46. Basin boundary for the limit cycle and chaos for the food-chain model (Rai and Upadhyay [24]) when $a_1 = 1.75,\ b_1 = 0.05,\ a_2 = 1.0,\ c = 0.7,\ w = 1.0,\ w_1 = 2.0,\ w_2 = 1.5,$ $w_3 = 3.75,\ D = 10.0,\ D_1 = 10.0,\ D_2 = 10.0,\ D_3 = 20.$

Basin boundaries for the chaotic attractor are plotted in Fig. 1.7.

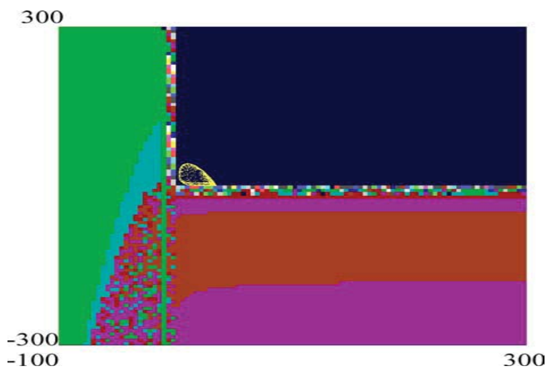


Fig.1.7. Basin boundaries for the chaotic attractor for Problem 46.
(From Rai, V., Upadhyay, R. K., Chaotic population dynamics and biology of the top-predator. *Chaos, Solitons Fractals*, 21, 1195–1204, Copyright 2004, Elsevier. Reprinted with permission).

CHAPTER 2
Exercise 2.1

1. Without loss of generality, let the initial time be $t = 0$. The solution is given by $P(t) = P_0 e^{rt}$. At time $t = 0$, we have $P = P_0$. Let at time $t = T$, the population is tripled, that is $P(T) = 3P_0$. Hence, $3P_0 = P_0 e^{rT}$, or $T = \ln 3/r$. The time taken for the population to triple its size is $T = \ln 3/r$.

2. We have $r = 0.09, K = 900, t_0 = 0, P_0 = 90$. The value of the constant A is obtained as $A = [(P_0 - K)/P_0] = -9$. The solution is given by

$$P(t) = \frac{K}{1 - Ae^{-rt}} = \frac{900}{1 + 9e^{-0.09t}}. \quad P(90) = \frac{900}{1 + 9e^{-8.1}} \approx 897.$$

3. (i) Equilibrium solutions are $P_1 = 0$ and $P_2 = 500$. Now,

$$f'(P) = 0.5\left(1 - \frac{P}{250}\right), \qquad f'(P_1) = 0.5 > 0, \qquad f'(P_2) = -0.5 < 0.$$

Hence, the equilibrium point $P_1 = 0$ is unstable and the equilibrium point $P_2 = 500$ is stable. Thus, solutions initiating in a neighbourhood of $K = 500$ approach K as $t \to \infty$ while no solution starting in a neighbourhood of $P = 0$ remains close to zero in the future.

(ii) $P(t) = 500/[1 + 9e^{-0.5t}]$.

4. $\dfrac{dP}{dt} = (0.0025)P\left(1 - \dfrac{P}{100}\right)$. The solution is $P(t) = \dfrac{K}{1 - Ae^{-rt}} = \dfrac{100}{1 - Ae^{-0.0025t}}$,

where $A = -11.5\,e^{4.9875}$. The estimated populations in the years 2005 and 2100 are 8.18 and 10.15 billions respectively.

5. Separating the variables and solving the differential equation, we get $R/(R-1) = ce^{\alpha t}$, a, c are arbitrary constants to be determined. Applying the conditions that at 8 hours, R = 0.08, and at 12 hours R = 0.05, we obtain $a = 0.25\ln(11.5)$, and $c = -e^{-12a} \approx -1/1520.9$. Now, when R = 0.9, we obtain t = 15.5985 or approximately 3.36 P.M.

6. We have $\dfrac{d^2P}{dt^2} = \left(\dfrac{r}{\alpha}\right)^2 P\left[1-\left(\dfrac{P}{K}\right)^{\alpha}(1+\alpha)\right]\left[1-\left(\dfrac{P}{K}\right)^{\alpha}\right]$

$$= \left(\frac{r}{\alpha}\right)^2 P[1-t(2+\alpha)+t^2(1+\alpha)], \qquad \text{where } t = \left(\frac{P}{K}\right)^{\alpha}.$$

Setting $P''(t) = 0$, we get P = 0, t = 1, and $t = 1/(1+\alpha)$, that is, P = 0, P = K, and P = $K(1+\alpha)^{-1/\alpha}$. The point of inflection is $P = K(1+\alpha)^{-1/\alpha}$.

For $\alpha = 1$, the point of inflection is P = K/2. Taking the limit as $\alpha \to 0$, we find that the point of inflection moves to P = K/e.

7. Comparing with (2.12), we have $r = 10^{-3}, \gamma = 10^{-9}, P_0 = 10^5$. The solution is given by

$$P(t) = \frac{rP_0}{\gamma P_0 + (r-\gamma P_0)e^{-rt}} = \frac{10^6}{1+9\exp(-0.001t)}.$$

The limiting value of the population is 10^6.

8. Comparing with the model for the density dependent growth (2.12), we have r = 0.09, $\gamma = 0.0009$, $P_0 = 2500$. The solution of the model is obtained as

$$P(t) = \frac{rP_0}{\gamma P_0 + (r-\gamma P_0)e^{-rt}}$$

$$= \frac{(0.09)(2500)}{(0.0009)(2500) + \{0.09 - 0.0009(2500)\}e^{-0.09t}} = \frac{225}{2.25 - 2.16e^{-0.09t}}.$$

The limiting value of the population is 100. The equilibrium points are 0 and 100. The point 0 is unstable and the point 100 is stable. All other solutions move away from P = 0, towards P =100.

9. $P(t) = P_0 \exp[(r_0/\alpha)(1-e^{-\alpha t})]$.

10. Assume that no student leaves the campus throughout the duration of viral fever. Now,
Number of students infected at a point of time = p,
Number of students who are not infected at the same point of time = 2500 – p.
Since, the rate at which viral fever spreads is proportional not only to the number of people affected, but also to the number of people who are not yet exposed to it, we obtain the model as

$$\frac{dp}{dt} = ap(2500-p), \ \ p(0) = 1, \ a \text{ is an arbitrary constant.}$$

Separating the variables and solving, we get $p(t) = 2500/(1+2499e^{-2500at})$. Since, $p(5) = 50$, we obtain $a = \ln(51)/12500$. Hence, $p(15) \approx 2454$ students.

11. The equilibrium points are the solutions of the equation $P(6-P)-h = 0$. The equilibrium points are given by $P_1 = 3-\sqrt{9-h}$, $P_2 = 3+\sqrt{9-h}$, $h < 9$. We have

$$f'(P) = 6-2P, \quad f'(P_1) = 2\sqrt{9-h} > 0, \quad f'(P_2) = -2\sqrt{9-h} < 0.$$

The equilibrium point P_1 is unstable whereas the equilibrium point P_2 is asymptotically stable. The two equilibrium points coincide when $h = 9$. The critical value of h is $h_c = 9$. When $h > 9$, there are no equilibrium points and the population tends to extinction. If $P > 3$, $f'(P) < 0$. In this case, we get stable solutions. Hence, the initial value of the population should satisfy $P_0 > 3$. That is, the population may be extinct if $P_0 < 3$, even though the condition $h < h_c$ is satisfied.

12. The equation is a harvesting model with constant harvesting.

(i) The equilibrium points are

$$N = \frac{1}{2r}[rK \pm \sqrt{r^2K^2 - 4rhK}] = \frac{K}{2}[1 \pm \sqrt{1-p}], \quad \text{where } p = \frac{4h}{rK}.$$

Let, $$N_1 = \frac{K}{2}[1 - \sqrt{1-p}], \text{ and } N_2 = \frac{K}{2}[1 + \sqrt{1-p}].$$

Real solutions are obtained only when $p < 1$, or $h < (rK/4)$. We find

$$f'(N) = r[1-(2/K)N], \; f'(N_1) = r\sqrt{1-p} > 0, \quad f'(N_2) = -r\sqrt{1-p} < 0.$$

The equilibrium point N_1 is unstable whereas the equilibrium point N_2 is asymptotically stable. The two equilibrium points coincide when $p = 1$, that is when $4h = rK$, or $h = rK/4$ and $N = K/2$. The critical value of h is $h_c = rK/4$. When $h > rK/4$, there are no equilibrium points and the population tends to extinction.

(ii) If $2N > K$, $f'(N) < 0$. In this case, we get stable solutions. Hence, the initial value of the population should satisfy $n_0 > (K/2)$. That is, the population may become extinct if $n_0 < (K/2)$, even though the condition $h < h_c$ is satisfied.

13. Equilibrium point is $N_1 = Ke^{-qE/\alpha}$. $N = 0$ can not be taken as an equilibrium point even though the limit of the right hand side exists. Now,

$$f'(N) = \alpha[\ln(K/N) - 1] - qE, \quad \text{and} \quad f'(N_1) = -\alpha < 0.$$

Therefore, N_1 is asymptotically stable. Note that $f'(N=0)$ is not defined.

The sustainable yield is given by

$$Y(qE) = qEN_1 = qEKe^{-qE/\alpha} = \alpha\, Kue^{-u} = Y(u), \quad \text{where } u = qE/\alpha.$$

Also, $Y'(u) = \alpha K[1-u]e^{-u}$, $Y''(u) = \alpha K[u-2]e^{-u}$. Setting $Y'(u) = 0$, we get the stationary point as $u = 1$. When $u = 1$, $Y''(u) < 0$. The maximum occurs for $u = 1$, or $qE = \alpha$. Maximum sustainable yield $= Y(u=1) = \alpha K/e$.

14. The location of the steady state varies with the length of the delay and the form of the characteristic equation changes due to the direct inclusion of the delay in the parameters and the indirect changes resulting from the varying location of the steady state. The model has the trivial steady state (0, 0) and the nontrivial steady state is given by

$$be^{-\mu\tau}e^{-a\overline{P}} = d, \quad \text{or} \quad \overline{P} = \frac{1}{a}\ln\left(\frac{be^{-\mu\tau}}{d}\right).$$

In particular, if $\tau > (1/\mu)\ln(b/d)$, there is no positive steady state. In this case, given that the initial value is positive, we have

$$\frac{dP}{dt} \le be^{-\mu\tau}P(t-\tau) - dP(t), \quad \text{with} \quad be^{-\mu\tau} < d.$$

The solution goes to zero and the trivial steady state is globally stable.

15. $P^* = F(P^*)$, gives $P^* = K(1+r)$ which depends on r. At $P = P^*$, $F' = [1/(1+r)] < 1$ for all $r > 0$. Asymptotically stable for all r.

16. We find the solution of the equation $P^* = F(P^*)$, that is of

$$P^* = P^* \exp\left[\lambda\left(1 - \frac{P^*}{K}\right)\right], \text{ or } \exp\left[\lambda\left(1 - \frac{P^*}{K}\right)\right] = 1.$$

The solution is given by $P^* = K$, for all λ.

Now, $$F(P) = P\exp\left[\lambda\left(1 - \frac{P}{K}\right)\right], \quad \frac{dF}{dP} = \exp\left[\lambda\left(1 - \frac{P}{K}\right)\right]\left[1 - \frac{\lambda P}{K}\right].$$

At $P = P^* = K$, we get $\dfrac{dF}{dP} = [1 - \lambda]$.

Equilibrium is stable for $|1-\lambda| < 1$, or $0 < \lambda < 2$. It is unstable for $\lambda < 0$ and $\lambda > 2$. Neighboring trajectories approach the equilibrium point asymptotically for $0 < \lambda < 1$, and with damped oscillations for $1 < \lambda < 2$.

Now, $$\frac{dF}{dP} = 0, \text{ when } P = \frac{K}{r}. \quad \frac{d^2F}{dP^2} = \left(-\frac{r}{K}\right)\exp\left[r\left(1 - \frac{P}{K}\right)\right]\left[2 - \frac{rP}{K}\right].$$

For $$P = \frac{K}{r}, \quad \frac{d^2F}{dP^2} = \left(-\frac{r}{K}\right)\exp(r-1) < 0.$$

Maximum of the trajectory occurs at $P = K/r$. The maximum value $= (K/r)\exp(r-1)$.

(i) For $K = 500$, $r = 1$, $P_0 = 50$, we have the following sequence of values: 122.98, 261.40, 421.26, 493.11, 499.95, 500, (see Fig.2.1).

(ii) For $K = 500$, $r = 1$, $P_0 = 660$, we get the sequence of values: 479.26, 499.56, 500, (see Fig. 2.1).

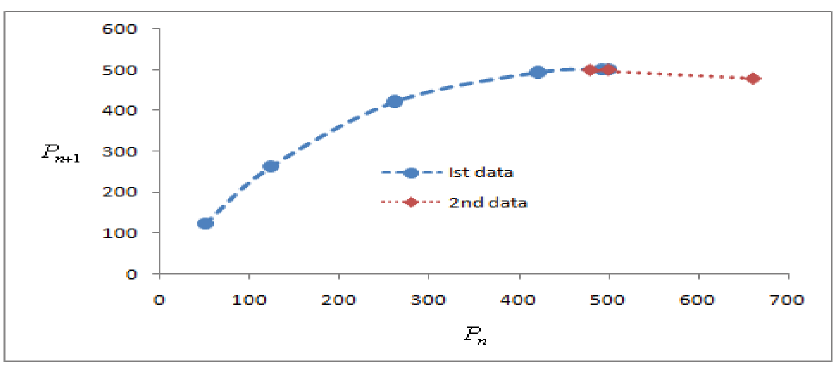


Fig.2.1. Discrete solution values for the data sets (i) and (ii).

17. Choose $V_i = (N_i - K)^2$. $V_i \geq 0$ and has a minimum $V = 0$ at $N_i = K$.

The increment ΔV_i on the trajectory is given by

$\Delta V_i = K^2 n_i [e^{r(1-n_i)} - 1][n_i e^{r(1-n_i)} + n_i - 2]$, where $N_i = Kn_i$.

$\Delta V_i \leq 0$ when (i) $e^{r(1-n_i)} < 1$, and $[e^{r(1-n_i)} + 1] \geq [2/n_i]$;

or (ii) $e^{r(1-n_i)} \geq 1$, and $[e^{r(1-n_i)} + 1] < [2/n_i]$. We obtain $\Delta V_i \leq 0$ for $0 < r < 2$ and for all n_i. $\Delta V_i = 0$ only for $n_i = N^*/K$. Hence, equilibrium is globally asymptotically stable.

18. Equilibrium point is $x^* = 1$. $|F'(1)| = |1-r| < 1$ gives $0 < r < 2$. Asymptotically stable for $0 < r < 2$. Maximum occurs at $x = 1/r$ and the maximum value is $\exp(r-1)/r$.

19. For steady state, we solve $P^* = F(P^*) = [(\lambda P^*)/(1+bP^{*2})]$, to get $P^* = 0$, and $P^* = \sqrt{(\lambda-1)/b}, \lambda > 1$. We have

$$\frac{dF}{dP} = \frac{\lambda(1-bP_n^2)}{(1+bP_n^2)^2}.$$

Hence, the eigen values corresponding to $P^* = 0$ and $P^* = \sqrt{(\lambda-1)/b}$, are λ and $(2-\lambda)/\lambda$ respectively. For $\lambda = 1$, we have only a trivial equilibrium point.

Setting $dF/dP = 0$, we get $P = 1/\sqrt{b}$. For this value of P, $F''(P) < 0$. Hence, we obtain the maximum at $P = 1/\sqrt{b}$. The maximum value is $F(1/\sqrt{b}) = \lambda/(2\sqrt{b})$.

Exercise 2.2

1. Setting $F(X,Y) = 0, G(X,Y) = 0,$ we obtain the equilibrium point as $X^* = 212/3$, $Y^* = 2332/15$. Note that (0, 0) is not an equilibrium point. The elements of the Jacobian matrix of the system evaluated at an equilibrium point (X^*, Y^*) are

$$a_{11} = 1 - \frac{X}{50} - \frac{8.0Y^2}{(X+5Y)^2}, \quad a_{12} = -\frac{1.6X^2}{(X+5Y)^2},$$

$$a_{21} = \frac{3Y^2}{(X+5Y)^2}, \quad a_{22} = \frac{0.6X^2}{(X+5Y)^2} - 0.05.$$

At the equilibrium point (X^*, Y^*): We obtain $a_{11} = -0.682222$, $a_{12} = -0.011111$, $a_{21} = 0.100833$, $a_{22} = -0.045833$. The characteristic equation is $\lambda^2 + 0.72806\lambda + 0.032389 = 0$. The coefficients in the equation are positive. By Routh-Hurwitz criterion, the eigen values are negative or have negative real parts. The equilibrium point (X^*, Y^*) is asymptotically stable.

2. We have $F(P,Z) = (a_1 - b_1P - c_1Z)$, and $G(P,Z) = [a_2 - c_2(Z/P)]$.

Conditions (i), (ii), (iii), (v), (vi) and (vii) are satisfied. Equality condition in (iv) is satisfied. The requirement (viii), G (C, 0) = 0 gives $a_2 = 0$, which violates the assumption that a_2 is positive. Hence, Kolmogorov theorem cannot be applied.

3. We have $F(X,Y) = a_1 - b_1X - \dfrac{wY}{X+D}$, $\quad G(X,Y) = -a_2 + \dfrac{w_1X}{X+D_1}$.

Conditions (i), (ii), (iv), (v), (vi), (vii) are satisfied. Equality condition in (iii) is satisfied.

(viii) $G(C,0) = -a_2 + \dfrac{w_1C}{C+D_1} = 0$, gives $C = \dfrac{a_2D_1}{w_1 - a_2}$.

$C > 0$ gives the condition $w_1 > a_2$.

(ix) $B > C$ gives the condition $\dfrac{a_1}{b_1} > \dfrac{D_1a_2}{(w_1 - a_2)}$, or $\quad w_1 - a_2 > \dfrac{D_1a_2}{K}$.

The two species system (2.80), (2.81) qualifies as a Kolmogorov system when the conditions $w_1 > a_2$, $K(w_1 - a_2) > D_1a_2$ are satisfied.

4. The elements of the Jacobian matrix of the system are

$$a_{11} = a_1 - 2b_1 X - \frac{wDY}{(X+D)^2}, \quad a_{12} = -\frac{wX}{X+D},$$

$$a_{21} = \frac{w_1 D_1 Y}{(X+D_1)^2}, \qquad a_{22} = -a_2 + \frac{w_1 X}{X+D_1}.$$

At the equilibrium point $E_0(0,0)$: We obtain $a_{11} = a_1, a_{12} = 0, a_{21} = 0, a_{22} = -a_2$. The eigen values are $\lambda_1 = a_1 > 0$, and $\lambda_2 = -a_2 < 0$. The equilibrium point is unstable. Since $\mathrm{Re}(\lambda) \neq 0$ for both eigen values, the fixed point is hyperbolic. Since the eigen values are real and are of opposite signs, we find that E_0 is a hyperbolic saddle point which repels in the x-direction and attracts in y-direction.

At the equilibrium point $E_1(K,0)$: We obtain

$$a_{11} = a_1 - 2b_1 K, \; a_{12} = -\frac{wK}{(K+D)}, \; a_{21} = 0, \; a_{22} = -a_2 + \frac{w_1 K}{K+D_1}, \quad K = \frac{a_1}{b_1}.$$

The eigen values are $\lambda_1 = a_{11} = -a_1 < 0$, and

$$\lambda_2 = a_{22} = -a_2 + \frac{w_1 a_1}{b_1(K+D_1)} = \frac{a_1(w_1 - a_2) - a_2 b_1 D_1}{a_1 + b_1 D_1} > 0$$

using the result from Kolmogorov condition (ix). The equilibrium point is unstable. The fixed point $E_1(K,0)$ is also a hyperbolic saddle point which attracts in the x-direction and repels in y-direction.

At the equilibrium point $E^*(X^*, Y^*)$: We obtain

$$a_{11} = a_1 - 2b_1 X^* - \frac{wDY^*}{(X^*+D)^2}, \qquad a_{12} = -\frac{wX^*}{X^*+D},$$

$$a_{21} = \frac{w_1 D_1 Y^*}{(X^*+D_1)^2}, \quad a_{22} = G(X,Y) = -a_2 + \frac{w_1 X}{X+D_1} = 0.$$

The eigen values of J are the roots of $\lambda^2 - a_{11}\lambda - a_{12}a_{21} = 0$. By Routh-Hurwitz theorem, the necessary and sufficient conditions for the eigen values to be negative or have negative real parts are $(-a_{11}) > 0, (-a_{12}a_{21}) > 0$. That is

$$(-a_{12}a_{21}) = \left(\frac{wX^*}{X^*+D}\right)\left(\frac{w_1 D_1 Y^*}{(X^*+D_1)^2}\right) > 0, \text{ which is true.}$$

$$(-a_{11}) = -\left[a_1 - 2b_1 X^* - \frac{wDY^*}{(X^*+D)^2}\right] = \frac{wY^*}{(X^*+D)} \cdot \frac{D}{(X^*+D)} - (a_1 - 2b_1 X^*)$$

$$= (a_1 - b_1 X^*)\frac{D}{(X^*+D)} - (a_1 - 2b_1 X^*) \qquad (\text{from } F(X,Y) = 0)$$

$$= -(a_1 - b_1 X^*)\frac{X^*}{(X^*+D)} + b_1 X^* = b_1 X^*\left[1 + \frac{X^* - K}{(X^*+D)}\right]$$

$$= b_1 X^*\left[\frac{2X^* + D - K}{(X^*+D)}\right].$$

Now, $(-a_{11}) > 0$, if $2X^* + D - K > 0$. Substituting the expression for X^*, we get

$$2\left(\frac{a_2 D_1}{w_1 - a_2}\right) + D - K > 0, \quad \text{or} \quad 2b_1\left(\frac{a_2 D_1}{w_1 - a_2}\right) + b_1 D - a_1 > 0. \qquad \text{(A)}$$

The equilibrium point $E^*(X^*, Y^*)$ is locally asymptotically stable if the condition (A) is satisfied.

5. The elements of the Jacobian matrix of the system are

$$a_{11} = 1 - 2u - \frac{h_P v}{(h_P + u)^2}, \quad a_{12} = -\frac{u}{h_P + u},$$

$$a_{21} = \frac{b h_P v}{(h_P + u)^2}, \qquad a_{22} = \frac{bu}{(h_P + u)} - c - \frac{2 f v h_Z^2}{(h_Z^2 + v^2)^2}.$$

At the equilibrium point $E_0(0,0)$: We obtain $a_{11} = 1, a_{12} = 0, a_{21} = 0, a_{22} = -c.$ The eigen values are $\lambda_1 = 1,$ and $\lambda_2 = -c < 0.$ The equilibrium point is a hyperbolic saddle point.

At the equilibrium point $E_1(1,0)$: We obtain

$$a_{11} = -1, \; a_{12} = -\frac{1}{(1 + h_P)}, \; a_{21} = 0, \; a_{22} = \frac{b}{h_P + 1} - c.$$

The eigen values are $\lambda_1 = a_{11} = -1 < 0,$ and $\lambda_2 = a_{22} = \dfrac{b}{h_P + 1} - c = \dfrac{(b-c) - c h_p}{h_P + 1}.$

But (from text), $b > c$, $0 < c h_P < b - c$, $0 < h_P < 1.$ Therefore, the equilibrium E_1 is a saddle point with stable manifold locally in the u-direction and with unstable manifold locally in the v-direction.

At the equilibrium point $E^*(u^*, v^*)$: We have

$$a_{11} = 1 - 2u - \frac{h_P v}{(h_P + u)^2} = 1 - u - u - \frac{h_P v}{(h_P + u)^2},$$

$$= \frac{v}{(h_P + u)} - u - \frac{h_P v}{(h_P + u)^2} = u\left[\frac{v}{(h_P + u)^2} - 1\right].$$

$$a_{12} = -\frac{u}{h_P + u}, \; a_{21} = \frac{b h_P v}{(h_P + u)^2}, \; a_{22} = \frac{bu}{(h_P + u)} - c - \frac{2 f v h_Z^2}{(h_Z^2 + v^2)^2} = \frac{f v (v^2 - h_Z^2)}{(h_Z^2 + v^2)^2},$$

where all the quantities are evaluated at $(u^*, v^*).$ The eigen values of J are the roots of $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0.$ Using the Routh-Hurwitz theorem, we find the conditions for local stability as

$$A = -(a_{11} + a_{22}) = u\left[1 - \frac{v}{(h_P + u)^2}\right] + \frac{f v (h_Z^2 - v^2)}{(h_Z^2 + v^2)^2} > 0,$$

$$B = a_{11}a_{22} - a_{12}a_{21} = u\left[1 - \frac{v}{(h_P + u)^2}\right]\left[\frac{f v (h_Z^2 - v^2)}{(h_Z^2 + v^2)^2}\right] + \frac{b h_P u v}{(h_P + u)^3} > 0,$$

where all the quantities are evaluated at $(u^*, v^*).$ Sufficient conditions are $v^* < (h_P + u^*)^2,$ and $(v^*)^2 < h_Z^2.$ If the above conditions are satisfied, then both predator and prey species coexist, and they settle down at its equilibrium point.

6. The non-zero equilibrium points are the solution of the equations

$$a_1 - b_1 X - \frac{wY}{(\alpha + \beta Y + \gamma X)} = 0, \qquad -a_2 + \frac{w_1 X}{(\alpha + \beta Y + \gamma X)} = 0.$$

We have $\dfrac{a_1 - b_1 X}{wY} = \dfrac{1}{\alpha + \beta Y + \gamma X} = \dfrac{a_2}{w_1 X}$. Solving the right equality, we obtain

$$Y = \frac{1}{a_2 \beta}[(w_1 - \gamma a_2)X - \alpha a_2]. \qquad (2.1)$$

Using the first and third terms in the equality and substituting the expression for Y, we obtain $(a_1 - b_1 X)w_1 X = \dfrac{w}{\beta}[(w_1 - \gamma a_2)X - \alpha a_2]$.

Simplifying, we obtain $\beta w_1 b_1 X^2 + [w(w_1 - \gamma a_2) - \beta w_1 a_1]X - w\alpha a_2 = 0$.

The roots of this equation are $X = [p \pm \sqrt{p^2 + q}]/(2\beta w_1 b_1)$,

where $p = \beta w_1 a_1 + w\gamma a_2 - ww_1$, $q = 4\beta w_1 b_1 w\alpha a_2$. Irrespective of the sign of p, the root in the first quadrant is $X^* = [p + \sqrt{p^2 + q}]/(2\beta w_1 b_1)$.

The value of Y^* is given by (2.1) (the chosen parameter values should satisfy $Y^* > 0$).

7. We have $F(X,Y) = a_1 - b_1 X - \dfrac{wY}{(\alpha + \beta Y + \gamma X)}$, and $G(X,Y) = -a_2 + \dfrac{w_1 X}{(\alpha + \beta Y + \gamma X)}$.

Conditions (i), (ii), (iii), (iv), (v), (vii) are satisfied.

(vi) $F(0, A) = 0$, gives $A = \dfrac{a_1 \alpha}{(w - a_1 \beta)} > 0$ if $w > a_1 \beta$.

(viii) $G(C, 0) = 0$, gives $C = \dfrac{a_2 \alpha}{(w_1 - a_2 \gamma)} > 0$ if $w_1 > a_2 \gamma$.

(ix) $B > C$ gives $\dfrac{a_1}{b_1} > \dfrac{a_2 \alpha}{(w_1 - a_2 \gamma)}$.

Summarizing, we get the conditions as $w > a_1 \beta$, $w_1 > a_2 \gamma$ and $\dfrac{a_1}{b_1} > \dfrac{a_2 \alpha}{(w_1 - a_2 \gamma)}$.

An oscillatory predator-prey dynamics (time series) exhibited by the model system for the given set of parameter values, $a_1 = 2.5$, $b_1 = 0.05$, $w = 0.85$, $\alpha = 0.45$, $\beta = 0.2$, $\gamma = 0.6$, $a_2 = 0.95$, and $w_1 = 1.65$, is presented in Fig. 2.2.

8. The equilibrium points are $(0, 0)$, $(a_1 / b_1, 0)$ and (X^*, Y^*) (see Problem 6).

The elements of the Jacobian matrix of the system are

$$a_{11} = a_1 - 2b_1 X - \frac{w(\alpha + \beta Y)Y}{(\alpha + \beta Y + \gamma X)^2}, \quad a_{12} = -\frac{w(\alpha + \gamma X)X}{(\alpha + \beta Y + \gamma X)^2},$$

$$a_{21} = \frac{w_1(\alpha + \beta Y)Y}{(\alpha + \beta Y + \gamma X)^2}, \quad a_{22} = -a_2 + \frac{w_1(\alpha + \gamma X)X}{(\alpha + \beta Y + \gamma X)^2}.$$

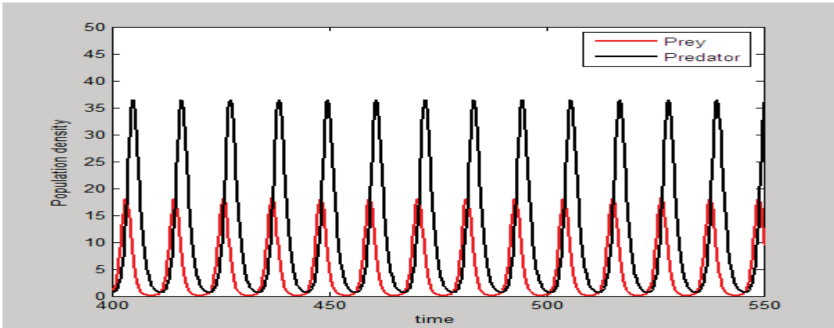


Fig.2.2. Time-series displaying oscillatory dynamics in the model (2.71)-(2.72).

At the equilibrium point $(0, 0)$: We obtain $a_{11} = a_1$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = -a_2$. The eigen values are $\lambda_1 = a_1$, and $\lambda_2 = -a_2 < 0$. The equilibrium point (0, 0) is unstable. Since, $\mathrm{Re}(\lambda) \neq 0$ for both eigen values, the fixed point is hyperbolic. Since, the eigen values are real and are of opposite signs, we find that (0, 0) is a hyperbolic saddle point which repels in the *x*-direction and attracts in *y*-direction.

At the equilibrium point (*K*, 0), *where* $K = a_1 / b_1$.

We obtain $a_{11} = -a_1,\ a_{12} = -\dfrac{wK}{\alpha + \gamma K},\ a_{21} = 0,\ a_{22} = \dfrac{w_1 K}{\alpha + \gamma K} - a_2.$

The eigen values are $\lambda_1 = -a_1$, and $\lambda_2 = a_{22} = (w_1 K / (\alpha + \gamma K)) - a_2$. If $\lambda_2 < 0$, that is $[w_1 K / (\alpha + K\gamma)] < a_2$, that is, $[w_1 a_1 / (b_1 \alpha + a_1 \gamma)] < a_2$, then the equilibrium point $(K, 0)$ is asymptotically stable. Otherwise, $(K, 0)$ is unstable. It depends on the values of the parameters $w_1, a_1, a_2, b_1, \alpha, \gamma$. If $\lambda_2 > 0$, that is, $[w_1 a_1 / (b_1 \alpha + a_1 \gamma)] > a_2$, then the eigen values are real and are of opposite signs and the fixed point $(K, 0)$ is a hyperbolic saddle point.

At the equilibrium point (*X**, *Y**): The expressions for *X**, *Y** are

$$X^* = [p + \sqrt{p^2 + q}]/(2\beta w_1 b_1);\quad p = \beta w_1 a_1 + w\gamma a_2 - w w_1,\ q = 4\beta w_1 b_1 w \alpha a_2.$$

$$a_2 \beta\, Y^* = [(w_1 - \gamma a_2) X^* - \alpha a_2].$$

The eigen values of *J* are the roots of $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11} a_{22} - a_{12} a_{21}) = 0$. Using the Routh-Hurwitz theorem, the necessary and sufficient conditions are given by $-(a_{11} + a_{22}) > 0$, and $(a_{11} a_{22} - a_{12} a_{21}) > 0$. We have (dropping *)

$$\frac{a_1 - b_1 X}{wY} = \frac{1}{\alpha + \beta Y + \gamma X} = \frac{a_2}{w_1 X}. \text{ (see Problem 6)}$$

$$\alpha + \beta Y = \alpha + \frac{1}{a_2}[(w_1 - \gamma a_2) X - \alpha a_2] = \frac{1}{a_2}(w_1 - \gamma a_2) X..$$

$$a_{11} = a_1 - 2 b_1 X - \frac{w(\alpha + \beta Y) Y}{(\alpha + \beta Y + \gamma X)^2}$$

$$= a_1 - b_1 X - b_1 X - \left[\frac{w(w_1 - \gamma a_2) X}{a_2}\right]\left[\frac{(a_1 - b_1 X) a_2}{w w_1 X}\right] = \frac{\gamma a_2}{w_1}(a_1 - b_1 X) - b_1 X.$$

$$a_{22} = -a_2 + \frac{w_1(\alpha + \gamma X) X}{(\alpha + \beta Y + \gamma X)^2} = -a_2 + \frac{a_2(\alpha + \gamma X)}{(\alpha + \beta Y + \gamma X)} = -\frac{a_2 \beta}{w}(a_1 - b_1 X).$$

$$a_{11}+a_{22}=\frac{\gamma a_2}{w_1}(a_1-b_1X)-b_1X-\frac{a_2\beta}{w}(a_1-b_1X)$$
$$=a_1a_2\left[\frac{\gamma}{w_1}-\frac{\beta}{w}\right]-b_1X\left[\frac{\gamma a_2}{w_1}+1-\frac{\beta a_2}{w}\right]$$
$$=a_1a_2\left[\frac{\gamma w-\beta w_1}{ww_1}\right]+\frac{b_1X}{ww_1}[a_2(\beta w_1-\gamma w)-ww_1].$$

Sufficient conditions for $[-(a_{11}+a_{22})]>0$ are

$\gamma w-\beta w_1<0,$ and $a_2(\beta w_1-\gamma w)-ww_1<0.$ Now,

$$(a_{11}a_{22}-a_{12}a_{21})=\left[\frac{\gamma a_2}{w_1}(a_1-b_1X)-b_1X\right]\left[-\frac{a_2\beta}{w}(a_1-b_1X)\right]$$
$$+\frac{wa_2^2}{w_1^3X}(\alpha+\gamma X)(w_1-\gamma a_2)(a_1-b_1X)$$
$$=\frac{a_2\beta}{ww_1}(a_1-b_1X)[b_1X(w_1+\gamma a_2)-\gamma a_1a_2]+\text{second term}.$$

Sufficient conditions for $(a_{11}a_{22}-a_{12}a_{21})>0$ are

$(w_1-\gamma a_2)>0, (a_1-b_1X)>0,$ and $b_1X(w_1+\gamma a_2)-\gamma a_1a_2>0,$

that is, $(w_1-\gamma a_2)>0,$ and $\dfrac{\gamma a_1a_2}{(w_1+\gamma a_2)}<b_1X<a_1.$

We require the above conditions to be satisfied for asymptotic stability. However, it is possible to derive alternate conditions by simplifying in a different way.

9. The equilibrium points are the solutions of the equations

$$u\left[\left(1-\frac{u}{K}\right)-\frac{v}{(u^2/\alpha)+u+1}\right]=0,\quad v\left[\frac{\beta u}{(u^2/\alpha)+u+1}-\gamma\right]=0.$$

Two of the equilibrium points are (0, 0) and (K, 0). From the second equation, we get $[1/\{(u^2/\alpha)+u+1\}]=(\gamma/\beta u).$ Using this result in the first equation, we get $v^*=(\beta/\gamma K)(K-u^*)u^*.$

Simplifying the equations $(K-u)(u^2+\alpha u+\alpha)-vK\alpha=0,\ \alpha\beta u-\gamma(u^2+\alpha u+\alpha)=0,$ we obtain $(K-u)[u^2+\alpha u+\alpha-(\alpha\beta/\gamma)u]=0.$ The first root gives u = K, which gives the equilibrium point (K, 0). Setting $S=\alpha[1-(\beta/\gamma)],$ we obtain the solutions of $u^2+Su+\alpha=0$ as $u^*=[-S\pm\sqrt{S^2-4\alpha}\,]/2.$ Hence, $v^*=(\beta/\gamma K)[K-u^*]u^*.$

The non-trivial solutions exist if $S^2>4\alpha,$ and $u^*<K,$ that is if $[1-(\beta/\gamma)]^2>(4/\alpha),$ and $u^*<K.$ If $S<0,$ that is $\beta>\gamma,$ we obtain two positive equilibrium points. If $S>0,$ that is $\beta<\gamma,$ we have $u^*<0,$ and there are no positive equilibrium points.

10. We have $F(u,v)=\left(1-\dfrac{u}{K}\right)-\dfrac{v}{(u^2/\alpha)+u+1},$ and $G(u,v)=\dfrac{\beta u}{(u^2/\alpha)+u+1}-\gamma.$

(i) $(\partial F/\partial v)<0$, (v) $F(0,0)>0$, (vi) $F(0,A)=0, A>0$, (vii) $F(B,0)=0,\quad B=K>0,$ are satisfied. Equality in condition (iii), $(\partial G/\partial v)=0$ is satisfied.

(iv) $u\left(\dfrac{\partial G}{\partial u}\right)+v\left(\dfrac{\partial G}{\partial v}\right)>0,$ gives $u\beta[1-(u^2/\alpha)]>0.$ Hence, we get the condition $u^2<\alpha.$

(ii) $u\left(\dfrac{\partial F}{\partial u}\right)+v\left(\dfrac{\partial F}{\partial v}\right)<0,$ gives (after simplification) the condition

$$vK[(u^2/\alpha)-1]<u[(u^2/\alpha)+u+1]^2.$$

Since $u^2<\alpha,$ the left hand side is negative and the inequality is satisfied as all other quantities are positive.

(viii) $G(C,0)=0,\ C>0,$ gives the quadratic equation for C as $\gamma\, C^2+(\gamma-\beta)\alpha\, C+\alpha\gamma$ $=0.$ The roots of the equation are $2C=[\{(\beta/\gamma)-1\}\pm\alpha\sqrt{\{(\beta/\gamma)-1\}^2-(4/\alpha)}].$

Both the roots are real and positive if $\beta>\gamma,$ and $\{(\beta/\gamma)-1\}^2>(4/\alpha).$

(ix) $B>C$ gives $K>C$, that is $K>$ (larger root of C).

Summarizing, we get the conditions as $u^2<\alpha,\ \ \beta>\gamma,\ \ \{(\beta/\gamma)-1\}^2>(4/\alpha),$

$$K\ >\ 0.5[\{(\beta/\gamma)-1\}+\alpha\sqrt{\{(\beta/\gamma)-1\}^2-(4/\alpha)}].$$

11. A stable equilibrium solution for the given model system exhibited for a typical set of parameter values, $K=1,\ \alpha=3,\ \beta=2.3$ and $\gamma=0.3$ is presented in Fig.2.3. We obtain the non-zero equilibrium solution as $(u^*, v^*) = (0.1511, 0.9836)$.

12. We have $\quad F(X,Z)=A\left(1-\dfrac{X}{K}\right)-\dfrac{BZ}{(D+dX+Z)},\quad G(X,Z)=Z\left(c-\dfrac{w_3}{(X+D_3)}\right).$

Conditions (i), (ii), (v), (vii), (viii) are satisfied.

(iii) $\dfrac{\partial G}{\partial Z}<0,$ gives $c-\dfrac{w_3}{X+D_3}<0,$ or $X<\dfrac{w_3-cD_3}{c}.$

Since $X>0,$ we obtain the condition $c<(w_3/D_3).$

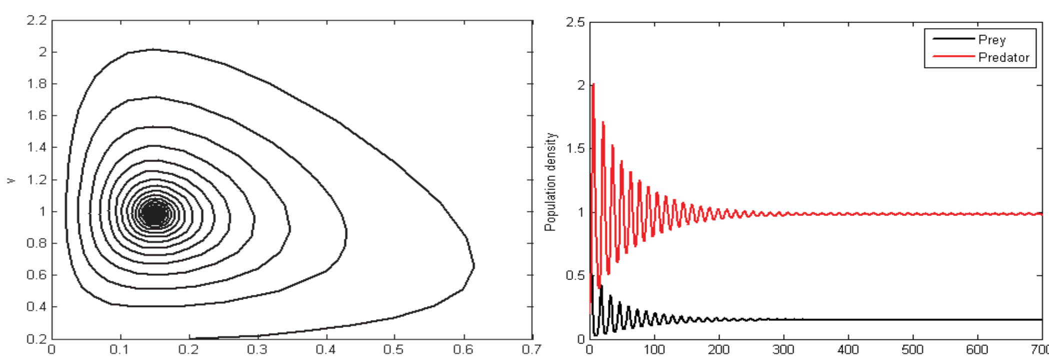


Fig.2.3. Phase plot and time series for the model system (2.75)-(2.76) for $K=1$, $\alpha=3,\ \beta=2.3$ and $\gamma=0.3.$

(iv) $X\left(\dfrac{\partial G}{\partial X}\right)+Z\left(\dfrac{\partial G}{\partial Z}\right)>0,$ gives $\quad X\left[\dfrac{w_3 Z}{(X+D_3)^2}\right]+Z\left[c-\dfrac{w_3}{X+D_3}\right]>0\,.$

A sufficient condition is $\ Z\left[c-\dfrac{w_3 D_3}{(X+D_3)^2}\right]>0,$ or $\ c>\dfrac{w_3 D_3}{(X+D_3)^2}\,.$

(vi) From $F(0, A^*) = 0$, we obtain $A^* = AD/(B - A)$. The condition $A^* > 0$, gives the requirement $B > A$.

(ix) The condition $B^* > C^*$, gives $K > C^*$. From (iii), we have $C^* < [(w_3 - cD_3)/c]$. We may choose $K > [(w_3 - cD_3)/c]$.

Summarizing the results, Kolmogorov theorem gives the following conditions.

(a) Combining (iii), and (iv), we get $\dfrac{w_3 D_3}{(X + D_3)^2} < c < \dfrac{w_3}{(X + D_3)}$.

If X_m is the maximum value of X, we can choose $c < w_3/(X_m + D_3)$.

(b) $B > A$, (c) $K > [(w_3 - cD_3)/c]$.

13. We have $F(Z,U) = A\left(1 - \dfrac{Z}{K_1}\right) - \dfrac{w_3 U}{(Z + D_3)}$, and $G(Z,U) = c - \dfrac{w_4 U}{Z}$.

Conditions (i), (ii), (iii), (v), (vi) are satisfied. Equality condition in (iv) is satisfied. The requirement (vii) gives $B^* = K_1 > 0$. The requirement (ix) gives $K_1 > C^*$. But condition (viii) is violated. We obtain $G(C^*, 0) = c$. The condition $G(C, 0) = 0$, $C > 0$ is violated since $c \neq 0$. Hence, Kolmogorov theorem cannot be applied.

14. The oscillatory predator-prey dynamics exhibited by Holling-Tanner model (2.84), (2.85) for the given set of parameter values is given in Fig.2.4.

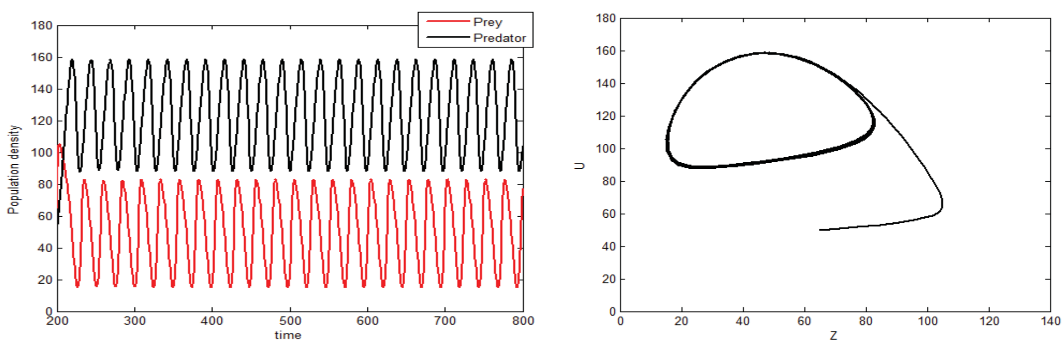


Fig.2.4. Oscillatory predator-prey dynamics exhibited by Holling-Tanner model.

15. Note that (0, 0) is not an equilibrium point. (K, 0) is an equilibrium point. The second equation gives $U = cZ/w_4$. Substituting in the first equation, we get

$$\frac{A}{K}(K - Z) - \frac{w_3 c Z}{\alpha_1 w_4 + \beta_1 c Z + w_4 \gamma_1 Z} = 0,$$

or $\quad AK\alpha_1 w_4 + Z[AK(\beta_1 c + w_4\gamma_1) - A\alpha_1 w_4 - w_3 K c] - AZ^2(\beta_1 c + w_4 \gamma_1) = 0,$

or $\quad Z^2 + (p - K)Z - q = 0, \quad$ where $p = \dfrac{\alpha_1 w_4 + (w_3 K c/A)}{(\beta_1 c + w_4 \gamma_1)}, q = \dfrac{K\alpha_1 w_4}{(\beta_1 c + w_4 \gamma_1)}$.

Irrespective of the sign of $(p - K)$, the positive root is given by

$$2Z^* = \left[-(p - K) + \sqrt{(p - K)^2 + 4q}\right].$$ We have $U^* = cZ^*/w_4$.

An oscillatory predator-prey dynamics exhibited by the model system for the given set of parameter values, $A = 2$, $K = 100$, $w_3 = 2.1$, $\alpha_1 = 0.45$, $\beta_1 = 0.2$, $\gamma_1 = 0.6$, $c = 0.95$ and $w_4 = 1.65$, is presented in Fig. 2.5.

16. We have $F(Z,U) = A\left(1-\dfrac{Z}{K}\right) - \dfrac{w_3 U}{(\alpha_1+\beta_1 U+\gamma_1 Z)}$, and $G(Z,U) = c - \dfrac{w_4 U}{Z}$.

Conditions (i), (ii), (iii), (v), (vii) are satisfied. Equality condition in (iv) is satisfied.

(vi) $F(0, A^*) = 0$, gives $A^* = \dfrac{A\alpha_1}{(w_3 - A\beta_1)} > 0$, if $w_3 > A\beta_1$.

(viii) $G(C,0) = c \neq 0$. The condition is not satisfied. (ix) $B > C$ is also not satisfied. Hence, Kolmogorov theorem cannot be applied.

17. The equilibrium point is $X_1^* = (K_1 - K_2 b_1)/(1-b_1 b_2)$, $X_2^* = (K_2 - K_1 b_2)/(1-b_1 b_2)$. Since $X_1^* > 0$, and $X_2^* > 0$, we obtain the conditions $b_1 < (K_1/K_2) < (1/b_2)$, and $b_1 b_2 < 1$. The second condition is implied in the first condition. (Positivity holds also when the inequalities are reversed). The elements of the Jacobian matrix are (dropping the superfix *)

$$a_{11} = (r_1/K_1)[K_1 - 2X_1 - b_1 X_2] = -(r_1 X_1/K_1),\ a_{12} = -(r_1 b_1 X_1)/K_1,$$
$$a_{21} = -(r_2 b_2 X_2)/K_2,\ a_{22} = (r_2/K_2)[K_2 - b_2 X_1 - 2X_2] = -(r_2 X_2/K_2).$$

The characteristic equation is

$$\lambda^2 + \lambda[(r_1 X_1/K_1) + (r_2 X_2)/K_2] + [(r_1 X_1/K_1)(r_2 X_2)/K_2](1-b_1 b_2) = 0.$$

Applying the Routh-Hurwitz criterion, we find that the positive equilibrium point is asymptotically stable when $b_1 b_2 < 1$. The required condition is $b_1 < (K_1/K_2) < (1/b_2)$.

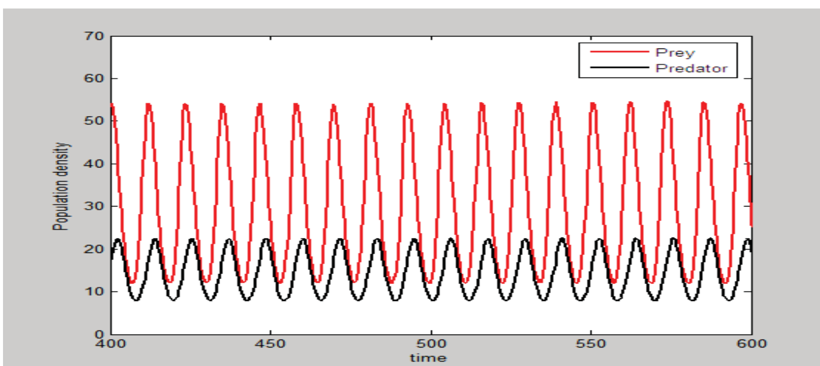


Fig.2.5. Time-series displaying oscillatory predator-prey dynamics exhibited by the modified *HT* model (2.90)-(2.91).

18. The positive equilibrium point $E^*(X_1^*,\ X_2^*)$ is the solution of the equations

$$a_1 - b_1 t_1 - c_1 t_2 = 0, \text{ and } a_2 - b_2 t_1 - c_2 t_2 = 0, \text{ where } t_1 = \ln X_1,\ t_2 = \ln X_2.$$

We obtain $t_1 = \dfrac{a_1 c_2 - a_2 c_1}{b_1 c_2 - b_2 c_1}$, $t_2 = \dfrac{b_1 a_2 - b_2 a_1}{b_1 c_2 - b_2 c_1}$, $X_1 = e^{t_1}$, $X_2 = e^{t_2}$.

The elements of the Jacobian matrix are

$$a_{11} = a_1 - b_1(1+\ln X_1) - c_1 \ln X_2 = -b_1,\ a_{12} = -c_1 X_1/X_2,$$
$$a_{21} = -b_2 X_2/X_1,\ a_{22} = a_2 - b_2 \ln X_1 - c_2(1+\ln X_2) = -c_2.$$

The characteristic equation is $\lambda^2 + \lambda(b_1 + c_2) + (b_1 c_2 - c_1 b_2) = 0.$

Routh-Hurwitz criterion gives the necessary and sufficient conditions for the roots to be negative or have negative real parts, as $b_1 + c_2 > 0,$ and $b_1c_2 - c_1b_2 > 0.$ The positive equilibrium point is asymptotically stable when $(b_1 / b_2) > (c_1 / c_2).$

For the given set of parameter values $(b_1 / b_2) = 3/4,$ and $(c_1 / c_2) = 2/3.$ The condition is satisfied and the equilibrium point is asymptotically stable. The equilibrium point is $(X_1^*, X_2^*) = (e^7, e^{-5}).$

19. For $r = 1.5,\ \alpha = 3,$ we get $(1/3) < u^* < 1.$ u^* is a solution of

$$f(u^*) = 1 + \frac{r(u^*-1)}{\alpha u^*} - e^{r(u^*-1)} = 1.5 - \frac{1}{2u^*} - e^{1.5(u^*-1)} = 0.$$

Newton-Raphson's method applied with the initial approximation taken as 0.5, gives the sequence of iterates as 0.478602777, 0.480756737, 0.47971041, 0.480048641, 0.48004895. With $u^* = 0.48004895,$ we get $v^* = r(1-u^*) = 0.779926575.$ We obtain

$$p = 4.5u^* - 1.5 = 0.660220275,\ A + B + C = r(1 - u^* p) = 1.0245929 > 0,$$
$$A - B + C = 2 - r + u^*(2\alpha - rp) = 2.904886625 > 0,$$
$$A - C = 1 - u^*(\alpha - rp) = 0.035260224 > 0.$$

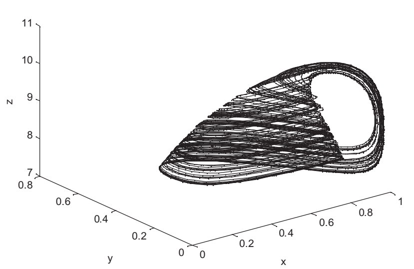
By Miller's theorem or Jury test, the equilibrium point is asymptotically stable.

20. The equilibrium points are obtained as (0, 0), and (4/9, 5/3). The elements of the Jacobian matrix J are $a_{11} = 2.5 - 3N - 0.5P,\ a_{12} = -0.5N,\ a_{21} = 1.8P,\ a_{22} = 0.2 + 1.8N.$ At (0, 0), the eigen values of J are 2.5 and 0.2. The system is unstable. At (4/9, 5/3), the eigen values of J are 1.0 and 1/3. The system is unstable.

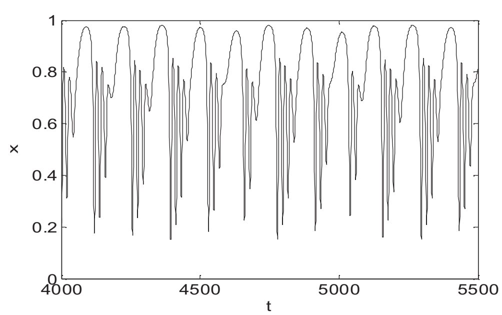
Chapter 3

Exercise 3.1

1. MATLAB 7.0 is used to compute the phase plane diagram to generate the chaotic attractor and time series. Chaotic attractor and the temporal evolution for (i) *t* vs *x,* (ii) *t* vs *y,* (iii) *t* vs *z* are plotted in Figs. 3.1 (a), (b), (c) and (d).



(a)



(b)