CHAPTER 2 HOMEWORK PROBLEMS

SOLUTIONS

2.1. Solve part (c) of Example Problem 2.1 by hand, based on the specified rotation angles listed. In each case calculate that the magnitude of the transformed force vector. Confirm your calculations using program 3DROTATE.

<u>Solution</u>:

From Example Problem 2.1, the transformed force vector is given by:

$$\begin{Bmatrix} F_{x''} \\ F_{y''} \\ F_{z''} \end{Bmatrix} = \begin{Bmatrix} (\cos\theta)F_x + (\sin\theta)F_y \\ (-\cos\beta\sin\theta)F_x + (\cos\beta\cos\theta)F_y + (\sin\beta)F_z \\ (\sin\beta\sin\theta)F_x + (-\sin\beta\cos\theta)F_y + (\cos\beta)F_z \end{Bmatrix}$$

The force components are specified in the $x-y-z$ coordinate system by the problem statement:

$$\overline{F} = 1000\,\hat{i} + 200\,\hat{j} + 600\,\hat{k}\ (N) \qquad \text{(or)} \qquad \{\overline{F}\} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 200 \\ 600 \end{Bmatrix} N$$

Hence:

$$\begin{Bmatrix} F_{x''} \\ F_{y''} \\ F_{z''} \end{Bmatrix} = \begin{Bmatrix} (\cos\theta)(1000N) + (\sin\theta)200N \\ (-\cos\beta\sin\theta)(1000N) + (\cos\beta\cos\theta)(200N) + (\sin\beta)(600N) \\ (\sin\beta\sin\theta)(1000N) + (-\sin\beta\cos\theta)(200N) + (\cos\beta)(600N) \end{Bmatrix}$$

(a) $\theta = 60°, \beta = -45°$: $\quad \overline{F} = 673.2\,\hat{i}'' - 965.9\,\hat{j}'' - 117.4\,\hat{k}''\ (N)$

(b) $\theta = 60°, \beta = 45°$: $\quad \overline{F} = 673.2\,\hat{i}'' - 117.4\,\hat{j}'' + 965.9\,\hat{k}''\ (N)$

(c) $\theta = -60°,\ \beta = -45°$: $\quad \overline{F} = 326.8\,\hat{i}'' + 258.8\,\hat{j}'' + 1107\,\hat{k}''\ (N)$

(d) $\theta = -60°,\ \beta = 45°$: $\quad \overline{F} = 326.8\,\hat{i}'' + 1107\,\hat{j}'' - 258.8\,\hat{k}''\ (N)$

(e) $\theta = -45°,\ \beta = 60°$: $\quad \overline{F} = 565.7\,\hat{i}'' + 954.9\,\hat{j}'' - 434.8\,\hat{k}''\ (N)$

(f) $\theta = -45°,\ \beta = -60°$: $\quad \overline{F} = 565.7\,\hat{i}'' - 95.35\,\hat{j}'' + 1035\,\hat{k}''\ (N)$

(g) $\theta = 45°,\ \beta = 60°$: $\quad \overline{F} = 848.5\,\hat{i}'' + 236.8\,\hat{j}'' + 789.9\,\hat{k}''\ (N)$

(h) $\theta = 45°,\ \beta = -60°$: $\quad \overline{F} = 848.5\,\hat{i}'' - 802.5\,\hat{j}'' - 189.9\,\hat{k}''\ (N)$

In each case the magnitude is calculated using Eq 2.2 and found to be 1183 *N*. Identical results are returned by program 3DROTATE.

2.2. Consider an $x'''-y'''-z'''$ coordinate system, which is generated from an $x-y-z$ coordinate system by the following three rotations:

- a rotation of θ-degs about the original z-axis, which defines an intermediate $x'-y'-z'$ coordinate system (see Figure 2.2a), followed by

- a rotation of β-degs about the x'-axis, which defines an intermediate $x''-y''-z''$ coordinate system (see Figure 2.2b), followed by

- a rotation of ψ-degs about the y''-axis, which defines the final $x'''-y'''-z'''$ coordinate system.

Derive the direction cosines relating the $x'''-y'''-z'''$ and $x-y-z$ coordinate systems.

Solution:

The first two successive rotations (θ and β) are considered in Example Problem 2.1(a). When rotated to the $x''-y''-z''$ coordinate system, unit vectors $\overline{I}$, $\overline{J}$, and $\overline{K}$ can be written:

$$\overline{I} = (\cos\theta)\,\hat{i}'' + (-\cos\beta\sin\theta)\,\hat{j}'' + (\sin\beta\sin\theta)\,\hat{k}''$$
$$\overline{J} = (\sin\theta)\,\hat{i}'' + (\cos\beta\cos\theta)\,\hat{j}'' + (-\sin\beta\cos\theta)\,\hat{k}''$$
$$\overline{K} = (0)\,\hat{i}'' + (\sin\beta)\,\hat{j}'' + (\cos\beta)\,\hat{k}''$$

The direction cosines associated with a transformation from the $x''-y''-z''$ coordinate system to the $x'''-y'''-z'''$ coordinate system are given by:

$$c_{x'''x''} = \text{cosine(angle between } x'''\text{- and } x''\text{-axes)} = \cos\psi$$
$$c_{x'''y''} = \text{cosine(angle between } x'''\text{- and } y''\text{-axes)} = \cos(90°) = 0$$
$$c_{x'''z''} = \text{cosine(angle between } x'''\text{- and } z''\text{-axes)} = \cos(90° + \psi) = -\sin\psi$$

$$c_{y'''x''} = \text{cosine(angle between } y'''\text{- and } x''\text{-axes)} = \cos(90°) = 0$$
$$c_{y'''y''} = \text{cosine(angle between } y'''\text{- and } y''\text{-axes)} = \cos(0°) = 1$$
$$c_{y'''z''} = \text{cosine(angle between } y'''\text{- and } z''\text{-axes)} = \cos(90°) = 0$$

$$c_{z'''x''} = \text{cosine(angle between } z'''\text{- and } x''\text{-axes)} = \cos(90° - \psi) = \sin\psi$$
$$c_{z'''y''} = \text{cosine(angle between } z'''\text{- and } y''\text{-axes)} = \cos(90°) = 0$$
$$c_{z'''z''} = \text{cosine(angle between } z'''\text{- and } z''\text{-axes)} = \cos(\psi)$$

These direction cosines together with Eqs (2.6) can be used to rotate the unit vector $\overline{I}$ from the $x''-y''-z''$ coordinate system to the final $x'''-y'''-z'''$ coordinate system:

$$I_{x'''} = c_{x'''x''}I_{x''} + c_{x'''y''}I_{y''} + c_{x'''z''}I_{z''} = (\cos\psi)(\cos\theta) + (0)(-\cos\beta\sin\theta) + (-\sin\psi)(\sin\beta\sin\theta)$$
$$I_{x'''} = \cos\psi\cos\theta - \sin\psi\sin\beta\sin\theta$$

$$I_{y'''} = c_{y'''x''} I_{x''} + c_{y'''y''} I_{y''} + c_{y'''z''} I_{z''} = (0)(\cos\theta) + (1)(-\cos\beta \sin\theta) + (0)(\sin\beta \sin\theta)$$
$$I_{y'''} = -\cos\beta \sin\theta$$

$$I_{z'''} = c_{z'''x''} I_{x''} + c_{z'''y''} I_{y''} + c_{z'''z''} I_{z''} = (\sin\psi)(\cos\theta) + (0)(-\cos\beta \sin\theta) + (\cos\psi)(\sin\beta \sin\theta)$$
$$I_{z'''} = \sin\psi \cos\theta + \cos\psi \sin\beta \sin\theta$$

Therefore, in the final $x'''-y'''-z'''$ coordinate system the vector $\overline{I}$ is written:

$$\overline{I} = (\cos\psi \cos\theta - \sin\psi \sin\beta \sin\theta)\,\hat{i}''' + (-\cos\beta \sin\theta)\,\hat{j}''' + (\sin\psi \cos\theta + \cos\psi \sin\beta \sin\theta)\,\hat{k}''$$

In the original $x-y-z$ coordinate system $\overline{I}$ is a unit vector aligned with the x-axis : $\overline{I} \equiv (1)\,\hat{i}$. Therefore this result defines the direction cosines associated with the angle between the original x-axis and the final $x''-$, $y''-$, and $z''-$axes . That is:

$$c_{x'''x} = \cos\psi \cos\theta - \sin\psi \sin\beta \sin\theta$$
$$c_{y'''x} = -\cos\beta \sin\theta$$
$$c_{z'''x} = \sin\psi \cos\theta + \cos\psi \sin\beta \sin\theta$$

Using an identical process, the $\overline{J}$ and $\overline{K}$ unit vectors are rotated from the $x''-y''-z''$ coordinate system to the final $x'''-y'''-z'''$ coordinate system, resulting in:

$$\overline{J} = (\cos\psi \sin\theta + \sin\psi \sin\beta \cos\theta)\,\hat{i}''' + (\cos\beta \cos\theta)\,\hat{j}''' + (\sin\psi \sin\theta - \cos\psi \sin\beta \cos\theta)\,\hat{k}''$$
$$\overline{K} = (\sin\psi \cos\beta)\,\hat{i}''' + (\sin\beta)\,\hat{j}''' + (\cos\psi \cos\beta)\,\hat{k}''$$

Assembling these results, we find that the direction cosines relating the original $x-y-z$ coordinate system to the final $x'''-y'''-z'''$ coordinate system are:

$$\begin{bmatrix} c_{x'''x} & c_{x'''y} & c_{x'''z} \\ c_{y'''x} & c_{y'''y} & c_{y'''z} \\ c_{z'''x} & c_{z'''y} & c_{z'''z} \end{bmatrix} = \begin{bmatrix} \cos\psi \cos\theta - \sin\psi \sin\beta \sin\theta & \cos\psi \sin\theta + \sin\psi \sin\beta \cos\theta & -\sin\psi \cos\beta \\ -\cos\beta \sin\theta & \cos\beta \cos\theta & \sin\beta \\ \sin\psi \cos\theta + \cos\psi \sin\beta \sin\theta & \sin\psi \sin\theta - \cos\psi \sin\beta \cos\theta & \cos\psi \cos\beta \end{bmatrix}$$

2.3. The force vector discussed in Example Problem 2.1 is given by:

$$\overline{F} = 1000\hat{i} + 200\hat{j} + 600\hat{k}$$

Using Equation 2.6(c), express $\overline{F}$ in a new coordinate system defined by three successive rotations, as specified, using the direction cosines listed in Problem 2.2. In each case compare the magnitude of the transformed force vector to the magnitudes calculated in Example Problem 2.1. Solve these problems by hand, and then confirm your calculations using program 3DROTATE.

<u>Solution</u>:

Equation 2.6(c) is:

$$\begin{Bmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{Bmatrix} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{x'z} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

The force vector discussed in Example Problem 2.1 is:

$$\overline{F} = 1000\hat{i} + 200\hat{j} + 600\hat{k}$$

The direction cosines relating the two coordinate systems are:

$$\begin{bmatrix} c_{x'''x} & c_{x'''y} & c_{x'''z} \\ c_{y'''x} & c_{y'''y} & c_{y'''z} \\ c_{z'''x} & c_{z'''y} & c_{z'''z} \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta - \sin\psi\sin\beta\sin\theta & \cos\psi\sin\theta + \sin\psi\sin\beta\cos\theta & -\sin\psi\cos\beta \\ -\cos\beta\sin\theta & \cos\beta\cos\theta & \sin\beta \\ \sin\psi\cos\theta + \cos\psi\sin\beta\sin\theta & \sin\psi\sin\theta - \cos\psi\sin\beta\cos\theta & \cos\psi\cos\beta \end{bmatrix}$$

Hence,

$$\begin{Bmatrix} F_{x'''} \\ F_{y'''} \\ F_{z'''} \end{Bmatrix} = \begin{bmatrix} \cos\psi\cos\theta - \sin\psi\sin\beta\sin\theta & \cos\psi\sin\theta + \sin\psi\sin\beta\cos\theta & -\sin\psi\cos\beta \\ -\cos\beta\sin\theta & \cos\beta\cos\theta & \sin\beta \\ \sin\psi\cos\theta + \cos\psi\sin\beta\sin\theta & \sin\psi\sin\theta - \cos\psi\sin\beta\cos\theta & \cos\psi\cos\beta \end{bmatrix} \begin{Bmatrix} 1000 \\ 200 \\ 600 \end{Bmatrix}$$

Results for the specified rotations are shown below:

(a) $\theta = 60°$, $\beta = -45°$, $\psi = 25°$: $\overline{F} = 659.7\hat{i}''' + -965.9\hat{j}''' + 178.1\hat{k}'''$

(b) $\theta = 60°$, $\beta = -45°$, $\psi = -25°$: $\overline{F} = 560.5\hat{i}''' - 965.9\hat{j}''' - 390.9\hat{k}'''$

(c) $\theta = -60°$, $\beta = -45°$, $\psi = 25°$: $\overline{F} = -171.8\hat{i}''' - 258.8\hat{j}''' - 1142\hat{k}'''$

(d) $\theta = 60°$, $\beta = 45°$, $\psi = 25°$: $\overline{F} = 201.9\hat{i}''' - 117.4\hat{j}''' - 1160\hat{k}'''$

(e) $\theta = -60°$, $\beta = -45°$, $\psi = 25°$: $\overline{F} = -171.8\hat{i}''' - 258.8\hat{j}''' - 1142\hat{k}'''$

In each case the magnitude is calculated using Eq 2.2 and found to be 1183 *N*. Identical results are returned by program 3DROTATE.

2.4. Solve Example Problem 2.3 by hand, using the specified rotation angles.

Solution:

Direction cosines relating the $x - y - z$ and $x''-y''-z''$ coordinate systems following successive rotations θ and β were determined as a part of Example Problem 2.1 and are:

$$\begin{bmatrix} c_{x''x} & c_{x''y} & c_{x''z} \\ c_{y''x} & c_{y''y} & c_{y''z} \\ c_{z''x} & c_{z''y} & c_{z''z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\beta\sin\theta & \cos\beta\cos\theta & \sin\beta \\ \sin\beta\sin\theta & -\sin\beta\cos\theta & \cos\beta \end{bmatrix}$$

The stress tensor to be transformed is:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 50 & -10 & 15 \\ -10 & 25 & 30 \\ 15 & 30 & -5 \end{bmatrix} (ksi)$$

Expanding Eq (2.13), each stress component is given by:

$$\begin{aligned} \sigma_{x''x''} = {} & c_{x''x}\, c_{x''x}\, \sigma_{xx} + c_{x''x}\, c_{x''y}\, \sigma_{xy} + c_{x''x}\, c_{x''z}\, \sigma_{xz} \\ & + c_{x''y}\, c_{x''x}\, \sigma_{yx} + c_{x''y}\, c_{x''y}\, \sigma_{yy} + c_{x''y}\, c_{x''z}\, \sigma_{yz} \\ & + c_{x''z}\, c_{x''x}\, \sigma_{zx} + c_{x''z}\, c_{x''y}\, \sigma_{zy} + c_{x''z}\, c_{x''z}\, \sigma_{zz} \end{aligned}$$

$$\begin{aligned} \sigma_{x''y''} = {} & c_{x''x}\, c_{y''x}\, \sigma_{xx} + c_{x''x}\, c_{y''y}\, \sigma_{xy} + c_{x''x}\, c_{y''z}\, \sigma_{xz} \\ & + c_{x''y}\, c_{y''x}\, \sigma_{yx} + c_{x''y}\, c_{y''y}\, \sigma_{yy} + c_{x''y}\, c_{y''z}\, \sigma_{yz} \\ & + c_{x''z}\, c_{y''x}\, \sigma_{zx} + c_{x''z}\, c_{y''y}\, \sigma_{zy} + c_{x''z}\, c_{y''z}\, \sigma_{zz} \end{aligned}$$

$$\begin{aligned} \sigma_{x''z''} = {} & c_{x''x}\, c_{z''x}\, \sigma_{xx} + c_{x''x}\, c_{z''y}\, \sigma_{xy} + c_{x''x}\, c_{z''z}\, \sigma_{xz} \\ & + c_{x''y}\, c_{z''x}\, \sigma_{yx} + c_{x''y}\, c_{z''y}\, \sigma_{yy} + c_{x''y}\, c_{z''z}\, \sigma_{yz} \\ & + c_{x''z}\, c_{z''x}\, \sigma_{zx} + c_{x''z}\, c_{z''y}\, \sigma_{zy} + c_{x''z}\, c_{z''z}\, \sigma_{zz} \end{aligned}$$

$$\begin{aligned} \sigma_{y''y''} = {} & c_{y''x}\, c_{y''x}\, \sigma_{xx} + c_{y''x}\, c_{y''y}\, \sigma_{xy} + c_{y''x}\, c_{y''z}\, \sigma_{xz} \\ & + c_{y''y}\, c_{y''x}\, \sigma_{yx} + c_{y''y}\, c_{y''y}\, \sigma_{yy} + c_{y''y}\, c_{y''z}\, \sigma_{yz} \\ & + c_{y''z}\, c_{y''x}\, \sigma_{zx} + c_{y''z}\, c_{y''y}\, \sigma_{zy} + c_{y''z}\, c_{y''z}\, \sigma_{zz} \end{aligned}$$

$$\begin{aligned} \sigma_{y''z''} = {} & c_{y''x}\, c_{z''x}\, \sigma_{xx} + c_{y''x}\, c_{z''y}\, \sigma_{xy} + c_{y''x}\, c_{z''z}\, \sigma_{xz} \\ & + c_{y''y}\, c_{z''x}\, \sigma_{yx} + c_{y''y}\, c_{z''y}\, \sigma_{yy} + c_{y''y}\, c_{z''z}\, \sigma_{yz} \\ & + c_{y''z}\, c_{z''x}\, \sigma_{zx} + c_{y''z}\, c_{z''y}\, \sigma_{zy} + c_{y''z}\, c_{z''z}\, \sigma_{zz} \end{aligned}$$

$$\begin{aligned} \sigma_{z''z''} = {} & c_{z''x}\, c_{z''x}\, \sigma_{xx} + c_{z''x}\, c_{z''y}\, \sigma_{xy} + c_{z''x}\, c_{z''z}\, \sigma_{xz} \\ & + c_{z''y}\, c_{z''x}\, \sigma_{yx} + c_{z''y}\, c_{z''y}\, \sigma_{yy} + c_{z''y}\, c_{z''z}\, \sigma_{yz} \\ & + c_{z''z}\, c_{z''x}\, \sigma_{zx} + c_{z''z}\, c_{z''y}\, \sigma_{zy} + c_{z''z}\, c_{z''z}\, \sigma_{zz} \end{aligned}$$

Substituting the known direction cosines and simplifying:

$$\sigma_{x''x''} = (\cos\theta)^2\ \sigma_{xx} + 2(\cos\theta\ \sin\theta)\,\sigma_{xy} + (\sin\theta)^2\ \ \sigma_{yy}$$

$$\sigma_{x''y''} = -(\cos\theta\cos\beta\sin\theta)\sigma_{xx} + \cos\beta(\cos^2\theta - \sin^2\theta)\,\sigma_{xy} + (\cos\theta\sin\beta)\,\sigma_{xz}$$
$$+ (\sin\theta\cos\beta\cos\theta)\,\sigma_{yy} + (\sin\theta\sin\beta)\,\sigma_{yz}$$

$$\sigma_{x''z''} = (\cos\theta\sin\beta\sin\theta)\,\sigma_{xx} + \sin\beta(\sin^2\theta - \cos^2\theta)\,\sigma_{xy} + (\cos\theta\cos\beta)\,\sigma_{xz}$$
$$- (\sin\theta\sin\beta\cos\theta)\,\sigma_{yy} + (\sin\theta\cos\beta)\,\sigma_{yz}$$

$$\sigma_{y''y''} = (\cos\beta\sin\theta)^2\ \sigma_{xx} - 2(\cos^2\beta\sin\theta\cos\theta)\,\sigma_{xy} - 2(\cos\beta\sin\theta\sin\beta)\,\sigma_{xz}$$
$$+ (\cos\beta\cos\theta)^2\ \sigma_{yy} + 2(\cos\beta\cos\theta\sin\beta)\,\sigma_{yz} + (\sin\beta)^2\ \ \sigma_{zz}$$

$$\sigma_{y''z''} = -(\cos\beta\sin\beta\sin^2\theta)\sigma_{xx} + 2(\cos\beta\sin\theta\sin\beta\cos\theta)\,\sigma_{xy} + \sin\theta(\sin^2\beta - \cos^2\beta)\,\sigma_{xz}$$
$$- (\cos\beta\sin\beta\cos^2\theta)\,\sigma_{yy} + \cos\theta(\cos^2\beta - \sin^2\beta)\,\sigma_{yz} + (\sin\beta\cos\beta)\,\sigma_{zz}$$

$$\sigma_{z''z''} = (\sin\beta\sin\theta)^2\ \ \sigma_{xx} - 2(\sin^2\beta\sin\theta\cos\theta)\,\sigma_{xy} + 2(\sin\beta\sin\theta\cos\beta)\,\sigma_{xz}$$
$$+ (\sin\beta\cos\theta)^2\ \ \sigma_{yy} - 2(\sin\beta\cos\theta\cos\beta)\,\sigma_{yz} + (\cos\beta)^2\ \sigma_{zz}$$

These results are used to solve each problem below:

(a) $\theta = 20°$ $\qquad$ $\beta = -35°$:

$$\sigma_{x''x''} = (\cos 20°)^2\ (50) + 2(\cos 20°)\,(\sin 20°)\,(\text{-}10) + (\sin 20°)^2\ \ (25)$$
$$\sigma_{x''x''} = 40.65 ksi$$

$$\sigma_{x''y''} = -(\cos 20°\cos(\text{-}35°)\sin 20°)(50) + \cos(\text{-}35°)(\cos^2 20° - \sin^2 20°)\,(\text{-}10)$$
$$+ (\cos 20°\sin(\text{-}35°))\,(15) + (\sin 20°\cos(\text{-}35°)\cos 20°)\,(25)$$
$$+ (\sin 20°\sin(\text{-}35°))\,(30)$$
$$\sigma_{x''y''} = -26.83 ksi$$

$$\sigma_{x''z''} = (\cos 20°\sin(\text{-}35°)\sin 20°)\,(50) + \sin(\text{-}35°)(\sin^2 20° - \cos^2 20°)\,(\text{-}10)$$
$$+ (\cos 20°\cos(\text{-}35°))\,(15) - (\sin 20°\sin(\text{-}35°)\cos 20°)\,(25)$$
$$+ (\sin 20°\cos(\text{-}35°))\,(30)$$
$$\sigma_{x''z''} = 10.95 ksi$$

$$\sigma_{y''y''} = (\cos(-35°)\sin 20°)^2\ (50) - 2(\cos^2(-35°)\sin 20°\cos 20°)\ (-10)$$
$$-2(\cos(-35°)\sin 20°\sin(-35°))\ (15) + (\cos(-35°)\cos 20°)^2\ (25)$$
$$+2(\cos(-35°)\cos 20°\sin(-35°))\ (30) + (\sin(-35°))^2\ \ (-5)$$
$$\sigma_{y''y''} = -0.2640 ksi$$

$$\sigma_{y''z''} = -(\cos(-35°)\sin(-35°)\sin^2 20°)(50) + 2(\cos(-35°)\sin 20°\sin(-35°)\cos 20°)\ (-10)$$
$$+\sin 20°(\sin^2(-35°) - \cos^2(-35°))\ (15) - (\cos(-35°)\sin(-35°)\cos^2 20°)\ (25)$$
$$+\cos 20°(\cos^2(-35°) - \sin^2(-35°))\ (30) + (\sin(-35°)\cos(-35°))\ (-5)$$
$$\sigma_{y''z''} = 26.38 ksi$$

$$\sigma_{z''z''} = (\sin(-35°)\sin 20°)^2\ \ (50) - 2(\sin^2(-35°)\sin 20°\cos 20°)\ (-10)$$
$$+2(\sin(-35°)\sin 20°\cos(-35°))\ (15) + (\sin(-35°)\cos 20°)^2\ \ (25)$$
$$-2(\sin(-35°)\cos 20°\cos(-35°))\ (30) + (\cos(-35°))^2\ (-5)$$
$$\sigma_{z''z''} = 29.62 ksi$$

$$\begin{bmatrix} \sigma_{x''x''} & \sigma_{x''y''} & \sigma_{x''z''} \\ \sigma_{y''x''} & \sigma_{y''y''} & \sigma_{y''z''} \\ \sigma_{z''x''} & \sigma_{z''y''} & \sigma_{z''z''} \end{bmatrix} = \begin{bmatrix} 40.65 & -26.83 & 10.95 \\ -26.83 & -0.2640 & 26.38 \\ 10.95 & 26.38 & 29.62 \end{bmatrix} (ksi)$$

(b) $\theta = -20°$ $\qquad \beta = 35°$

$$\begin{bmatrix} \sigma_{x''x''} & \sigma_{x''y''} & \sigma_{x''z''} \\ \sigma_{y''x''} & \sigma_{y''y''} & \sigma_{y''z''} \\ \sigma_{z''x''} & \sigma_{z''y''} & \sigma_{z''z''} \end{bmatrix} = \begin{bmatrix} 53.50 & 2.506 & 2.927 \\ 2.506 & 44.09 & -1.053 \\ 2.927 & -1.053 & -27.59 \end{bmatrix} (ksi)$$

(c) $\theta = -20°$ $\qquad \beta = -35°$

$$\begin{bmatrix} \sigma_{x''x''} & \sigma_{x''y''} & \sigma_{x''z''} \\ \sigma_{y''x''} & \sigma_{y''y''} & \sigma_{y''z''} \\ \sigma_{z''x''} & \sigma_{z''y''} & \sigma_{z''z''} \end{bmatrix} = \begin{bmatrix} 53.50 & -1.893 & 3.356 \\ -1.893 & -18.53 & 23.85 \\ 3.356 & 23.85 & 35.03 \end{bmatrix} (ksi)$$

2.5. Use Eq 2.12(a) to obtain an expression (in expanded form) for the specified stress component (in each case the expanded expression will be similar to Eq 2.13).

Solution:

In expanded form:

2.5(a)

$$\begin{aligned}\sigma_{x'x'} = & c_{x'x}\, c_{x'x}\, \sigma_{xx} + c_{x'x}\, c_{x'y}\, \sigma_{xy} + c_{x'x}\, c_{x'z}\, \sigma_{xz} \\ & + c_{x'y}\, c_{x'x}\, \sigma_{yx} + c_{x'y}\, c_{x'y}\, \sigma_{yy} + c_{x'y}\, c_{x'z}\, \sigma_{yz} \\ & + c_{x'z}\, c_{x'x}\, \sigma_{zx} + c_{x'z}\, c_{x'y}\, \sigma_{zy} + c_{x'z}\, c_{x'z}\, \sigma_{zz}\end{aligned}$$

2.5(b)

$$\begin{aligned}\sigma_{x'y'} = & c_{x'x}\, c_{y'x}\, \sigma_{xx} + c_{x'x}\, c_{y'y}\, \sigma_{xy} + c_{x'x}\, c_{y'z}\, \sigma_{xz} \\ & + c_{x'y}\, c_{y'x}\, \sigma_{yx} + c_{x'y}\, c_{y'y}\, \sigma_{yy} + c_{x'y}\, c_{y'z}\, \sigma_{yz} \\ & + c_{x'z}\, c_{y'x}\, \sigma_{zx} + c_{x'z}\, c_{y'y}\, \sigma_{zy} + c_{x'z}\, c_{y'z}\, \sigma_{zz}\end{aligned}$$

2.5(c)

$$\begin{aligned}\sigma_{y'y'} = & c_{y'x}\, c_{y'x}\, \sigma_{xx} + c_{y'x}\, c_{y'y}\, \sigma_{xy} + c_{y'x}\, c_{y'z}\, \sigma_{xz} \\ & + c_{y'y}\, c_{y'x}\, \sigma_{yx} + c_{y'y}\, c_{y'y}\, \sigma_{yy} + c_{y'y}\, c_{y'z}\, \sigma_{yz} \\ & + c_{y'z}\, c_{y'x}\, \sigma_{zx} + c_{y'z}\, c_{y'y}\, \sigma_{zy} + c_{y'z}\, c_{y'z}\, \sigma_{zz}\end{aligned}$$

2.5(d)

$$\begin{aligned}\sigma_{y'z'} = & c_{y'x}\, c_{z'x}\, \sigma_{xx} + c_{y'x}\, c_{z'y}\, \sigma_{xy} + c_{y'x}\, c_{z'z}\, \sigma_{xz} \\ & + c_{y'y}\, c_{z'x}\, \sigma_{yx} + c_{y'y}\, c_{z'y}\, \sigma_{yy} + c_{y'y}\, c_{z'z}\, \sigma_{yz} \\ & + c_{y'z}\, c_{z'x}\, \sigma_{zx} + c_{y'z}\, c_{z'y}\, \sigma_{zy} + c_{y'z}\, c_{z'z}\, \sigma_{zz}\end{aligned}$$

2.5(e)

$$\begin{aligned}\sigma_{z'z'} = & c_{z'x}\, c_{z'x}\, \sigma_{xx} + c_{z'x}\, c_{z'y}\, \sigma_{xy} + c_{z'x}\, c_{z'z}\, \sigma_{xz} \\ & + c_{z'y}\, c_{z'x}\, \sigma_{yx} + c_{z'y}\, c_{z'y}\, \sigma_{yy} + c_{z'y}\, c_{z'z}\, \sigma_{yz} \\ & + c_{z'z}\, c_{z'x}\, \sigma_{zx} + c_{z'z}\, c_{z'y}\, \sigma_{zy} + c_{z'z}\, c_{z'z}\, \sigma_{zz}\end{aligned}$$

2.6. Use program *3DROTATE* to determine the stress invariants for the specified stress tensor, and compare to those determined in Example Problem 2.3. (Note: the stress tensor is similar to the one considered in Example Problem 2.3, except that the algebraic sign of all three normal stresses has been reversed.):

<u>Solution</u>:

Program *3DROTATE* was used to determine the stress invariants for the following stress tensor:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -50 & -10 & 15 \\ -10 & -25 & 30 \\ 15 & 30 & 5 \end{bmatrix} (ksi)$$

The stress invariants are:

$$\Theta = -70\, ksi$$
$$\Phi = -350\,(ksi)^2$$
$$\Psi = 47375\,(ksi)^3$$

Compared to the calculations made as a part of Example Problem 2.3:

- The algebraic sign of the first stress invariant has become negative
- The second stress invariant is unchanged
- The third stress invariant is wholly different

2.7. Use program 3DROTATE to determine the stress invariants for the specified stress tensor, and compare to those determined in Example Problem 2.3. (Note: the stress tensor is similar to the one considered in Example Problem 2.3, except that the algebraic sign of all three shear stresses has been reversed.):

<u>Solution</u>:

Program *3DROTATE* was used to determine the stress invariants for the following stress tensor:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 50 & 10 & -15 \\ 10 & 25 & -30 \\ -15 & -30 & -5 \end{bmatrix} (ksi)$$

The stress invariants are:

$$\Theta = 70\, ksi$$
$$\Phi = -350\, (ksi)^2$$
$$\Psi = -47375\, (ksi)^3$$

Compared to the calculations made as a part of Example Problem 2.3:

- The first stress invariant in unchanged
- The second stress invariant is unchanged
- The third stress invariant is wholly different

2.8. Use program 3DROTATE to determine the stress invariants for the specified stress tensor, and compare to those determined in Example Problem 2.3. (Note: the stress tensor is similar to the one considered in Example Problem 2.3, except that the algebraic sign of all stress components has been reversed.):

<u>Solution</u>:

Program *3DROTATE* was used to determine the stress invariants for the following stress tensor:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -50 & 10 & -15 \\ 10 & -25 & -30 \\ -15 & -30 & 5 \end{bmatrix} (ksi)$$

The stress invariants are:

$$\Theta = -70\, ksi$$
$$\Phi = -350\,(ksi)^2$$
$$\Psi = 65375\,(ksi)^3$$

Compared to the calculations made as a part of Example Problem 2.3:

- The algebraic sign of the first stress invariant has become negative
- The second stress invariant is unchanged
- The algebraic sign of the third stress invariant has become positive

2.9. Use program 3DROTATE to determine the strain invariants for the specified strain tensor, and compare to those determined in Example Problem 2.7. (Note: the strain tensor is similar to the one considered in Example Problem 2.7, except that the algebraic sign of all shear strain components has been reversed.):

<u>Solution</u>:

Program *3DROTATE* was used to determine the strain invariants for the following strain tensor:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 1000\mu m/m & -500\mu rad & -250\mu rad \\ -500\mu rad & 1500\mu m/m & -750\mu rad \\ -250\mu rad & -750\mu rad & 2000\mu m/m \end{bmatrix}$$

The strain invariants are:

$$\Theta_\varepsilon = 4500\ \mu m/m = .004500\ m/m$$
$$\Phi_\varepsilon = 5625x10^6\ (\mu m/m)^2 = 5.625x10^{-6}\ (m/m)^2$$
$$\Psi_\varepsilon = 1.656x10^9\ (\mu m/m)^3 = 1.656x10^{-9}\ (m/m)^3$$

Compared to the calculations made as a part of Example Problem 2.7:

- The first strain invariant in unchanged
- The second strain invariant is unchanged
- The third strain invariant is wholly different

2.10. Use program 3DROTATE to determine the strain invariants for the specified strain tensor, and compare to those determined in Example Problem 2.7. (Note: the strain tensor is similar to the one considered in Example Problem 2.7, except that the algebraic sign of all normal strain components has been reversed.):

Solution:

Program *3DROTATE* was used to determine the strain invariants for the following strain tensor:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -1000\mu m/m & 500\mu rad & 250\mu rad \\ 500\mu rad & -1500\mu m/m & 750\mu rad \\ 250\mu rad & 750\mu rad & -2000\mu m/m \end{bmatrix}$$

The strain invariants are:

$$\Theta_\varepsilon = -4500\ \mu m/m = -.004500\ m/m$$
$$\Phi_\varepsilon = 5625x10^6\ (\mu m/m)^2 = 5.625x10^{-6}\ (m/m)^2$$
$$\Psi_\varepsilon = -1.656x10^9 (\mu m/m)^3 = -1.656x10^{-9} (m/m)^3$$

Compared to the calculations made as a part of Example Problem 2.7:

- The algebraic sign of first strain invariant has become negative
- The second strain invariant is unchanged
- The third strain invariant is wholly different

2.11. Use program 3DROTATE to determine the strain invariants for the specified strain tensor, and compare to those determined in Example Problem 2.7. (Note: the strain tensor is similar to the one considered in Example Problem 2.7, except that the algebraic sign of all strain components has been reversed.):

<u>Solution</u>:

Program *3DROTATE* was used to determine the strain invariants for the following strain tensor:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -1000\mu m/m & -500\mu rad & -250\mu rad \\ -500\mu rad & -1500\mu m/m & -750\mu rad \\ -250\mu rad & -750\mu rad & -2000\mu m/m \end{bmatrix}$$

The strain invariants are:

$$\Theta_\varepsilon = -4500\ \mu m/m = -.004500\ m/m$$
$$\Phi_\varepsilon = 5625x10^6\ (\mu m/m)^2 = 5.625x10^{-6}\ (m/m)^2$$
$$\Psi_\varepsilon = -2.031x10^9\ (\mu m/m)^3 = -2.031x10^{-9}\ (m/m)^3$$

Compared to the calculations made as a part of Example Problem 2.7:

- The algebraic sign of first strain invariant has become negative
- The second strain invariant is unchanged
- The algebraic sign of the third strain invariant has become negative