1.1 Introduction

This manual is complementary to the book "Optimal and robust control: Advanced Topics with MATLAB®", CRC Press, referred in the following to as the textbook. It contains the solutions to the exercises given at the end of each chapter. Although some solutions are given in analytical forms, for most of them the use of MATLAB® is required. The detailed procedures are all explained in the textbook, so that major emphasis is here given to the solution flow.

1.2 Solutions of exercises of Chapter 2

1.2.1 Exercise 2.1

Exercise: Apply the vectorization method to solve the Lyapunov equation:

$$\mathrm{A}^T\mathrm{P} + \mathrm{PA} = -\mathrm{I}$$

with

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -5 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

Solution: The solution of this exercise follows MATLAB® exercise 2.3 of the textbook. We have first to calculate matrix $\mathrm{M} = \mathrm{I} \otimes \mathrm{A}^T + \mathrm{A}^T \otimes \mathrm{I}$ (with I 3×3 identity matrix) and then to solve $\mathrm{P} = -\mathrm{M}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. One gets:

$$\mathrm{P} = \begin{bmatrix} 1.7045 & 0.2973 & -0.3649 \\ 0.2973 & 0.1494 & -0.0506 \\ -0.3649 & -0.0506 & 0.4577 \end{bmatrix}$$

1.2.2 Exercise 2.2

Exercise: Given the nonlinear system $\dot{x} = x^3 - 8x^2 + 17x + u$ calculate the equilibrium points for $u = -10$ and study their stability.

Solution: The equilibrium points are given by: $\dot{x} = 0$, which yields:

$$x^3 - 8x^2 + 17x - 10 = 0$$

Since

$$x^3 - 8x^2 + 17x - 10 = x^3 - x^2 - 7x^2 + 7x + 10x - 10 =$$

$$= (x-1)(x^2 - 7x + 10) = (x-1)(x-2)(x-5)$$

the equilibrium points are $\bar{x}_1 = 1$, $\bar{x}_2 = 2$ and $\bar{x}_3 = 5$.

Their stability is now calculated by linearizing the system around each of these equilibrium, i.e., by evaluating $\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}}$.

Since $\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}_1=1} = 4$, $\bar{x}_1$ is unstable.

Since $\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}_2=2} = -3$, $\bar{x}_2$ is stable.

Since $\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}_3=5} = 12$, $\bar{x}_3$ is unstable.

1.2.3 Exercise 2.3

Exercise: Given system $G(s) = \frac{1}{s}e^{-sT}$ determine the set of T values which provide a stable closed-loop system.

Solution: The exercise can be solved considering a Padé approximation for the delay term, for instance a first-order one $e^{-sT} \simeq \frac{1-s\frac{T}{2}}{1+s\frac{T}{2}}$, so that $G(s) \simeq \frac{1-s\frac{T}{2}}{s(1+s\frac{T}{2})}$. With this approximation the closed-loop transfer function becomes:

$$W(s) = \frac{s(1-s\frac{T}{2})}{s^2\frac{T}{s} + s(1-\frac{T}{2}) + 1}$$

which is stable if $T < 2$. Higher-order Padé approximations can be studied with the same approach and then the Routh criterion can be applied to the denominator of the closed-loop transfer function to derive the stability condition with respect to T.

1.2.4 Exercise 2.4

Exercise: Study the stability of system with transfer function

$$G(s) = \frac{s^2 + 2s + 2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

with $1 \leq a_1 \leq 3$, $4 \leq a_2 \leq 7$, $1 \leq a_3 \leq 2$, $0.5 \leq a_4 \leq 2$.

Solution: We can study the stability of this *uncertain* system by applying the Kharitonov criterion. Let's consider therefore:

$$\begin{array}{l} D_1(s) = s^4 + s^3 + 7s^2 + 2s + 0.5 \\ D_2(s) = s^4 + 3s^3 + 4s^2 + s + 2 \\ D_3(s) = s^4 + 3s^3 + 7s^2 + s + 0.5 \\ D_4(s) = s^4 + s^3 + 4s^2 + 2s + 2 \end{array}$$

By calculating the roots of these polynomial (using MATLAB® command `roots`), we find that $D_1(s)$, $D_3(s)$ and $D_4(s)$ have all roots with negative real part, while $D_2(s)$ has two positive real part roots. Therefore, we cannot state that for any value of the parameters the uncertain system is stable.

1.2.5 Exercise 2.5

Exercise: Given the polynomial $p(s,a) = s^4 + 5s^3 + 8s^2 + 8s + 3$ with $a = \left[\begin{array}{cccc} 3 & 8 & 8 & 5 \end{array}\right]$, find $p(s,b)$ with $b = \left[\begin{array}{ccc} (b_0^-, b_0^+) & \ldots & (b_3^-, b_3^+) \end{array}\right]$ so that the polynomial class $p(s,b)$ is Hurwitz.

Solution:

1.2.6 Exercise 2.6

Exercise: Study the stability of the system $G(s) = \frac{s^2+3s+2}{s^4+q_1 s^3+5s^2+q_2 s+q_3}$ with parameters $q_1 \in [1,3]$, $q_2 \in [5,10]$, $q_3 \in [2,18]$.

Solution: As for exercise 2.4, we can apply the Kharitonov criterion. Consider:

$$\begin{array}{l} D_1(s) = s^4 + s^3 + 5s^2 + 10s + 2 \\ D_2(s) = s^4 + 3s^3 + 5s^2 + 5s + 18 \\ D_3(s) = s^4 + 3s^3 + 5s^2 + 5s + 2 \\ D_4(s) = s^4 + s^3 + 5s^2 + 10s + 18 \end{array}$$

The analysis of the roots of these polynomials reveal that only $D_3(s)$ has all the roots with negative real part. Therefore, there exist values of the parameters for which the system is unstable.

1.3 Solutions of exercises of Chapter 3

1.3.1 Exercise 3.1

Exercise: Calculate the Kalman decomposition for the system with state-space matrices:

$$\mathrm{A} = \begin{bmatrix} -2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 \\ -2 & 1 & -1 & -1 & 0 \\ -4 & 1 & 2 & -1 & -2 \end{bmatrix}; \mathrm{B} = \mathrm{C}^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: We first calculate the controllability and observability matrices:

$$\mathrm{M}_c = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & -3 & -1 & -9 & -19 \\ 1 & -4 & 14 & -12 & 84 \end{bmatrix}; \mathrm{M}_o^T = \begin{bmatrix} 1 & -6 & 32 & -104 & 384 \\ 1 & 4 & -26 & 94 & -362 \\ 1 & 4 & 10 & 34 & 94 \\ 1 & -2 & 4 & -8 & 16 \\ 1 & -2 & 4 & -8 & 16 \end{bmatrix}$$

Both have rank equal to 4 and in both cases the first four columns are linearly independent. Therefore, we can determine X_r as the subspace spanned by the first four columns of M_c and X_o as the subspace spanned by the first four columns of M_o^T. We then determine X_{nr} has the null space of M_c^T (MATLAB® command `Xnr=null(Mc')`) and X_{no} has the null space of M_o (MATLAB®