1.23. The statement $\exists x, Q(x)$ says that there is some value of x for which $P(x)$ is true. The statement $\sim \forall \sim Q(x)$ says that it is not true that $Q(x)$ fails for all x. These say the same thing.

1.24. The statement $\forall x \exists y, y > x$ says that, for any x, there is a y that is greater than x. In the real numbers, for instance, this is true. The statement $\exists y \forall x, y > x$ says that there is a y that is greater than all x. In the real numbers, for instance, this is false. So the two statements are quite different, and we cannot commute the quantifiers.

Chapter 2: Methods of Proof

2.1. If m, n are odd then, by definition, $m = 2k+1$ and $n = 2\ell+1$ for some $k, \ell \in \mathbb{Z}$. Multiplying, we obtain:

$$m \cdot n = (2k+1) \cdot (2\ell+1) = 2(2k\ell + k + \ell) + 1,$$

which is odd by definition.

2.2. If n is even then, by definition, $n = 2\ell$ for some $\ell \in \mathbb{Z}$. Multiplying, we obtain:

$$m \cdot n = m(2\ell) = 2(m\ell),$$

which is even by definition.

2.4. Writing things out, we obtain

$$\begin{aligned} 2 + 4 + \cdots + (2k-2) + 2k &= 2\left(1 + 2 + \cdots + (k-1) + k\right) \\ &= 2\,\frac{k(k+1)}{2} \qquad \text{by Prop. 2.4.2} \\ &= k^2 + k. \end{aligned}$$

2.5. Writing things out, we obtain

$$\begin{aligned}
&1+3+\cdots+(2k-3)+(2k-1)\\
&\quad= (2-1)+(4-1)+\cdots+((2k-2)-1)+(2k-1)\\
&\quad= (2+4\cdots+(2k-2)+2k)-k\\
&\quad= (k^2+k)-k \qquad \text{by Exercise 2.4}\\
&\quad= k^2.
\end{aligned}$$

2.6. Since, for $n \geq 1$, we have $2n \geq n+1$ mailboxes, we can apply Theorem 2.3.3 and conclude that some mailbox contains at least two pieces of mail. Exhausting all the posibilities, those two pieces of mail must be either both blue, both red, or one blue and one red.

2.7. We can write $m = 3^k$ and $n = 3^\ell$ for some $k, \ell \in \mathbb{N}$. Let us assume, without loss of generality, that $k \leq \ell$. Seeking a contradiction, assume that $m+n = 3^r$, for some $r \in \mathbb{N}$. Then we have:

$$m+n = 3^\ell + 3^k = 3^k(1+3^{\ell-k}) = 3^r, \quad \text{with} \quad \ell - k \geq 0.$$

Now note that since $3^k = m < m+n = 3^r$, it must be that $k < r$. Hence,

$$1+3^{\ell-k} = 3^{r-k}, \quad \text{with} \quad r-k \geq 0.$$

The right hand side of the last equality is either 1 or divisible by 3, whereas the left hand side is bigger than or equal to 2 and definitely not divisible by three. Thus, the equality must be false, which means that our original hypothesis $m+n=3^r$, for some $r \in \mathbb{N}$) was false.

2.8. Nothing is really special about 3. The same proof would work for any other number except 2. It does not work for 2 because if we take $l = k$ and $r = k+1$, the last equality above would read:

$$1+2^0 = 2^1,$$

which is definitely true, so we would not arrive to any contradiction. In fact, for $l = k$ and $r = k+1$, we always have $2^k + 2^l = 2^r$, so Exercise 2.7 does not apply.

2.11. Following the given scheme, we have $2q^2 = p^2$. If q has, say, r prime factors, then q^2 has $2r$ prime factors. Thus $2q^2$ has $2r+1$ prime factors. On the other hand, p^2 must have an even number of prime factors, and we arrive at a contradiction. Hence our original assumption $\sqrt{2} = p/q$ must be false.

2.12. Write $n = k^2$, and suppose that $n + 1 = \ell^2$ for some $l \in \mathbb{N}$. Then $1 = (n+1) - n = \ell^2 - k^2 = (\ell + k)(\ell - k)$. Then $\ell + k = 1$, and that is impossible for natural numbers ℓ and k.

2.13. We proved in Exercise 2.1 that the product of two odd numbers is odd. Therefore, if the product of two numbers is even, at least one of them must be even.

2.14. From Exercise 2.2, we know that if either m or n is even, then $n \cdot m$ must also be even regardless of the parity of the other number. Thus, for $n \cdot m$ to be odd, we must have that both numbers are odd.

2.15. If n is even, then $n-1$ is odd. Now, $n = (n-1)+1$, so n is the sum of two odds. If n is odd, then $n = n$, so n is the sum of one odd integer.

2.17. False: $1^2 + 2^2 = 5$, which is not a perfect square.

2.18. True: Suppose, seeking a contradiction, that there are no perfect squares in that list. This means that all the numbers in the list fall between two consecutive squares, i.e. there is k such that $k^2 < n < n+1 < \ldots < 2n+2 < (k+1)^2$. We then have:

$$2n + 2 < (k+1)^2 = k^2 + 2k + 1 < n + 2k + 1,$$

or

$$n + 1 < 2k.$$

Squaring both sides we obtain:

$$n^2 + 2n + 1 < 4k^2 < 4n,$$

or

$$(n-1)^2 = n^2 - 2n + 1 < 0,$$

which is impossible.

2.19. True: 6=1+2+3, or 28=1+2+4+7+14. In fact, these numbers have a name: Perfect numbers. Not much is known about perfect numbers. It is conjectured that there are no odd perfect numbers, mainly because nobody ever found an odd perfect number, but it has never been proved. It has also not been proved that there are infinitely many perfect numbers.

2.20. False: $2^2 - 1^2 = 3$, which is a prime.

2.21. False: $2^2 + 1^2 = 5$, which is a prime.

2.22. True: If $x > 0$ we have:

$$0 < 2x \quad \Leftrightarrow \quad x^2 + 1 < x^2 + 2x + 1 = (1 + x)^2.$$

2.23. False: Take $n = 2, a_1 = 1, a_2 = 4$. Then the inequality would read:

$$\frac{5}{2} = \frac{1+4}{2} \leq (1 \cdot 4)^{1/2} = 2,$$

which is clearly false.

2.24. True: Write the decimal expansion of these two numbers. Since the numbers are different, there will be a first digit in the decimal expansion which does not coincide. Take the highest of the two numbers and truncate its decimal expansion right after this digit. The number we obtain is rational (since its decimal expansion has finite length) and it clearly lies between the two numbers.

2.25. True: Write the rationals as fractions p_1/q, p_2/q with the same denominator q so that $p_1 + 2 \leq p_2$ (one can always achieve this by taking q big enough). Then either $\sqrt{{p_1}^2 + 1}$ or $\sqrt{{p_1}^2 + 2}$ is an irrational number that lies between p_1 and p_2 (cf. Exercise 2.12). Divide this irrational by q to obtain another irrational that lies between the two rationals.

2.26. True: We have to prove that $k^2 < m < n < (k+1)^2$ cannot happen. In other words, writing $m = \ell^3$, we want to prove that if $k^2 < \ell^3$, then $(k+1)^2 < (\ell+1)^3$. This is the same as

$$k^2 + 2k + 1 < \ell^3 + 3\ell^2 + 3\ell + 1,$$

or

$$k^2 + 2k < \ell^3 + 3\ell^2 + 3\ell.$$

Since $k^2 < \ell^3$ and $0 < 3\ell$, it sufiices to show

$$2k < 3\ell^2.$$

To prove this, note that since $k^2 < \ell^3$, we have $4k^2 < 4\ell^3 < 9\ell^4$. Taking square roots in both ends of the last equation we obtain the desired inequality.

2.27. We will use the alternative form of the principle of complete induction given in the text (see also Exercise 2.31). The property is clearly true for 2 (since 2 is prime). Assume that it is true for any $k < n$. We have to prove that it is true for $k = n$. If the only divisors of n are n itself and 1, we are done, since that would imply that n is itself a prime. Otherwise n has a divisor $d < n$. By the induction hypothesis, d must have some prime factor p. Now, since p divides d which divides n, p must also divide n. But then p will be a factor of n. Hence the property is also true for n.

2.28. For $k = 3$, we know that the property is true. Suppose that the property is true for $k = n - 1$. We need to prove that it is true for $k = n$. Take three consecutive vertices A, B, C of your n-gon and join A with C by a segment. This segment separates the n-gon in an $(n-1)$-gon and a triangle. Note that the sum of the interior angles of the $(n-1)$-gon plus the sum of the interior angles of the triangle equals the sum of the interior angles of the original n-gon. By the induction hipothesis, the sum of the interior angles of the $(n-1)$-gon is $((n-1)-2) \cdot 180°$. Thus, for the n-gon, we will obtain:

$$((n-1)-2) \cdot 180° + 180° = (n-2) \cdot 180°,$$

which is what we wanted to prove.

2.29. The property is true for $k = 3$, since $2^3 = 8 > 7 = 1 + 2 \cdot 3$. Assume that the property is true for $k = n - 1$. We want to show that it is true for $k = n$. In other words, we want to prove that

$$2^n > 1 + 2n.$$

Observing that $2^n = 2 \cdot 2^{n-1}$, we find an obvious place to apply the induction hypothesis:

$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &> 2 \cdot (1 + 2(n-1)) \quad \text{from the induction hypothesis} \\ &= (1+2n) + (2n-3) \\ &> 1 + 2n \quad \text{since } 2n - 3 > 0 \text{ for } n \geq 2. \end{aligned}$$

2.31. If one starts the induction process from a number $n_0 + 1$, $n_0 \geq 1$, then $P(1)$ might not be true (it might not even be defined), and we would not be able to use induction according to the statement in the text. But we can modify this statement using the following trick:

Define a property P' as follows:

$$P'(k) \text{ is true} \quad \Leftrightarrow \quad P(n_0 + k) \text{ is true} .$$

Then $P'(1)$ is true since $P(n_0 + 1)$ is true, and $P'(n-1) \Rightarrow P'(n)$ because $P(n_0 + n - 1) \Rightarrow P(n_0 + n)$ by hypothesis. But now we can apply induction (as stated in the text) to the property P', so that $P'(n)$ holds for any natural number n. This implies that $P(n_0 + n)$ holds for all $n \in \mathbb{N}$, or equivalently, $P(m)$ holds for all $n \geq n_0 + 1$.

2.32. For $n = 1$, we have:

$$1 + \frac{q}{1-q} = \frac{1-q+q}{1-q} = \frac{1}{1-q} \ ,$$

so the formula holds for $n = 1$. Assume that the formula is true for $n - 1$. We want to prove that it is true for n. Then we have:

$$\begin{aligned} 1 \ &+ \ \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} + \frac{q^3}{(1-q)(1-q^2)(1-q^3)} + \cdots \\ &+ \ \frac{q^{n-1}}{(1-q)(1-q^2)\cdots(1-q^{n-1})} + \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= \ \frac{1}{(1-q)(1-q^2)\cdots(1-q^{n-1})} + \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= \ \frac{1-q^n+q^n}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= \ \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \, , \end{aligned}$$

where we used the induction hypothesis in the first equality.

2.34. It is true for $n = 5$:

$$2^5 = 32 > 26 = 5^2 + 1.$$

Assume that it is true for $n - 1 \geq 5$. We must prove it for n. We can write:

$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &> 2((n-1)^2 + 1) \quad \text{by the induction hypothesis} \\ &= n^2 + 1 + (n^2 - 4n + 3) \\ &= n^2 + 1 + (n^2 - 4n + 4) - 1 \\ &= n^2 + 1 + (n-2)^2 - 1 \\ &> n^2 + 1 \quad \text{since } (n-2)^2 - 1 > 0 \text{ if } n > 4. \end{aligned}$$

2.35. Let $P(n)$ be the statement, "If $n + 1$ letters are placed into n mailboxes then some mailbox must contain two letters. When $n = 1$ the claim is that if we put two letters into one mailbox then some mailbox must contain two letters. Obvious. Now suppose that $P(n-1)$ has been proved. We have n mailboxes and we place $n + 1$ letters into n mailboxes. If the last mailbox contains two letters then we are done. If not, then the last box contains one or two letters. But then the first $n - 1$ mailboxes contain at least n letters. So the inductive hypothesis applies and one of them must contain two letters. That completes the inductive step, and the proof.

2.36. Assume that we have n mailboxes. Let $\ell(j)$ be the number of letters in box j. Now

$$\ell(1) + \ell(2) + \cdots + \ell(n) = n + 1\,,$$

since all the letters taken together total $n + 1$ letters. Dividing by n gives

$$\frac{\ell(1) + \ell(2) + \cdots + \ell(n)}{n} = \frac{n+1}{n} > 1\,.$$

So the average number of letters per box exceeds 1. This can only be true if some box contains more than 1 letter. Thus some box contains two letters.

2.37. It cannot be that just one letter is in the wrong envelope. Because then the letter that *should have been in that envelope* is also in the wrong envelope. So at least two letters would be in the wrong envelope. So the probability is 0.

2.38. Consider the set S all ordered pairs (ℓ, p) where ℓ is a line passing through (at least) two of the given points and p is a point not on that line (certainly p exists because the points are not all colinear). Define a function f on S by

$$f(\ell, p) = \text{distance of } \ell \text{ to } p.$$

Then f is a function with a finite domain, so there is a particular ordered pair (ℓ_0, p_0) which minimizes the function. Then ℓ_0 is the line that we seek. We invite the reader to check cases to verify this assertion.

2.39. Draw a pair of coordinate axes. Now draw a "smallest possible rectangle", with sides parallel to the coordinate axes, that contains the string. Let the lengths of the sides of the rectangle be a and b. Then, using the fact that a line is a distance-minimizing curve in the plane, we can see that $a + b \leq 1$. The rectangle of greatest area with this constraint on its perimeter is the square of side 1/2. It has area 1/4.

2.40. Let

$$B(x) = x \text{ is a boy under the age of 10.}$$

and

$$P(x) = x \text{ practices all pieces in his/her piano book every day.}$$

Then our statement is

$$\forall x, B(x) \Rightarrow P(x)\,.$$

We can rewrite this as

$$\sim \exists x, \sim (B(x) \Rightarrow P(x))\,.$$

2.41. The assertion is true for $n = 1$ by inspection.

Assume now that the assertion is verified for $n = j$. Then we have

$$\begin{aligned}\begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix}^{j+1} &= \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix}^{j} \\ &= \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a^j & 2ja^{j-1} \\ 0 & a^j \end{pmatrix} \\ &= \begin{pmatrix} a^{j+1} & 2(j+1)a^j \\ 0 & a^{j+1} \end{pmatrix}.\end{aligned}$$

That completes the inductive step.

2.42. The assertion is clear for $n = 1$. Now assume that it is true for $n = j$. We write

$$\begin{aligned}(j+1)^3 - (j+1) &= \left(j^3 + 3j^2 + 3j + 1\right) - (j+1) \\ &= j^3 + 3j^2 + 2j \\ &= (j^3 - j) + (3j^2 + 3j)\,.\end{aligned}$$

Now, by the inductive hypothesis, $j^3 - j$ is divisible by 6. Also

$$3j^2 + 3j = 3j(j+1)\,.$$

Since either j or $j + 1$ is divisible by 2, this last expression is also divisible by 6. Hence $(j + 1)^3 - (j + 1)$ is divisible by 6, and the induction is complete.

2.44. We will prove that, for any positive integer n,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}\,.$$

The claim is plainly true for $n = 1$. Now assume that it has been established for $n = j$. Then we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{j}} + \frac{1}{\sqrt{j+1}} \geq \sqrt{j} + \frac{1}{\sqrt{j+1}}\,.$$

So we need to show that

$$\sqrt{j} + \frac{1}{\sqrt{j+1}} \geq \sqrt{j+1}\,.$$

Multiplying both sides by $\sqrt{j+1}$, we see that this is the same as

$$\sqrt{j(j+1)} + 1 \geq j + 1$$

or

$$\sqrt{j(j+1)} \geq j\,.$$

Now squaring both sides gives the result. The induction is complete, and the result proved.

Chapter 3: Set Theory

3.1. (a) $S \cap U = \{1, 2, 3, 4\}$.

(c) $(S \cup U) \cap V = \{2, 4\}$.

(e) $(U \cup V \cup T) \setminus S = \{6, 7, 8, 9\}$.

(g) $(S \times V) \setminus (T \times U) = \{(1, 2), (1, 4), (1, 6), (1, 8), (2, 2), (2, 4), (2, 6), (2, 8), (3, 6), (3, 8), (4, 6), (4, 8), (5, 6), (5, 8)\}$.

3.2. It is empty: If $a \in S \times T$, then $a = (x, y)$, with $x \in S$ and $y \in T$. But T has no elements, so such a y cannot exist, so such an a cannot exist either.

3.4. (a) **$\subseteq$:** If $x \in {}^c(\cap_{\alpha \in A} S_\alpha)$, then $x \notin \cap_{\alpha \in A} S_\alpha$, which implies $\exists \alpha_0$ such that $x \notin S_{\alpha_0}$. But this means that $x \in {}^cS_{\alpha_0}$, so certainly $x \in \cup_{\alpha \in A} {}^cS_\alpha$.

$\supseteq$: If $x \in \cup_{\alpha \in A} {}^cS_\alpha$, then $\exists \alpha_0$ such that $x \in {}^cS_{\alpha_0}$. Thus, $x \notin S_{\alpha_0}$, so $x \notin \cap_{\alpha \in A} S_\alpha$. Therefore, $x \in {}^c(\cap_{\alpha \in A} S_\alpha)$.

(c) $\subseteq$: If $x \in T \cap (\cup_{\alpha \in A} S_\alpha)$ then $x \in T$ and $x \in \cup_{\alpha \in A} S_\alpha$. This implies that $x \in T$ and $x \in S_{\alpha_0}$ for some $\alpha_0 \in A$. So $x \in T \cap S_{\alpha_0}$, which implies $x \in \cup_{\alpha \in A}(T \cap S_\alpha)$.

$\supseteq$: If $x \in \cup_{\alpha \in A}(T \cap S_\alpha)$ then $x \in T \cap S_{\alpha_0}$ for some $\alpha_0 \in A$. Thus, $x \in T$ and $x \in S_{\alpha_0}$. Therefore, $x \in T$ and $x \in \cup_{\alpha \in A} S_\alpha$, so $x \in T \cap (\cup_{\alpha \in A} S_\alpha)$.