \to 2$ as $x \to \infty$.
If $y_0 > 2$, then $y \to 2$ as $x \to \infty$.

(iv) tx-plane; see graph

2.3.15(i)



2.3.15(iv)



16. $y' = -y^2(4-y)(9-y^2)$

Roots	0	4	3	−3
Multiplicity	2	1	1	1

Highest power and coefficient: $-y^2(-y)(-y^2) = -y^5$

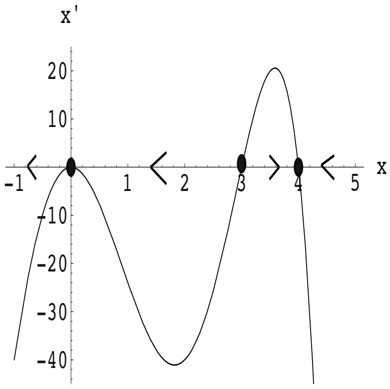
(i) $x - x'$-plane; see graph

(ii) $y = -3, 4$ are stable; $y = 0$ is half-stable; $y = 3$ is unstable.

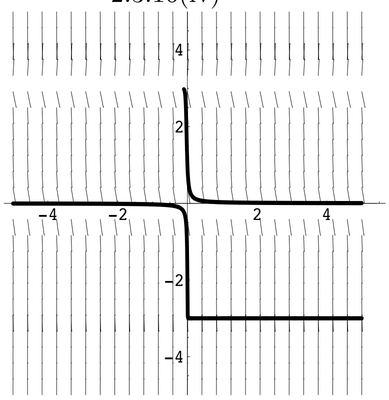
(iii) If $y_0 < -3$, $y \to -3$ as $x \to \infty$.
If $-3 < y_0 < 0$, $y \to -3$ as $x \to \infty$.
If $0 < x < 3$, $y \to 0$ as $x \to \infty$.
If $3 < x < 4$, $y \to 4$ as $x \to \infty$.
If $x > 4$, $y \to 4$ as $x \to \infty$.

(iv) tx-plane; see graph

2.3.16(i)



2.3.16(iv)



17. $x' = x^5(1-x)(1-x^3)$, $x^5(1-x)(1-x^3) = 0$

Roots	0	1	two normal
Multiplicity	5	2	ignore

End behavior: $x^5(-x)(-x^3) = x^9$

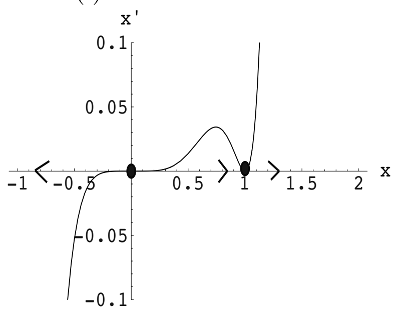
(i) $x - x'$-plane; see graph

(ii) By the phase line diagram, $= 1$ is a half-stable equilibrium point; $= 0$ is an unstable equilibrium point.

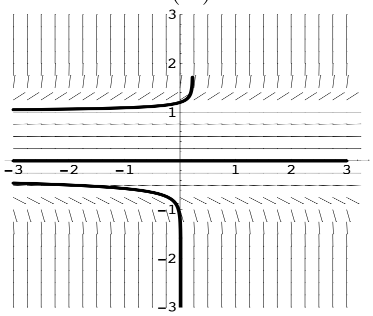
(iii) For $x > 1$, $\to \infty$ as $t \to \infty$.
For $0 < x < 1$, $\to 1$ as $t \to \infty$.
For $x < 0$, $\to -\infty$ as $t \to \infty$.

(iv) tx-plane; see graph

2.3.17(i)



2.3.17(iv)



18. $x' = x(x-3)(1+x^3)(1-x^2)^2$, $x(x-3)(1+x^3)(1-x^2)^2 = 0$

Roots	0	3	-1	two non-real	± 1
Multiplicity	1	1	1	ignore	2

End behavior: $x(x)(x^3)(-x^2)^2 = x^9$

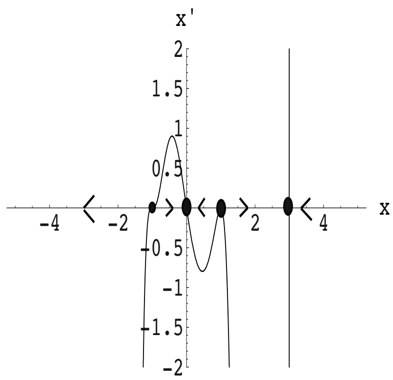
(i) $x - x'$-plane; see graph

(ii) By the phase line diagram, $x = 3$ is an unstable equilibrium point; $x = 1$ is a half-stable equilibrium point; $x = 0$ is a stable equilibrium point; $x = -1$ is an unstable equilibrium point.

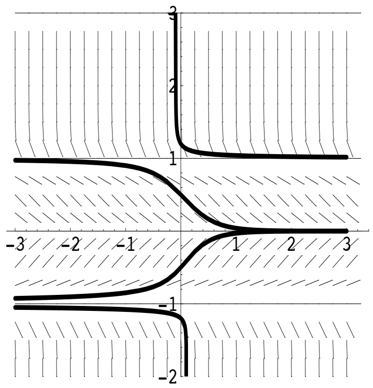
(iii) If $x > 3$, $\to \infty$ as $t \to \infty$.
If $1 < x < 3$, $\to 1$ as $t \to \infty$.
If $0 < x < 1$, $\to 0$ as $t \to \infty$.
If $-1 < x < 0$, $\to 0$ as $t \to \infty$.
If $x < -1$, $\to -\infty$ as $t \to \infty$.

(iv) tx-plane; see graph

2.3.18(i)



2.3.18(iv)



19. $x' = x^2(1-2x)^3(x^2-1)$, $x^2(1-2x)^3(x^2-1) = 0$

Roots	0	$\frac{1}{2}$	± 1
Multiplicity	2	3	1

End behavior: $x^2(-x)^3(x^2) = -x^7$

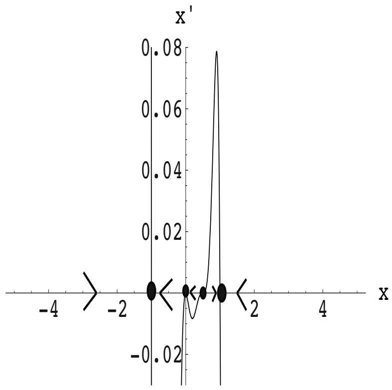
(i) $x - x'$-plane; see graph

(ii) By the phase line diagram, $x = 1$ is a stable equilibrium point; $x = \frac{1}{2}$ is an unstable equilibrium point; $x = -1$ is a stable equilibrium point.

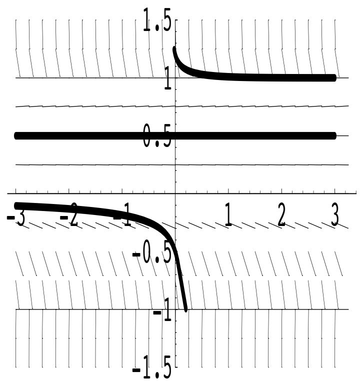
(iii) If $x > 1$, $\to 1$ as $t \to \infty$.
If $\frac{1}{2} < x < 1$, $\to 1$ as $t \to \infty$.
If $-1 < x < \frac{1}{2}$, $\to -1$ as $t \to \infty$.
If $x < -1$, $\to -1$ as $t \to \infty$.

(iv) tx-plane; see graph

2.3.19(i)



2.3.19(iv)



20. $x' = x^3(x^2+5)(x-4)^2(x+5)$

Roots	0	4	-5
Multiplicity	3	2	1

End behavior: $x^3x^2x^2x = x^8$

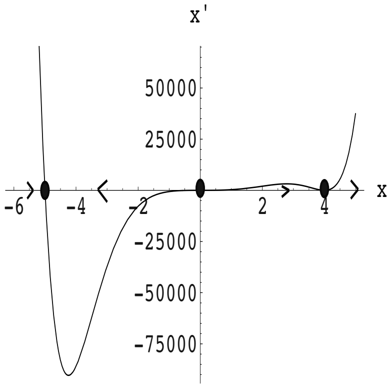
(i) $x - x'$-plane; see graph

(ii) By the phase line diagram, $x = 1$ is a stable equilibrium point; $x = \frac{1}{2}$ is an unstable equilibrium point; $x = -1$ is a stable equilibrium point.

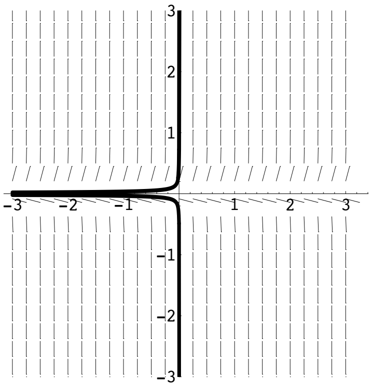
(iii) If $x > 1$, $\to 1$ as $t \to \infty$.
If $\frac{1}{2} < x < 1$, $\to 1$ as $t \to \infty$.
If $-1 < x < \frac{1}{2}$, $\to -1$ as $t \to \infty$.
If $x < -1$, $\to -1$ as $t \to \infty$.

(iv) tx-plane; see graph

2.3.20(i)



2.3.20(iv)



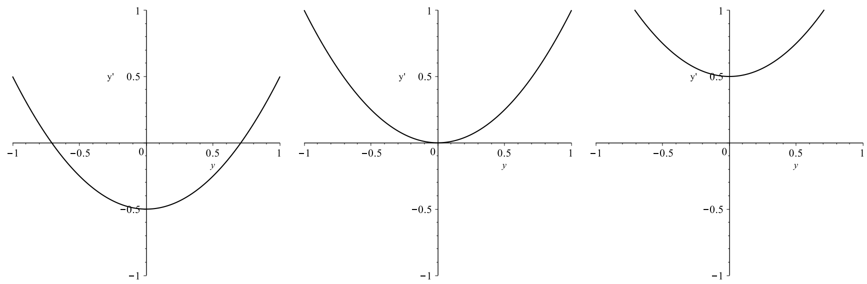
21. $y^* = \pm 1,\ f'(y) = -2y, f'(-1) = 2 \Rightarrow$ Unstable$f'(1) = -2 \Rightarrow$ Stable

22. $y^* = \pm 1,\ f'(y) = -2y, f'(-1) = 2 \Rightarrow$ Unstable$f'(1) = -2 \Rightarrow$ Stable

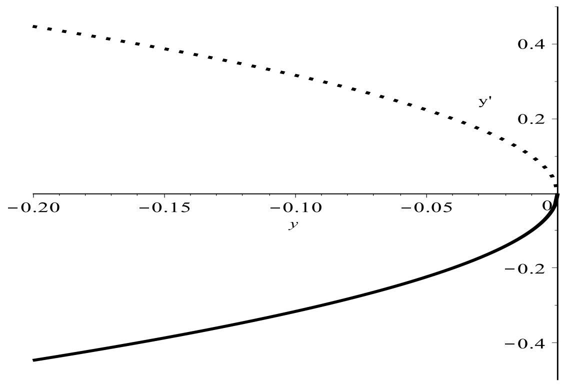
23. $y^* = -1,\ f'(y) = 3y^2, f'(-1) = 3 \Rightarrow$ Unstable

24. $y^* = 0,\ f'(y) = -3y^2, f'(0) = 0 \Rightarrow$ Inconclusive. However, from the phase line diagram, $y^* = 0$ is stable.

25. (a) $y' = r + y^2$

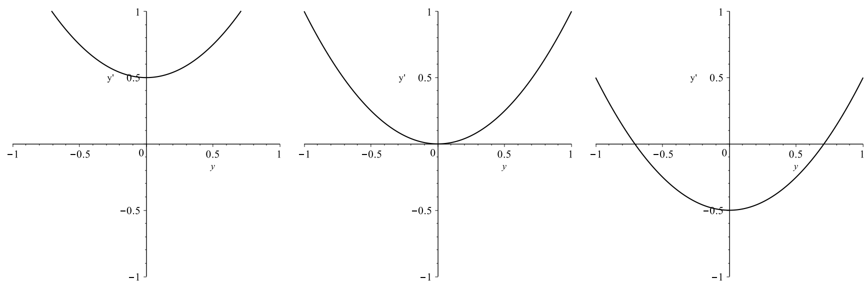


25: Phase line for $y' = r + y^2:\ r < 0,\ r = 0,\ r > 0$

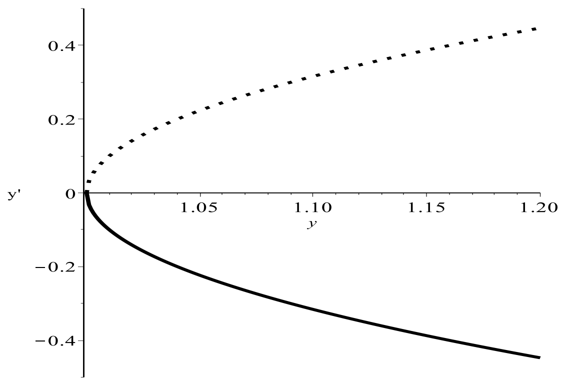


25: Saddlenode bifurcation for $y' = r + y^2$

26. (a) $y' = 1 - r + y^2$

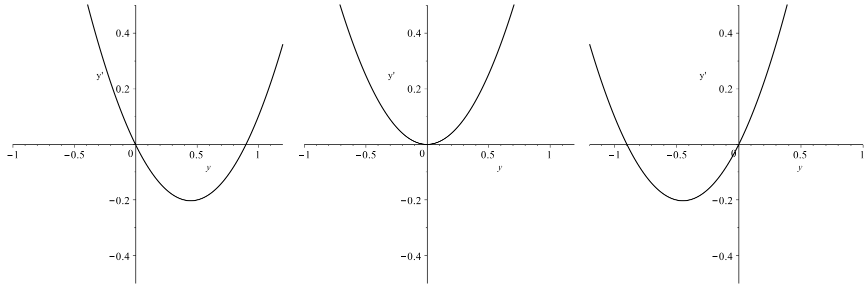


26: Phase line for $y' = 1 - r + y^2$: $r < 1$, $r = 1$, $r > 1$

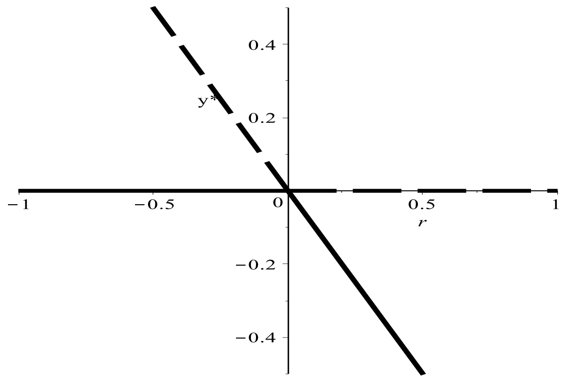


26: Saddlenode bifurcation for $y' = 1 - r + y^2$

27. (a) $y' = ry + y^2$

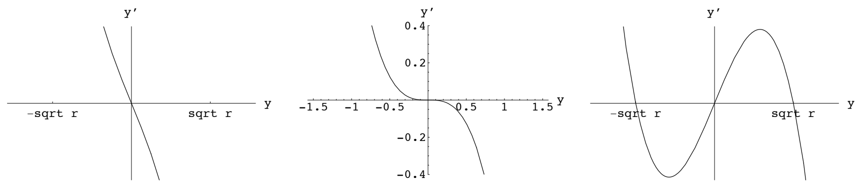


27: Phase line for $y' = ry + y^2$: $r < 0,\ r = 0,\ r > 0$

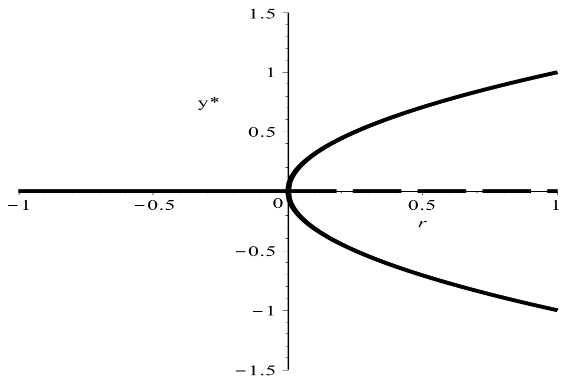


27: Transcritical bifurcation for $y' = ry + y^2$

28. (a) $y' = ry - y^3$

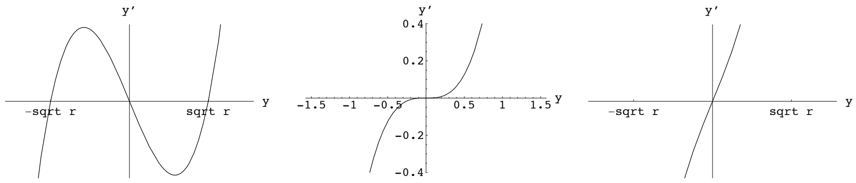


28a: Phase line for $y' = ry - y^3$: $r < 0,\ r = 0,\ r > 0$

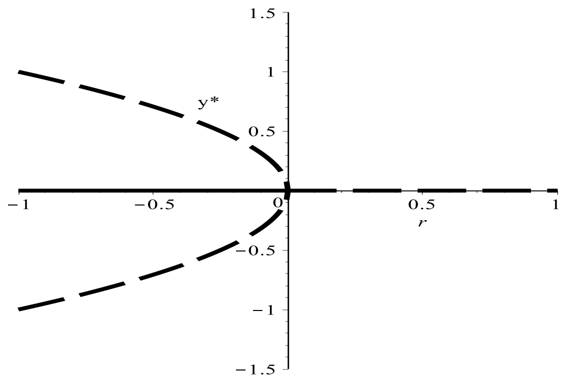


28a: Pitchfork bifurcation for $y' = ry - y^3$

(b) $y' = ry + y^3$



28b: Phaseline for $y' = ry + y^3:\ r < 0,\ r = -,\ r > 0$



28b: Pitchfork bifurcation for $y' = ry + y^3$

29. If we Taylor expand the function about y^* and keep the lowest order non-zero term, we see that we have $y' = f^{(3)}(y^*)y^3$ as the approximate solution near the equilibrium point. Phase line analysis then shows the equilibrium point is stable.

2.4 Modeling in Population Biology

1. (a) $x = 0$ is half-stable, $x = 1$ is stable. For logistic equation, $x = 0$ is unstable and $x = k$ is stable. Yes, the are different.
   (b) For small x, logistic model growth is larger.
2. $x = 0$ - Stable, $x = 1$ - Unstable, $x = 6$ - Stable
3. $x = 0$ - Stable, $x = 2$ - Unstable, $x = 10$ - Stable
4. (a) $x = 0$ - Half-stable, $x = 2$ - Unstable, $x = 7$ - Stable
   (b) For small x, Allee effect model has larger growth rate.
5. (a) $x = 0$ - Half-stable, $x = 1$ - Unstable, $x = 4$ - Stable

(b) For small x, Allee effect model has larger growth rate.

6. (a) Exponential growth - Unlimited growth rate. No limitations placed on organisms.

   (b) Logistic model - Growth rate dependent on factors such as population amount or food availability.

   (c) Allee effect - Growth rate dependent on factors such as population amount or food availability as well as a sufficient population to sustain itself.

7. (a) $x = 0$ - Unstable, $x = a$ - Stable, $x = 5$ - Unstable

   (b) For $x_0 > 5$, bacteria grows uninhibited.

   (c) The parameter a could represent the strength of the immune system or ability of the body to fight off the given bacteria. A healthy person would likely have an a-value that is closer to 0 than to 5 because the lower value of a represents a lower value of the bacteria (that is stable).

8. (a) For $a > 0$,
   $x = 0$ - Unstable
   $x = 5 - \sqrt{a}$ - Stable
   $x = 5 + \sqrt{a}$ - Unstable
   $x = 10$ - Stable

   (b) For $a = 0$,
   $x = 0$ - Unstable
   $x = 5$ - Half-stable
   $x = 10$ - Stable

   (c) For $a < 0$,
   $x = 0$ - Unstable
   $x = 10$ - Stable

   (d) Saddle-node

   (e) Bacteria grows unchecked to a level of 10. No, the bacteria population eventually reaches and levels off at 10, which above the fatal level.

   (f) The parameter a could again represent the strength of the immune system or ability of the body to fight off the given bacteria.

9. $x' = x(1 - x)(x - 6)(x - 10)$

10. $x' = x(2x-1)(x-1)^2(x-2)(x-8)$ or $x' = x(2x-1)^2(x-1)(x-2)(x-8)$

11. (a) $x' = x^2(2 - x)^2(x - 4)$

    (b) $x = 0$ - Half-stable, $x = 2$ - Half-stable, $x = 4$ - Unstable

12. (a) $x' = rx(x-a)(x-1) \quad r, a > 0$
    (b) i. $0 < a < 1$, $x = 0$ - Unstable, $x = a$ - Stable, $x = 1$ - Unstable
        ii. $a = 1$, $x = 0$ - Unstable, $x = 1$ - Half-stable
        iii. $a > 1$, $x = 0$ - Unstable, $x = 1$ - Stable, $x = a$ - Unstable
    (c) When $a < 1$, the bacteria goes to the stable level a if it starts with a level less than 1 and grows without bound otherwise. When $a = 1$, the bacteria goes to the half-stable level $a = 1$ if it starts with a level less than 1 and grows without bound otherwise. When $a > 1$, the bacteria goes to the stable level of 1 if it starts with a level less than a and grows without bound otherwise.

13. $\frac{dN}{dt} = rN(1 - \frac{N}{K})$
    Let $x = \frac{N}{A}, \tau = \frac{t}{T} \to d\tau = \frac{1}{T}dt$
    $\frac{dN}{dt} = \frac{d(xA)}{d\tau}\frac{d\tau}{dt} = r(xA)(1 - x\frac{A}{K})$
    $Ax'\frac{1}{T} = rAx(1 - x\frac{A}{K})$.
    Let A=K and T = $\frac{1}{r} \Rightarrow x = \frac{N}{K}, \tau = rt$, then
    $x' = rTx(1 - x\frac{A}{K}) = x(1-x)$

14. Let $x = \frac{N}{A}, \tau = \frac{t}{T}$
    $Ax'\frac{1}{T} = R(xA)(1 - x\frac{A}{K}) - \frac{HxA}{B+xA}$.
    Then $x' = RTx(1 - x\frac{A}{K}) - \frac{THx}{B+xA}$
    Let $T = \frac{1}{R}, A = K \Rightarrow x' = x(1-x) - \frac{\frac{H}{R}x}{A(\frac{B}{A}+x)} \Rightarrow$ Let $h = \frac{H}{AR}, b = \frac{B}{A}$ so that $x = \frac{N}{K}, \tau = Rt$

15. $\frac{dg}{dt} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$
    $x = \frac{g}{A}, \tau = \frac{t}{T}$
    $Ax'\frac{1}{T} = k_1 s_0 - k_2 Ax + \frac{k_3(xA)^2}{k_4^2 + (xA)^2}$
    $x' = \frac{T}{A}k_1 s_0 - Tk_2 x + \frac{Tk_3 Ax^2}{k_4^2 + A^2x^2} = \frac{T}{A}k_1 S_0 - Tk_2 x + \frac{T\frac{k_3}{A}x^2}{\left(\dfrac{k_4^2}{A^2} + x^2\right)}$
    $T\frac{k_3}{A} = 1 \Rightarrow T = \frac{A}{k_3} = \frac{k_4}{k_3}$
    Let $A = k_4, T = \frac{k_4}{k_3}, r = Tk_2 = \frac{k_4 k_2}{k_3}, s = \frac{T}{A}k_1 S_0 = \frac{k_1}{k_3}S_0$

16. (a) Small N is approximately constant harvesting; large N is approximately constant yield
    (b) $x^* = 0, \dfrac{(1-b) + \sqrt{(b+1)^2 - 4h}}{2}$; $x^* = 0$ is always a biologically relevant equilibrium solution and will be stable if it is the only one; the other equilibrium will be stable when it is biologically relevant (i.e., when the radicand is positive)

2.5 Numerical Approximation: Euler and Runge-Kutta Methods

1. $dy/dx = x^3, \quad y(1) = 1$; explicit solution: $y = \frac{1}{4}(x^4 + 3)$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
1.0	1	1	1
1.1	1.1	1.116	1.1160
1.2	1.2331	1.2684	1.2684

2. $dy/dx = -y^2, \quad y(0) = 1$; explicit solution: $y = \frac{1}{x+1}$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0	1	1	1
0.1	0.9	0.909	0.909
0.2	0.819	0.8333	0.8333

3. $dy/dx = x^4 y, \quad y(1) = 1$; explicit solution: $y = e^{(x^5-1)/5}$

x_i	Euler y_i	Explicit $y(x_i)$	
1.0	1	1	1
1.1	1.1	1.1299	1.1299
1.2	1.2611	1.3467	1.3467

4. $\dfrac{dy}{dx} = -y^2 \cos x,\ h = 0.1,\ y(0) = 1$; explicit solution $y = \dfrac{1}{1 + \sin x}$.

$$
\begin{aligned}
& x_0 = 0 \\
& x_1 = 0 + (0.1)(1) = 0.1 \\
& x_2 = 0 + (0.1)(2) = 0.2 \\
\text{Euler}: \ & y_0 = 1 \\
& y_1 = 1 + (0.1)(-(1)^2 \cos(0)) = .9 \\
& y_2 = .9 + (0.1)(-(.9)^2 \cos(0.1)) = .8194
\end{aligned}
$$

Runge-Kutta:
y_1:

$$\begin{aligned}
k_1 &= f(0,1) = -(1)^2\cos 0 = -1\\
k_2 &= f\left(0+.05, 1+\frac{(.1)(-1)}{2}\right) = -(.95)^2\cos(.05) = -.901372\\
k_3 &= f\left(0+.05, 1+\frac{(.1)}{2}(-.902499)\right) = -.9107543\\
k_4 &= f(0+.1, 1+(.1)(-.911786)) = -.8220166\\
y_1 &= 1+\frac{(.1)}{6}(-1+2(-.901372)+2(-.910754)+(-.8220166))\\
&= .9092288
\end{aligned}$$

y_2:

$$\begin{aligned}
k_1 &= -.822567\\
k_2 &= -.745136\\
k_3 &= -.751797\\
k_4 &= -.68177\\
\Rightarrow y_2 &= .8342587
\end{aligned}$$

Explicit:

$$\begin{aligned}
y(0) &= \frac{1}{1+\sin 0} = 1\\
y(0.1) &= \frac{1}{1+\sin(0.1)} = .909228\\
y(0.2) &= \frac{1}{1+\sin(0.2)} = .834258
\end{aligned}$$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0.0	1	1	1
0.1	.9	.909228	.909228
0.2	.8194	.834258	.834258

5. $y' = \dfrac{\sin x}{y^3}$, $y(\pi) = 2$, $h = .1$.

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
π	2	2	2
$\pi + .1$	2	1.999375	1.99937
$\pi + .2$	1.99875	1.9975	1.99750

6. $dy/dx = ye^{-x}$, $y(0) = 1$; explicit solution: $y = \exp(1 - e^{-x})$

x_i	Euler y_i	Explicit $y(x_i)$	
1.0	1	1	
1.1	1.1	1.0998	1.0998
1.2	1.1995	1.1987	1.1987

7. $dy/dx = e^{-y}$, $y(0) = 2$; explicit solution: $y = \ln(x + e^2)$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0.0	2	2	2
0.1	2.0135	2.0134	2.0134
0.2	2.0269	2.0267	2.0267
0.3	2.0401	2.0398	2.0398
0.4	2.0531	2.0527	2.0527
0.5	2.0659	2.0655	2.0655
0.6	2.0786	2.0781	2.0781
0.7	2.0911	2.0905	2.0905
0.8	2.1034	2.1028	2.1028

8. $dy/dx = -xy^2$, $y(0) = 1$; explicit solution: $y = \dfrac{2}{2 + x^2}$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0.0	1	1	1
0.1	1	0.99502	0.99502
0.2	0.99	0.980392	0.980392
0.3	0.970398	0.95694	0.95694
0.4	0.942148	0.92593	0.92593
0.5	0.90664	.88889	.88889
0.6	0.86554	0.847457	0.847458
0.7	0.82059	0.803213	0.803213
0.8	.77346	0.75758	0.75758

9. $dy/dx = y + \cos x$, $y(0) = 0$; explicit solution: $y = \frac{1}{2}(\sin x - \cos x + e^x)$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0.0	0	0	0
0.1	0.1	.1050	0.1050
0.2	0.2095	.2200	0.2200
0.3	0.3285	.3450	0.3450
0.4	0.4568	0.4801	0.4801
0.5	0.5946	0.6253	0.6253
0.6	0.7418	.7807	0.7807
0.7	0.8986	0.9466	0.9466
0.8	1.0649	1.1231	1.1231

10. $dy/dx = y + \sin x$, $y(0) = 2$; explicit solution: $y = \frac{-1}{2}(\cos x + \sin x - 5e^x)$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0.0	2	2	2
0.1	2.2	2.2155	2.2155
0.2	2.4300	2.4641	2.4641
0.3	2.6929	2.7492	2.7492
0.4	2.9917	3.0743	3.0743
0.5	3.3298	3.4433	3.4433
0.6	3.7107	3.8603	3.8603
0.7	4.1383	4.3299	4.3299
0.8	4.6165	4.8568	4.8568

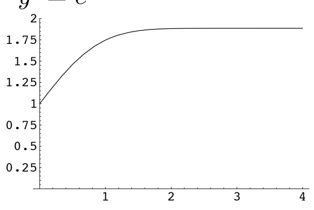
11. $dy/dx = x + y$, $y(0) = 0$; explicit solution: $y = -x - 1 + e^x$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0.0	0	0	0
0.1	0	0.0052	0.0052
0.2	.01	0.0214	0.0214
0.3	.031	0.0499	0.0499
0.4	.0641	0.0918	0.0918
0.5	.1105	0.1487	0.1487
0.6	.1716	0.2221	0.2221
0.7	.2487	0.3138	0.3138
0.8	.3436	0.4255	0.4255

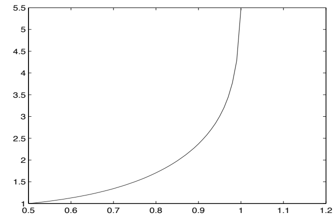
12. $dy/dx = (x+1)(y^2+1)$, $y(0) = 0$; explicit solution: $y = \tan(\frac{1}{2}x^2 + x)$

x_i	Euler y_i	RK4 y_i	Explicit $y(x_i)$
0.0	0	0	0
0.1	0.1	0.1054	0.1054
0.2	0.2111	0.2236	0.2236
0.3	0.3364	0.3594	0.3594
0.4	0.4812	0.5206	0.5206
0.5	0.6536	0.7215	0.7215
0.6	0.8677	0.9893	0.9893
0.7	1.1481	1.3837	1.3837
0.8	1.5422	2.0660	2.0660

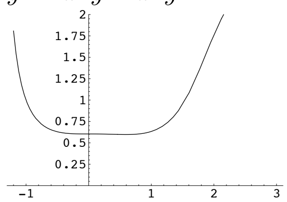
13. $y' = e^{-x^2}$



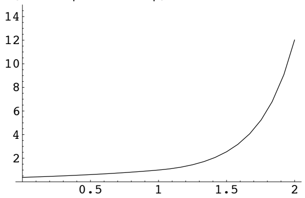
14. $y' = x^3 e^y + 3x^2 \sin y$



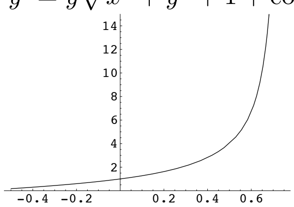
15. $y' = x^3 y - x^2 y^2$



16. $y' = |1 - x^2| y + x^3$



17. $y' = y\sqrt{x^2 + y^2 + 1} + \cos(xy)$



18.

h	(a.) $x(1)$	(b.) x(1)
1	100.75	101.249
0.5	100.75	101.249
0.25	100.75	101.249
0.1	100.75	101.249

2.6 An Introduction to Autonomous Second-Order Equations

1. $y = ce^{kx},\ y' = kce^{kx},\ y'' = k^2ce^{kx} \Rightarrow$
   $yy'' = ce^{kx} * k^2ce^{kx} = c^2k^2e^{2kx} = (y')^2$

2. If $v = \theta'$, then $v = \dfrac{C + g\cos\theta}{m}$ and
   $$\frac{\tanh^{-1}\left(\frac{(C-g)\tan(\theta/2)}{\sqrt{g^2-C^2}}\right)}{\sqrt{g^2 - C^2}} = \frac{t}{m} + K$$

3. Substitution gives $v\frac{dv}{dx} = -g\frac{R^2}{x^2}$. Separating, solving, and applying the IC gives $v = \pm\sqrt{2gR(\frac{R}{x} - 1) + v_0^2}$. The "+" corresponds to the object going away from center of earth; the object "escapes" if the radicand is always nonnegative. Since $\frac{R}{x} \to 0$ as $x \to \infty \Rightarrow -2gR + v_0^2 \geq 0$.

4. (a) If $u(x) = \dfrac{dy}{dx}$, then we have $u' = \sqrt{1 + u^2}$.

   (b) This is separable, so $\displaystyle\int \frac{du}{\sqrt{1 + u^2}} = \sinh^{-1}(u) = x - x_0$
   $\Rightarrow u = \dfrac{e^{(x-x0)} - e^{-(x-x0)}}{2}$ for some constants $c,\ d$.

   (d) Then, if $u = \dfrac{dy}{dx} = \dfrac{e^{(x-x0)} - e^{-(x-x0)}}{2} = \sinh(x - x0)$,
   then $y = \cosh(x - x0) + C$.

Chapter 2 Review

1. False. Neither f nor $\frac{\partial f}{\partial y}$ is continuous everywhere.
2. False. It *may* not be unique.
3. False. RK use four function evaluations to calculate the next step.
4. False. Euler's method is not superior.
5. True.
6. True. Phase line analysis will work and gives information on long-term behavior.

7. (i) Solutions exist everywhere; (ii) solutions are unique everywhere.

8. (i) Solutions exist everywhere; (ii) solutions are unique when $xy \neq 1$.

9. (i) Solutions exist everywhere; (ii) solutions are unique everywhere except possibly along $y = 0$.

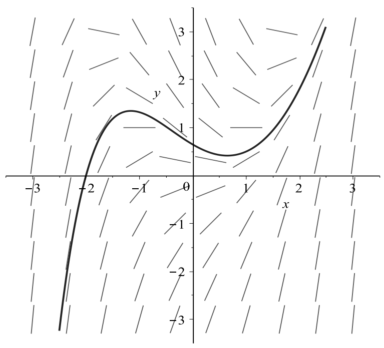
10. (i) Solutions exist everywhere except possibly when $x = \frac{\pi}{2} \pm n\pi$ for $n = 0, 1, 2, \cdots$; (ii) solutions are unique everywhere except possibly when $x = \frac{\pi}{2} \pm n\pi$ for $n = 0, 1, 2, \cdots$.

11. (i) Solutions exist everywhere except possibly when $y = \frac{\pi}{2} \pm n\pi$ for $n = 0, 1, 2, \cdots$; (ii) solutions are unique everywhere except possibly when $y = \frac{\pi}{2} \pm n\pi$ for $n = 0, 1, 2, \cdots$.

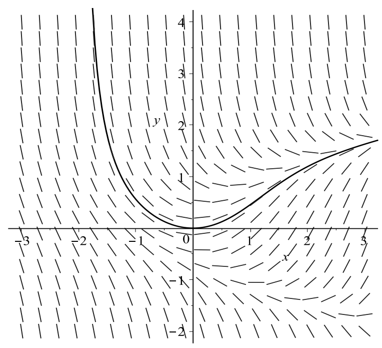
12. (i) Solutions exist everywhere except possibly when $x = \frac{\pi}{2} \pm n\pi$ for $n = 0, 1, 2, \cdots$; (ii) solutions are unique everywhere except possibly when $y = \frac{\pi}{2} \pm n\pi$ for $n = 0, 1, 2, \cdots$.

13. (i) Solutions exist everywhere except possibly along $x = -1$; (ii) solutions are unique except possibly along $x = -1$.

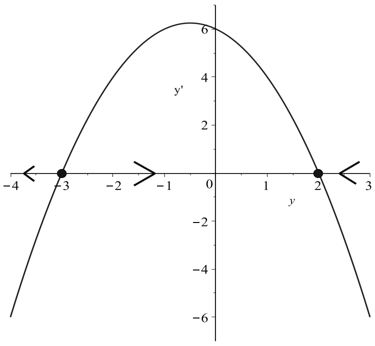
14. $y' = x^2 - y$ that passes through the initial condition $y(-2) = 0$

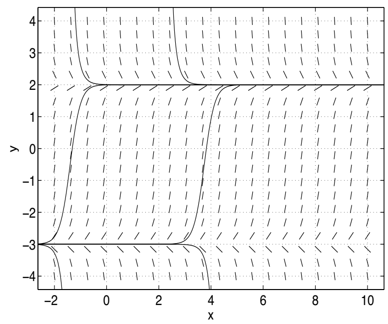


15. $y' = x - y^2$ that passes through the initial condition $y(0) = 0$

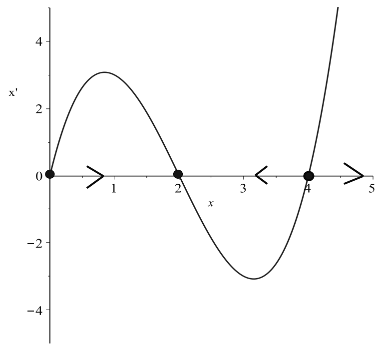


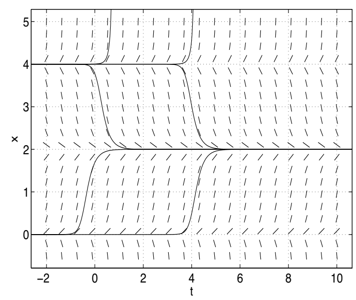
16. (a),(c) $y^* = -3$ is unstable; $y^* = 2$ is stable.
    (b),(d) see graphs
    Chap.2 Review 16(b) 16(d)



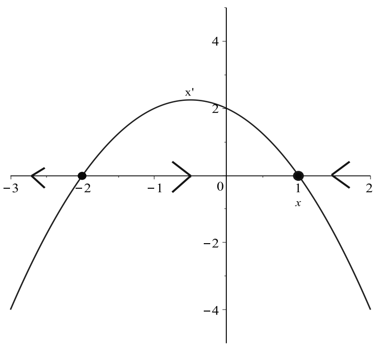


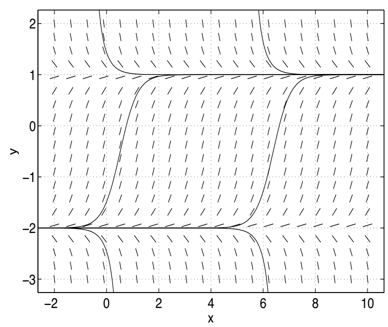
17. (a),(c) $x^* = a$ is stable; $x^* = 0, 4$ are unstable.
    (b),(d) see graphs ($a = 2$)
    Chap.2 Review 17(b) 17(d)





18. (a),(c) $y^* = 1$ is stable; $y^* = -2$ is unstable.
    (b),(d) see graphs
    Chap.2 Review 18(b) 18(d)

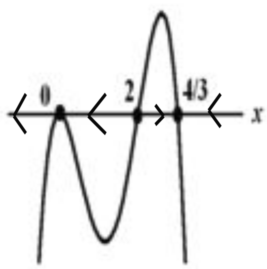




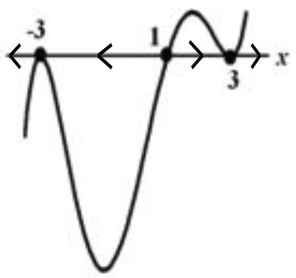
19. Left picture matches (d); right picture matches (b). Check pairs of points such as $(0,1)$, $(1,-1)$, $(1,0)$.

20. Left picture matches (a); right picture matches (d). Check pairs of points such as $(0,1)$, $(\frac{1}{2},1)$.

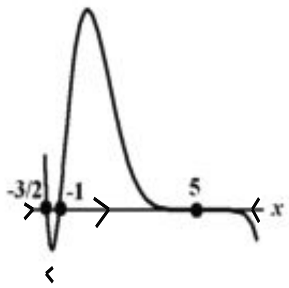
21. $x' = x^2(x-2)(4-3x)$
    0- half stable, 2 - stable, $\frac{4}{3}$ - unstable



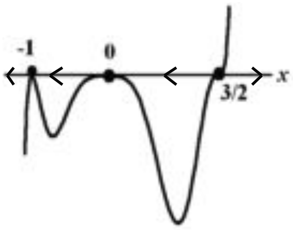
22. $x' = (x^2-9)^2(x-1)$
    -3- half stable, 3 - half stable, 1 - unstable



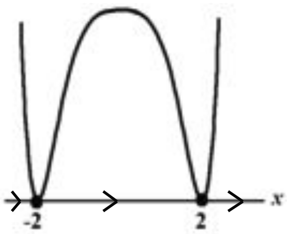
23. $x' = (1+x)(3+2x)(5-x)^7$
    $-\frac{3}{2}$- stable, 5 - stable, -1 - unstable



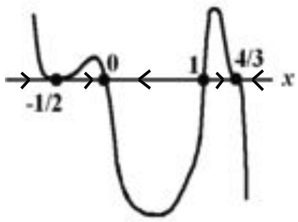
24. $x' = x^4(2x-3)^3(x^2+4)(x^2+2x+1)$
    -1- half stable, 0 - half stable, $\frac{3}{2}$ - unstable



25. $x' = (x^4-16)^2$
    $-\frac{4}{\sqrt{3}}$- unstable, -2 - stable, 2 - unstable, $\frac{4}{\sqrt{3}}$- stable

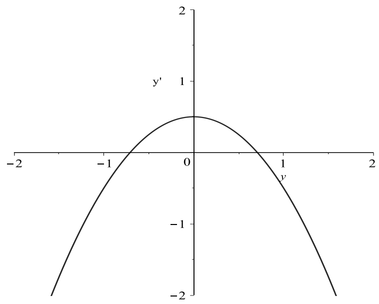


26. $x' = x(4-3x)^3(2x+1)^4(x^3-1)$
    $-.5$- half stable, 0 - stable, $\frac{4}{3}$ - stable, 1 - unstable

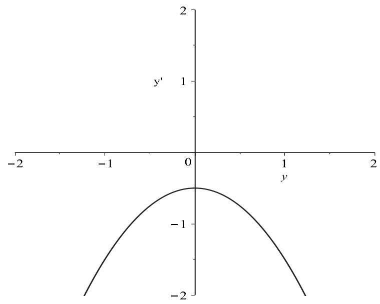


27. (a) $r > 1$, no equilibrium points.
    (b) $r \leq 1$, $-\sqrt{1-r}$ - unstable; $\sqrt{1-r}$ - stable

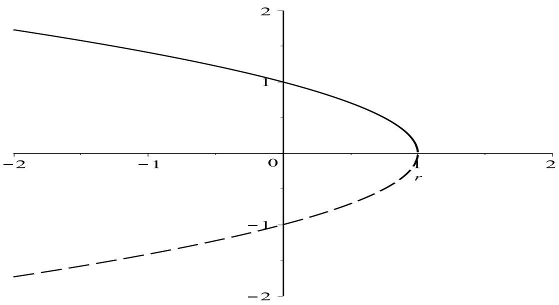
$r < 1$



$r > 1$

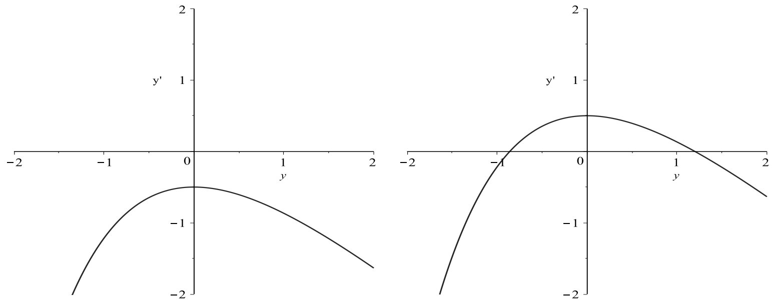


Bifurcation diagram for 27

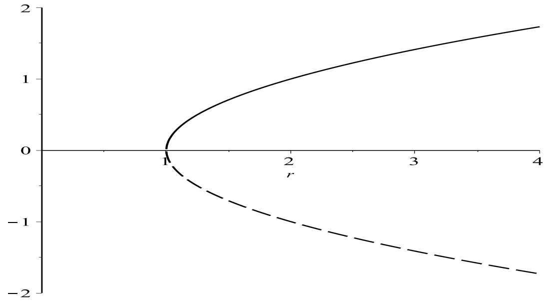


28. (a) $r < 1$, no equilibrium points.
    (b) $r \geq 1$, negative value - unstable; positive value - stable

$r < 1$ $r > 1$

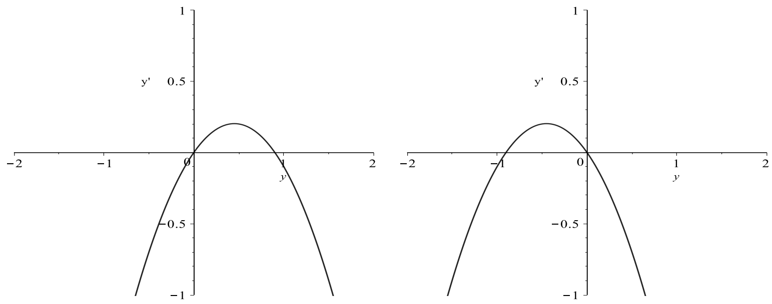


Bifurcation diagram for 28

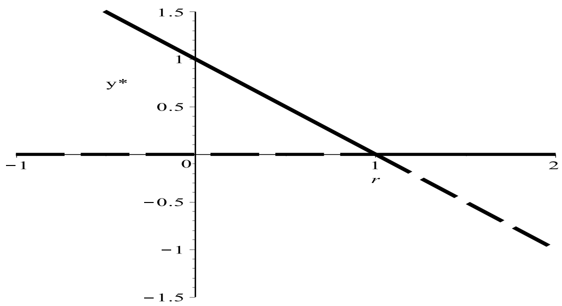


29. (a) $r = 1$, 0 - half stable
    (b) $r > 1$, 0 - unstable, positive value - stable
    (c) $r < 1$, 0 - stable, other value - unstable

$r < 1$ $\qquad$ $r > 1$



Bifurcation diagram for 29



30. $\frac{dN}{dt} = rN(1-\frac{N}{K}) - H$
Let $x = \frac{N}{A}, \tau = \frac{t}{T} \to d\tau = \frac{1}{T}dt$
$\frac{dN}{dt} = \frac{d(xA)}{d\tau}\frac{d\tau}{dt} = r(xA)(1-x\frac{A}{K}) - H$
$Ax'\frac{1}{T} = rAx(1-x\frac{A}{K}) - \frac{HT}{A}$.
Let A=K and T $= \frac{1}{r}, h = \frac{HT}{A}\frac{H}{Kr} \Rightarrow x = \frac{N}{K}, \tau = rt$, then
$x' = rTx(1-x\frac{A}{K}) - \frac{HT}{A} = x(1-x) - h$

31. $y' = \frac{1-y^2}{2x},\ y(1) = \pi$.

x_i	RK:y_i	Euler:y_x
1	3.1416	3.1416
1.1	2.7742	2.6981
1.2	2.5144	2.4127
1.3	2.3210	2.2118
1.4	2.1713	2.0621
1.5	2.0522	1.9459
1.6	1.9550	1.8531
1.7	1.8743	1.7770
1.8	1.8061	1.7135

32. $xy' + x^2 + xy - y = 0,\ y(1) = 1$.

x_i	RK:y_i	Euler:y_x
1	1	1
1.1	0.8906	0.9
1.5	0.3196	0.3494
2.0	-.5285	-.4981

33. $y' = \frac{y-y^2}{x+1},\ y(2) = 0$.
All answers are 0 because $y_0 = 0$ is an equilibrium point.

34. $(x+2y^3)y' = y, \quad y(-1) = 1$.

x_i	RK:y_i	Euler:y_x
-1	1	1
-0.9	1.0802	1.1
-0.5	1.2671	1.2919
0	1.4142	1.4383

35. $y' = \frac{1-y^2}{2x}$, $y(1) = \pi$.

x_i	RK:y_i	Euler:y_x
1.4142	1.0000	1.0000
1.5142	1.1251	1.1207
1.6142	1.2592	1.2495
1.7142	1.4030	1.3868
1.8142	1.5571	1.5332
1.9142	1.7222	1.6891
2.0142	1.8992	1.8552
2.1142	2.0889	2.0321
2.2142	2.2924	2.2207

36. $(1-x^2)y' + xy = 0$, $y(0) = 5$.

x_i	RK:y_i	Euler:y_x
0	5	5
0.1	4.9749	5.0
0.5	4.3301	4.4634
0.9	2.1777	2.5333

37. $(2xy^2 - y)^2 + xy' = 0$, $y(\pi) = 1$.

x_i	RK:y_i	Euler:y_i
3.24	.5678	.11153
3.34	.5110	.11150
3.44	.4724	.11148
3.54	.4435	.11146
3.64	.4206	.11144

38. $y^2 + 2(x-1)y' - 2y = 0$, $y(0) = -3$.

x_i	RK:y_i	Euler:y_x
0	-3	-3
0.1	-2.3479	-2.25
0.5	-.8572	-.7543
0.9	-.1279	-0.0999