Solutions to Exercises in Chapter 2

Solution to Exercise 2.1.

(a) At time $T = 1$ the terminal values of the bond and the stock are, respectively,

$$B_T = B_0(1+r) = 100(1+0.05) = 105$$
$$S_T = S_0(1+r_S) = \begin{cases} 100(1+0.1) = 110 & \text{with probability } 60\% \\ 100(1-0.05) = 95 & \text{with probability } 40\% \end{cases}$$

The terminal value of portfolio (x, y) is

$$\Pi_T = xS_T + yB_T = \begin{cases} 110x + 105y & \text{with probability } 60\% \\ 95x + 105y & \text{with probability } 40\% \end{cases}$$

The positions x and y of the target portfolio satisfy the following system of equations:

$$\begin{cases} 110x + 105y = 1000 \\ 95x + 105y = 1500 \end{cases} \implies \begin{cases} x = -\frac{100}{3} \cong -33.33 \\ y = \frac{400}{9} \cong 44.44 \end{cases}$$

(b) The initial value of this portfolio is

$$\Pi_0 = xS_0 + yB_0 = -\frac{100}{3} \cdot 100 + \frac{400}{9} \cdot 100 = \frac{10000}{9} \doteq \$1{,}111.11.$$

Therefore, the expected return is

$$\mathrm{E}[R_V] = \mathrm{E}\left[\frac{V_T}{V_0}\right] = \frac{9}{10} \cdot \frac{3}{5} + \frac{27}{20} \cdot \frac{2}{5} = \frac{54+54}{100} = \frac{108}{100} = 1.08.$$

The expected rate of return is $\mathrm{E}[r_V] = \mathrm{E}[R_V] - 1 = 8\%$.

Solution to Exercise 2.2. Suppose that $X_0 \leqslant Y_0$. Form the following portfolio of assets X and Y: sell short one share of Y, buy one share of X, keep the proceeds $Y_0 - X_0 \geqslant 0$ in cash. The total initial value of this portfolio is zero. At time T, sell the share of X, buy and return one share of Y. The final wealth, $X_T(\omega) - Y_T(\omega) + Y_0 - X_0$, is nonnegative for all $\omega \in \Omega$. In state ω', the wealth is strictly positive. Hence, it is an arbitrage portfolio. We arrive at a contradiction.

Solution to Exercise 2.3. At time 0, take a short position in investment V and a long position in investment W. The cash proceeds are $V_0 - W_0 > 0$. At time T close all positions. The terminal wealth equal to $W_T - V_T + V_0 - W_0$ is strictly positive with probability 1.

Solution to Exercise 2.4.

(a) We deal with a standard binomial model. The annual rates of return on the stock are $r_S^+ = 0.1$ and $r_S^- = -0.05$. The risk-neutral probabilities are

$$\begin{aligned}\tilde{p} = \widetilde{\mathbb{P}}(\text{the stock price goes up}) &= \frac{r - r_S^-}{r_S^+ - r_S^-} = \frac{0.05 - (-0.05)}{0.1 - (-0.05)} = \frac{0.1}{0.15} = \frac{2}{3},\\ 1 - \tilde{p} = \widetilde{\mathbb{P}}(\text{the stock price goes down}) &= 1 - \frac{2}{3} = \frac{1}{3}.\end{aligned}$$

(b) We have

$$\begin{aligned}\widetilde{\mathrm{E}}\left[\frac{S_1}{B_1}\right] &= \frac{S^+}{B_1}\tilde{p} + \frac{S^-}{B_1}(1-\tilde{p}) = \frac{S_0(1+r_S^+)}{B_0(1+r)}\tilde{p} + \frac{S_0(1+r_S^0)}{B_0(1+r)}(1-\tilde{p})\\ &= \frac{110}{105}\cdot\frac{2}{3} + \frac{95}{105}\cdot\frac{1}{3} = \frac{315}{3\cdot 105} = 1 = \frac{100}{100} = \frac{S_0}{B_0}.\end{aligned}$$

Solution to Exercise 2.5.

(a) Let us show that there is no portfolio (x, y) with $\Pi_0 = xS_0 + yB_0 = 50x + 10y = 0$ such that $\Pi_1(\omega^j) = xS_1(\omega^j) + yB_1 \geqslant 0$ holds for all $j = 1, 2, 3$ where at least one inequality is strict. Since $\Pi_0 = 0$, we have $y = -5x$. At time $t = 1$, we have

$$\begin{aligned}\Pi_1(\omega^1) &= 70x + 11y = 70x - 55x = 15x\\ \Pi_1(\omega^2) &= 55x + 11y = 55x - 55x = 0\\ \Pi_1(\omega^3) &= 40x + 11y = 40x - 55x = -15x.\end{aligned}$$

The only solution to the simultaneous inequalities $\Pi_1(\omega^j) \geqslant 0$, $j = 1, 2, 3$ is $x = 0$. Clearly, $(x, y) = (0, 0)$ is not an arbitrage portfolio since $\Pi_0^{(0,0)} = 0$ and $\Pi_1^{(0,0)} \equiv 0$. Therefore, the model is arbitrage free.

(b) The risk-neutral probabilities $\tilde{p}_j > 0$, $j = 1, 2, 3$, solve the following system of equations:

$$\begin{cases} \tilde{p}_1 \frac{S_1(\omega^1)}{B_1} + \tilde{p}_2 \frac{S_1(\omega^2)}{B_1} + \tilde{p}_3 \frac{S_1(\omega^3)}{B_1} = \frac{S_0}{B_0} \\ \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1 \end{cases} \iff \begin{cases} \frac{70}{11}\tilde{p}_1 + 5\tilde{p}_2 + \frac{40}{11}\tilde{p}_3 = 5 \\ \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1 \end{cases}$$

The solution is given by

$$\begin{cases} \tilde{p}_1 = \frac{1}{2} - \frac{t}{2} \\ \tilde{p}_2 = t \\ \tilde{p}_3 = \frac{1}{2} - \frac{t}{2} \end{cases}$$

where $t \in \mathbb{R}$ is a free parameter. The solution is strictly positive iff $t \in (0, 1)$.

Solution to Exercise 2.6. Assume that $S^d < B_T < S^u$ holds. Let us prove that there are no arbitrage opportunities. Consider a portfolio (x, y) with $\Pi_0 = 0$, where $\Pi_t = xS_t + yB_t$ for $t \in \{0, T\}$. If $x = 0$, then $y = 0$ and hence $\Pi_0 = 0$ and $\Pi_T \equiv 0$. Clearly, $(x, y) = (0, 0)$ is not an arbitrage portfolio. Let $x \neq 0$. Then, we have $y = -x\frac{S_0}{B_0} = -x$ since $\Pi_0 = 0$ and $S_0 = B_0$. The terminal value of the portfolio, $\Pi_T = xS_T + yB_T$, is given by

$$\Pi_T = x(S_T - B_T) = \begin{cases} \Pi_T^u \equiv x(S^u - B_T) & \text{with probability } p_1 \\ \Pi_T^m \equiv x(S^m - B_T) & \text{with probability } p_2 \\ \Pi_T^d \equiv x(S^d - B_T) & \text{with probability } p_3 \end{cases}$$

If $x > 0$, then $\Pi_T^u > 0$ and $\Pi_T^d < 0$. If $x < 0$, then $\Pi_T^u < 0$ and $\Pi_T^d > 0$. So there is no arbitrage portfolio for any value of x.

Now assume that $B_T \leqslant S^d$ holds. Clearly the portfolio $(x, y) = (1, -1)$ is an arbitrage portfolio. Indeed, the initial value of this portfolio, Π_0, is zero, the terminal value Π_T is nonnegative with probability one. Moreover, Π_T is strictly positive with nonzero probability since $\Pi_T^u > 0$. If $B_T \geqslant S^u$ holds, then $(x, y) = (-1, 1)$ is an arbitrage portfolio.

Solution to Exercise 2.7.

(a) This two-period model consists of three single-period binomial sub-models: one sub-model for the time interval $[0, 1]$ and two sub-models for the time interval $[1, 2]$. Consider a single-period binomial sub-model generated by scenarios ω^3 and ω^4 from time 1 to time 2. At time 1, the stock is priced at 40. The stock price remains the same (if ω^3 occurs) or changes from 45 to 40 (if ω^4 occurs). The return on the stock for the time interval $[1, 2]$ is given by

$$R_2^S(\omega) = \begin{cases} 0 & \text{if } \omega = \omega^3 \\ -\frac{1}{9} & \text{if } \omega = \omega^4 \end{cases}$$

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The return on the risk-free bond for the interval $[1, 2]$ is equal to $R_2^B = \frac{60-55}{55} = \frac{1}{11}$. Since the latter is greater than the return on the stock, the sub-model admits arbitrage. One the other hand, the other two single-period binomial sub-models are arbitrage free. Therefore, we can construct the following arbitrage strategy $\{(x_t, y_t)\}_{t=0,1}$. At time 0, set $x_0 = 0$ and $y_0 = 0$. The portfolio value is zero over the time interval $[0, 1]$: $\Pi_0 = \Pi_1(\omega) = 0$ for all ω. At time 1, we have two cases. If $S_1 = 60$, i.e. $\omega \in \{\omega^1, \omega^2\}$, then we set $x_1 = y_1 = 0$. If $S_1 = 45$, i.e. $\omega \in \{\omega^3, \omega^4\}$, then we sell short the stock: $x_1 = -1$. In the latter case, to ensure that the self-financing condition holds, we invest the proceeds in the bank account: $y_1 = \frac{9}{11}$. As a result, we have the following strategy:

$$\begin{aligned} &x_0 = 0, \quad x_1(\omega^1) = x_1(\omega^2) = 0, \quad x_1(\omega^3) = x_1(\omega^4) = -1; \\ &y_0 = 0, \quad y_1(\omega^1) = y_1(\omega^2) = 0, \quad y_1(\omega^3) = y_1(\omega^4) = \frac{9}{11}. \end{aligned}$$

The portfolio value process is as follows:

$$\Pi_0 = 0, \quad \Pi_1 \equiv 0, \quad \Pi_2(\omega) = \begin{cases} 0 & \text{if } \omega \in \{\omega^1, \omega^2\}, \\ \frac{45}{11} & \text{if } \omega = \omega^3, \\ \frac{100}{11} & \text{if } \omega = \omega^4. \end{cases}$$

This is an arbitrage strategy.

(b) If no short selling of the risky stock is allowed, then we should have $x_0 \geqslant 0$ and $x_1(\omega) \geqslant 0$ for all ω. Therefore, the arbitrage strategy constructed in (a) is not allowed. Clearly, an arbitrage opportunity may arise iff $S_1 = 45$. As follows from the proof of the arbitrage theorem for the binomial model, only a strategy with a short position in stock can produce an arbitrage profit.

Solution to Exercise 2.8. Suppose that transaction costs of 5% apply whenever stock is purchased or sold. Consider an investor who follows the strategy constructed in Solution 2.7. At time 1, one share of stock is sold short if $S_1 = 45$. The transaction costs of $\$45 \cdot 0.05 = \$2.25$ is paid. If the remaining amount of $45 - 2.25 = 42.75$ dollars were invested risk-free, it would worth $42.75 \cdot \frac{60}{55} \cong 46.64$ at time 2. However, closing the short position in stock would cost $\$45 \cdot (1 + 0.05) = \$47.25$ (including the transaction costs of 5%) in scenario ω^3, making the final wealth negative. Thus, there does not exists an arbitrage strategy.

Solution to Exercise 2.9. Given the bond and stock prices in Exercise 2.7, except that

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$S_2(\omega^2) = S_2(\omega^3) = 50$, we have

$$\begin{aligned}
\mathrm{E}\left[\frac{S_1}{B_1}\right] &= \frac{60}{55}\mathbb{P}(S_1 = 60) + \frac{45}{55}\mathbb{P}(S_1 = 45) \\
&= \frac{60}{55}\left(\mathbb{P}(\omega^1) + \mathbb{P}(\omega^2)\right) + \frac{45}{55}\left(\mathbb{P}(\omega^3) + \mathbb{P}(\omega^4)\right) \\
&= \frac{60}{55}\left(\frac{17}{33} + \frac{5}{33}\right) + \frac{45}{55}\left(\frac{10}{33} + \frac{1}{33}\right) \\
&= \frac{60 \cdot 22 + 45 \cdot 11}{33 \cdot 55} = 1 = \frac{S_0}{B_0}, \\
\mathrm{E}\left[\frac{S_2}{B_2} \mid S_1 = 60\right] &= \frac{70}{60}\mathbb{P}(S_2 = 70 \mid S_1 = 60) + \frac{50}{60}\mathbb{P}(S_2 = 50 \mid S_1 = 60) \\
&= \frac{70}{60}\left(\frac{\mathbb{P}(\omega^1)}{\mathbb{P}(\omega^1) + \mathbb{P}(\omega^2)}\right) + \frac{50}{60}\left(\frac{\mathbb{P}(\omega^2)}{\mathbb{P}(\omega^1) + \mathbb{P}(\omega^2)}\right) \\
&= \frac{70}{60} \cdot \frac{17}{22} + \frac{50}{60} \cdot \frac{5}{22} = \frac{70 \cdot 17 + 50 \cdot 5}{60 \cdot 22} = \frac{60}{55} = \frac{S_1}{B_1}, \\
\mathrm{E}\left[\frac{S_2}{B_2} \mid S_1 = 45\right] &= \frac{50}{60}\mathbb{P}(S_2 = 50 \mid S_1 = 45) + \frac{40}{60}\mathbb{P}(S_2 = 40 \mid S_1 = 45) \\
&= \frac{50}{60}\left(\frac{\mathbb{P}(\omega^3)}{\mathbb{P}(\omega^3) + \mathbb{P}(\omega^4)}\right) + \frac{40}{60}\left(\frac{\mathbb{P}(\omega^4)}{\mathbb{P}(\omega^3) + \mathbb{P}(\omega^4)}\right) \\
&= \frac{50}{60} \cdot \frac{10}{11} + \frac{40}{60} \cdot \frac{1}{11} = \frac{50 \cdot 10 + 40}{60 \cdot 11} = \frac{45}{55} = \frac{S_1}{B_1}.
\end{aligned}$$

Solution to Exercise 2.10. The self-financing condition for one period takes the form:

$$S_1(x_1 - x_0) + B_1(y_1 - y_0) = 0$$

where $x_1 = x$ and $y_1 = y$ are fixed. Let find all possible x_0 and y_0 that satisfy the above condition. The time-1 stock price S_1 takes one of two possible values:

$$S_1 = S_0(1 + r_S^-) = 100 \cdot 0.9 = 90 \quad \text{or} \quad S_1 = S_0(1 + r_S^+) = 100 \cdot 1.2 = 120\,.$$

Hence, x_0 and y_0 solve the system of two linear equations:

$$\begin{cases} 90(x - x_0) + 11(y - y_0) = 0 \\ 120(x - x_0) + 11(y - y_0) = 0 \end{cases} \iff \begin{cases} x = x_0 \\ y = y_0 \end{cases}$$

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Solution to Exercise 2.11. Let x and y denote the number of units of S and B, respectively. We have the following system of equations:

$$\begin{cases} xS_0 + yB_0 = 10000 \\ xS_0 = 2yB_0 \end{cases} \iff \begin{cases} x + y = 100 \\ x = 2y \end{cases} \iff \begin{cases} x = \frac{200}{3} \\ y = \frac{100}{3} \end{cases}$$

At time 1, we have

$$\begin{aligned} \Pi_1(\omega^+) &= xS_1(\omega^+) + yB_1 = \frac{200}{3} \cdot 115 + \frac{100}{3} \cdot 105 \doteq \$11{,}166.67, \\ \Pi_1(\omega^-) &= xS_1(\omega^-) + yB_1 = \frac{200}{3} \cdot 90 + \frac{100}{3} \cdot 105 = \$9500. \end{aligned}$$

Solution to Exercise 2.12. Using the Maclaurin series of $f(x) = \mathrm{e}^x$, we have the following approximation of the probability $\tilde{p}_N$ as a function of $\delta \equiv \delta_N = \frac{T}{N}$:

$$\begin{aligned} \tilde{p}_N &= \frac{(1 + \delta r + \mathcal{O}(\delta^2)) - \left(1 - \sigma\sqrt{\delta} + \frac{\sigma^2\delta}{2} + \mathcal{O}(\delta^{3/2})\right)}{\left(1 + \sigma\sqrt{\delta} + \frac{\sigma^2\delta}{2} + \mathcal{O}(\delta^{3/2})\right) - \left(1 - \sigma\sqrt{\delta} + \frac{\sigma^2\delta}{2} + \mathcal{O}(\delta^{3/2})\right)} \\ &= \frac{\sigma\sqrt{\delta} + (r - \sigma^2/2)\delta + \mathcal{O}(\delta^{3/2})}{2\sigma\sqrt{\delta} + \mathcal{O}(\delta^{3/2})} = \frac{\sigma + (r - \sigma^2/2)\sqrt{\delta} + \mathcal{O}(\delta)}{2\sigma + \mathcal{O}(\delta)}. \end{aligned}$$

As $N \to \infty$, we have that $\delta \to 0$ and $\tilde{p}_N \to \frac{\sigma}{2\sigma} = \frac{1}{2}$. The limiting values of the mathematical expectation and the variance of L_N (calculated under $\widetilde{\mathbb{P}}_N$ with the upward probability $\tilde{p}_N$) are as follows:

$$\begin{aligned} \mathrm{E}_{\widetilde{\mathbb{P}}_N}[L_N] &= (2\tilde{p}_N - 1)\frac{\sigma T}{\sqrt{\delta}} = \left(\frac{(r - \sigma^2/2)\sqrt{\delta} + \mathcal{O}(\delta)}{\sigma + \mathcal{O}(\delta)}\right)\left(\frac{\sigma T}{\sqrt{\delta}}\right) \\ &= \frac{(r - \sigma^2/2)T + \mathcal{O}(\sqrt{\delta})}{1 + \mathcal{O}(\delta)} \to (r - \sigma^2/2)T, \text{ as } \delta \to 0; \\ \mathrm{Var}_{\widetilde{\mathbb{P}}_N}[L_N] &= \tilde{p}_N(1 - \tilde{p}_N)4\sigma^2 T \\ &= \left(\frac{\sigma + (r - \sigma^2/2)\sqrt{\delta} + \mathcal{O}(\delta)}{2\sigma + \mathcal{O}(\delta)}\right)\left(\frac{\sigma - (r - \sigma^2/2)\sqrt{\delta} + \mathcal{O}(\delta)}{2\sigma + \mathcal{O}(\delta)}\right)4\sigma^2 T \\ &= \left(\frac{\sigma^2 + \mathcal{O}(\delta)}{\sigma^2 + \mathcal{O}(\delta)}\right)\sigma^2 T \to \sigma^2 T, \text{ as } \delta \to 0. \end{aligned}$$

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Solution to Exercise 2.13. Using the formula $\mathrm{E}[g(Z)] = \int_{-\infty}^{\infty} g(x) f_Z(x)\,\mathrm{d}x$ with $g(x) = \mathrm{e}^{ax+b}$ and $f_Z(x) = n(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2}$ gives

$$\begin{aligned}
\mathrm{E}\left[\mathrm{e}^{aZ+b}\right] &= \int_{-\infty}^{\infty} \mathrm{e}^{ax+b}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{ax+b-\frac{x^2}{2}}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{a^2}{2}+b-\frac{(x-a)^2}{2}}\,\mathrm{d}x = \mathrm{e}^{\frac{a^2}{2}+b}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-a)^2}{2}}\,\mathrm{d}x = \mathrm{e}^{\frac{a^2}{2}+b},
\end{aligned}$$

where we use the fact that the function $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-a)^2}{2}}$ is the PDF of $Norm(a,1)$ and hence the integral of it equals 1. Now, let us calculate the variance of Z:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \mathrm{E}\left[\mathrm{e}^{2aZ+2b}\right] - \mathrm{E}\left[\mathrm{e}^{aZ+b}\right]^2 \\
&= \mathrm{e}^{\frac{(2a)^2}{2}+2b} - \left(\mathrm{e}^{\frac{a^2}{2}+b}\right)^2 = \mathrm{e}^{2a^2+2b} - \mathrm{e}^{a^2+2b} = \mathrm{e}^{a^2+2b}(\mathrm{e}^{a^2}-1).
\end{aligned}$$

Solution to Exercise 2.14. The probability distribution of $S(5)$ is given by

$$S(5) \stackrel{d}{=} 100\,\mathrm{e}^{0.02\cdot 5+0.2\sqrt{5}Z} = 100\,\mathrm{e}^{0.1+0.2\sqrt{5}Z}, \quad \text{where } Z \sim Norm(0,1).$$

(a) Using the fact that $\mathrm{E}\left[\mathrm{e}^{aZ+b}\right] = \mathrm{e}^{\frac{a^2}{2}+b}$ gives

$$\mathrm{E}[S(5)] = 100\,\mathrm{E}\left[\mathrm{e}^{0.02\cdot 5+0.2\sqrt{5}Z}\right] = 100\,\mathrm{e}^{0.1+(0.2\sqrt{5})^2/2} = 100\,\mathrm{e}^{0.2} \cong 738.905610.$$

(b) Since $\ln(S(5)/100) = 0.1 + 0.2\sqrt{5}Z \sim Norm(0.1, 0.2)$, we have

$$\begin{aligned}
\mathbb{P}(S(5) > 100) &= \mathbb{P}(S(5)/100 > 1) = \mathbb{P}(\ln(S(5)/100) > 0) = \mathbb{P}(0.1 + 0.2\sqrt{5}Z > 0) \\
&= \mathbb{P}(Z > -\sqrt{5}/10) = \mathbb{P}(-Z < \sqrt{5}/10) = \mathcal{N}(\sqrt{5}/10) \cong 0.588468.
\end{aligned}$$

Here, we use the fact that if Z is a standard normal random variable, then so is $-Z$.

(c) We have

$$\begin{aligned}
\mathbb{P}(S(5) < 110) &= \mathbb{P}(S(5)/100 < 1.1) = \mathbb{P}(\ln(S(5)/100) < \ln 1.1) \\
&= \mathbb{P}(0.1 + 0.2\sqrt{5}Z < \ln 1.1) = \mathbb{P}(Z < (\ln 1.1 - 0.1)\sqrt{5}) \\
&= \mathcal{N}((\ln 1.1 - 0.1)\sqrt{5}) \cong 0.495816.
\end{aligned}$$

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Solution to Exercise 2.15. As shown in Exercise 2.13, $\mathrm{E}\left[\mathrm{e}^{aZ+b}\right] = \mathrm{e}^{\frac{a^2}{2}+b}$. Thus,

$$\begin{aligned}
\mathrm{E}\left[\mathrm{e}^{-rT}S(T)\right] &= \mathrm{E}\left[\mathrm{e}^{-rT}S(0)\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}\right] \\
&= S(0)\mathrm{E}\left[\mathrm{e}^{-\sigma^2T/2+\sigma\sqrt{T}Z}\right] = S(0)\mathrm{e}^{-\sigma^2T/2+(\sigma\sqrt{T})^2/2} \\
&= S(0)\mathrm{e}^{-\sigma^2T/2+\sigma^2T/2} = S(0)\mathrm{e}^0 = S(0).
\end{aligned}$$

Solution to Exercise 2.16.

(a) Denote $x = \ln(u_N)$. Since $u_N \cdot d_N = 1$, we have $\ln(d_N) = -x$. The nonlinear system with two unknowns, x and $p \equiv p_N$, takes the following form:

$$\begin{cases} xp - x(1-p) = \mu\delta_N \\ x^2p + x^2(1-p) = \sigma^2\delta_N \end{cases} \iff \begin{cases} x(2p-1) = \mu\delta_N \\ x^2 = \sigma^2\delta_N \end{cases} \iff \begin{cases} p = \frac{\mu\delta_N}{2\sigma\sqrt{\delta_N}} + \frac{1}{2} \\ x = \sigma\sqrt{\delta_N} \end{cases}$$

The final solution is given by

$$u_N = \frac{1}{d_N} = \mathrm{e}^x = \mathrm{e}^{\sigma\sqrt{\delta_N}} \quad \text{and} \quad p_N = \frac{\mu\sqrt{\delta_N}}{2\sigma} + \frac{1}{2}.$$

(b) Denote $x = \ln(u_N)$ and $y = \ln(d_N)$. The unknowns x and y satisfy the following system of equations:

$$\begin{cases} x + y = 2\mu\delta_N \\ x^2 + y^2 = 2\sigma^2\delta_N \end{cases}$$

There are two solutions:

$$(x, y) = \left(\mu\delta_N \pm \sqrt{\sigma^2\delta_N - \mu^2\delta_N^2},\ \mu\delta_N \mp \sqrt{\sigma^2\delta_N - \mu^2\delta_N^2}\right).$$

Assuming that $d_N < u_N$, we have

$$d_N = \mathrm{e}^{\mu\delta_N - \sqrt{\sigma^2\delta_N - \mu^2\delta_N^2}}, \quad u_N = \mathrm{e}^{\mu\delta_N + \sqrt{\sigma^2\delta_N - \mu^2\delta_N^2}}.$$

The solution exists iff $\delta_N < \frac{\sigma^2}{\mu^2}$.

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Solutions to Exercises in Chapter 3

Solution to Exercise 3.1. Let us verify for each function that the first derivative is positive and the second derivative is negative:

$$\begin{aligned}
u_1'(x) &= (\ln x)' = \frac{1}{x} > 0 \quad \text{for} \quad x > 0, \\
u_1''(x) &= \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0 \quad \text{for} \quad x \neq 0, \\
u_2'(x) &= (1 - \mathrm{e}^{-ax})' = a\mathrm{e}^{-ax} > 0 \quad \text{for} \quad x \in \mathbb{R} \quad \text{since} \quad a > 0, \\
u_2''(x) &= (a\mathrm{e}^{-ax})' = -a^2\mathrm{e}^{-ax} < 0 \quad \text{for} \quad x \in \mathbb{R}.
\end{aligned}$$

Therefore, $u_1(x)$ and $u_2(x)$ are utility functions defined on the intervals $(0, \infty)$ and $(-\infty, \infty)$, respectively.

Solution to Exercise 3.2. Let us verify for each function that the first derivative is positive and the second derivative is negative:

$$\begin{aligned}
u_3'(x) &= (x^a)' = ax^{a-1} > 0 \quad \text{for} \quad x > 0 \quad \text{since} \quad a > 0, \\
u_3''(x) &= (ax^{a-1})' = a(a-1)x^{a-2} < 0 \quad \text{for} \quad x > 0 \quad \text{since} \quad 0 < a < 1, \\
u_4'(x) &= (x - bx^2)' = 1 - 2bx > 0 \quad \text{for} \quad x < \frac{1}{2b}, \\
u_4''(x) &= (1 - 2bx)' = -2b < 0 \quad \text{for} \quad x \in \mathbb{R} \quad \text{since} \quad b > 0.
\end{aligned}$$

Therefore, $u_3(x)$ and $u_4(x)$ are utility functions defined on the intervals $(0, \infty)$ and $(-\infty, \frac{1}{2b})$, respectively.

Solution to Exercise 3.3. The value of investor's portfolio in one year is given by

$$V_1 = \begin{cases} 2aW + (1-a)(1+r)W & \text{with probability } p, \\ (1-a)(1+r)W & \text{with probability } 1-p. \end{cases}$$

Let us assume that $0 \leqslant r < 1$.

(a) The expected utility of the terminal wealth is

$$\mathrm{E}[u(V_1)] = \ln(2aW + (1-a)(1+r)W)p + \ln((1-a)(1+r)W)(1-p).$$