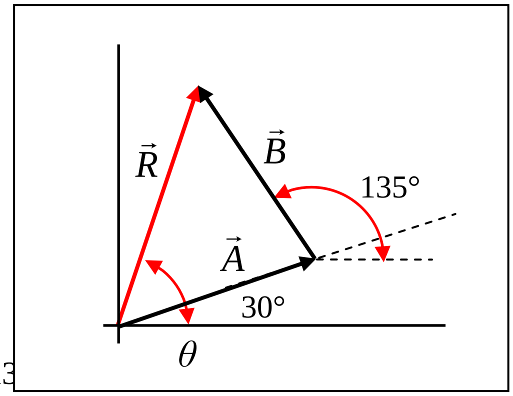
Solutions to Chapter 2 Problems

1. a. See the figure to the right.



b. $\vec{A} = 10(\cos 30°\,\hat{i} + \sin 30°\,\hat{j}) = 8.66\,\hat{i} + 5.00\,\hat{j}$

$\vec{B} = 10(\cos 135°\,\hat{i} + \sin 135°\,\hat{j}) = -7.07\,\hat{i} + 7.07\,\hat{j}$

c. $\vec{R} = (8.66 - 7.07)\,\hat{i} + (5.00 + 7.07)\,\hat{j} = 1.59\,\hat{i} + 12.07\,\hat{j}$

$R = \sqrt{1.59^2 + 12.07^2} = 12.17$

d. $\hat{R} = \dfrac{\vec{R}}{R} = \dfrac{1.59\,\hat{i} + 12.07\,\hat{j}}{12.17} = 0.131\,\hat{i} + .991\,\hat{j}$ Check: .13

e. $\tan\theta = \dfrac{12.07}{1.59} = 7.59$ $\theta = \tan^{-1} 7.59 = 82.5°$

2. a. If $\vec{A}$ and $\vec{B}$ are perpendicular, their dot product is equal to 0.

$\vec{A}\cdot\vec{B} = (3\hat{i} + 4\,\hat{j})\cdot(-4\hat{i} + 3\,\hat{j}) = 3(-4) + 4(3) = 0$

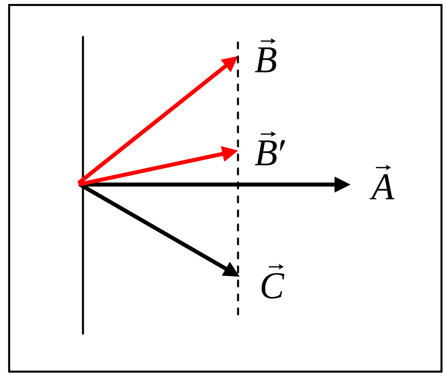
b. $\vec{C}\cdot\vec{D} = (\vec{A} + \vec{B})\cdot(\vec{A} - \vec{B}) = A^2 - B^2$. But $A = |\vec{A}| = \sqrt{3^2 + 4^2} = 5$, and similarly, $B = 5$,

so $A^2 - B^2 = 0$, proving $\vec{C}\cdot\vec{D} = 0$ and $\vec{C} \perp \vec{D}$.

3. $(\vec{U} + \vec{V})\cdot(\vec{U} - \vec{V}) = 0 = U^2 + \vec{V}\cdot\vec{U} + \vec{U}\cdot\vec{V} - V^2 = U^2 - V^2$ so $U = V$.

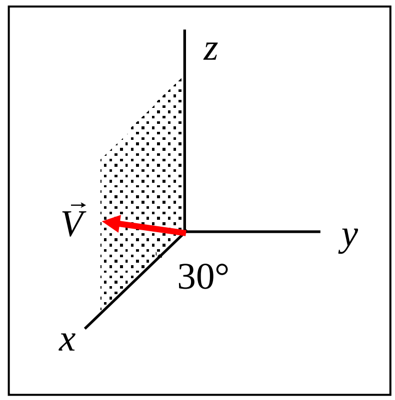
4. $\vec{A}\cdot\vec{B} = 2\cdot 3 - 5\cdot 6 + 4\cdot 6 = 6 - 30 + 24 = 0$. Therefore, $\vec{A} \perp \vec{B}$.

5. It is *not* permissible to cancel $\vec{A}$. If we did, we would conclude that $\vec{B}=\vec{C}$. But any vector $\vec{B}$ whose component parallel to $\vec{A}$ is the same as the component of $\vec{C}$ parallel to $\vec{A}$ will satisfy the equation. In the figure, $\vec{A}\cdot\vec{B}=\vec{A}\cdot\vec{B}'=\vec{A}\cdot\vec{C}$, but clearly $\vec{B}\neq\vec{B}'\neq\vec{C}$.



6. Let the vector be $\vec{V}=V_x\,\hat{i}+V_y\,\hat{j}+V_z\,\hat{k}$. Since $\vec{V}\perp\hat{j}$, then $V_y=0$. The vector lies entirely in the *x*-*z* plane:

$$\vec{V}=10(\cos 30°\,\hat{i}+\sin 30°\,\hat{k})=8.66\,\hat{i}+5.00\,\hat{k}$$

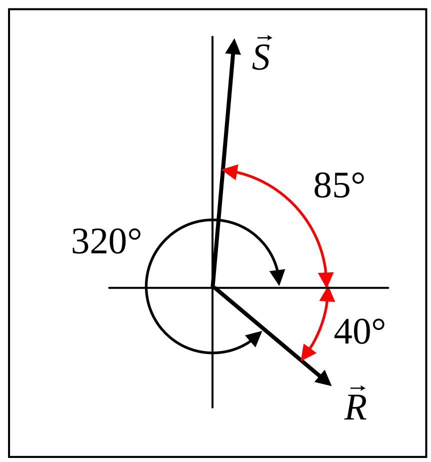


7. The vectors are shown in the figure to the right. We could do this problem in 2 ways. The easiest way is to use the definition of the dot product: $\vec{R}\cdot\vec{S}=RS\cos\theta$, where $\theta=125°$ is the angle between the two vectors. Then,

$$\vec{R}\cdot\vec{S}=(4.5)(7.3)\cos 125°=-18.8$$

Or we could write each vector in component form, and then take the dot product:

$$\vec{R}=3.45\,\hat{i}-2.89\,\hat{j}$$
$$\vec{S}=0.64\,\hat{i}+7.27\,\hat{j}\,.$$

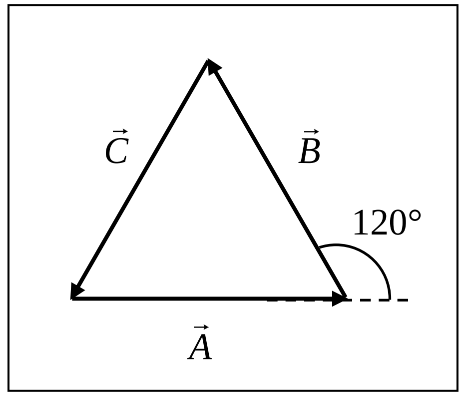


So $\vec{R}\cdot\vec{S}=(3.45)(0.64)-(2.89)(7.27)=2.21-21.0=-18.8$. The first method is obviously preferable.

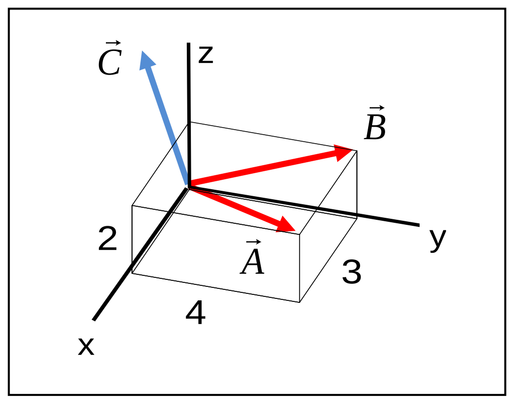
8. If $|\vec{A}+\vec{B}|$ has the same magnitude as $|\vec{A}-\vec{B}|$, then $(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})=(\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$, or ,

$A^2+B^2+2\vec{A}\cdot\vec{B}=A^2+B^2-2\vec{A}\cdot\vec{B}$. Therefore, $\vec{A}\cdot\vec{B}=0$. Vector $\vec{B}$ is a vector that is perpendicular to $\vec{A}$. There is not enough information given to determine its magnitude or its exact direction.

9. Draw the figure. Since $\vec{A}+\vec{B}+\vec{C}=0$, the three vectors must form a triangle when attached tip-to-tail. Since $A=B=C$, the triangle must be equilateral. The angle between $\vec{A}$ and $\vec{B}$ is 120°.



10. a. See the figure to the right.



b. $A=\sqrt{3^2+4^2+2^2}=5.4$, $B=\sqrt{4^2+2^2}=4.5$

c. $\vec{A}\cdot\vec{B}=3\cdot 0+4\cdot 4+2\cdot 2=20$

d. $\cos\theta=\dfrac{\vec{A}\cdot\vec{B}}{AB}=\dfrac{20}{(5.4)(4.5)}=0.823$, $\theta=34.6°$

e. $\vec{C}=-(\vec{A}+\vec{B})=-3\hat{i}-8\hat{j}-4\hat{k}$

f. $\hat{A}=\dfrac{\vec{A}}{A}=\dfrac{3\hat{i}+4\hat{j}+2\hat{k}}{5.4}=0.56\hat{i}+0.74\hat{j}+0.37\hat{k}$

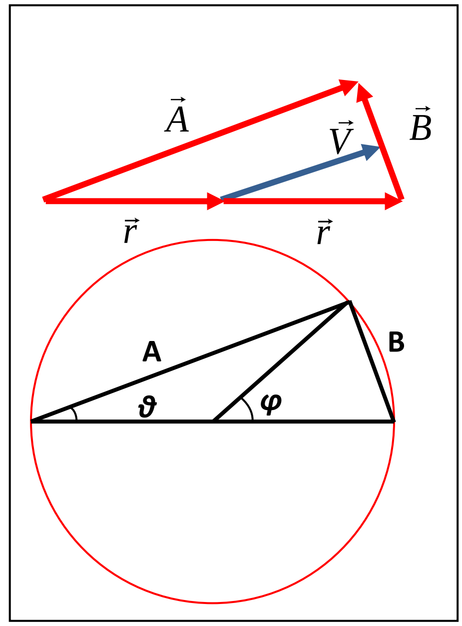
Check: $\sqrt{.56^2+.74^2+.37^2}=1.00$

g. Since $B_x=0$, any vector parallel to the *x*-axis will be perpendicular to $\vec{B}$: $\vec{V}=\pm\hat{i}$ are two *unit* vectors perpendicular to $\vec{B}$. There are also vectors perpendicular to $\vec{B}$ that lie in the *y*-*z* plane, such as the vector $\vec{C}$ shown. $\vec{C}=b\,\hat{j}+c\,\hat{k}$, with $4b+2c=0$. In fact, any vector lying in the plane containing $\vec{C}$ and the unit vector $\hat{i}$ will be perpendicular to $\vec{B}$.

11. Take the dot product of $\vec{R}$ with unit vector $\hat{i}$: $\vec{R}\cdot\hat{i}=R\cos\theta_x=a$, where we have used Equations 2.7 – 2.9. Similarly, $\vec{R}\cdot\hat{j}=R\cos\theta_y=b$, and $\vec{R}\cdot\hat{k}=R\cos\theta_z=c$. Squaring and adding the three

results, $R^2\cos^2\theta_x + R^2\cos^2\theta_y + R^2\cos^2\theta_z = a^2+b^2+c^2$. But $a^2+b^2+c^2 = R^2$, so $\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$.

12. Define a horizontal vector $\vec{r}$ having a length equal to the radius of the circle. Also define *vectors* $\vec{A}$ and $\vec{B}$ to correspond to sides *A* and *B* of the triangle. The vector $\vec{V}$ from the center of the circle to the midpoint of side *B* bisects angle φ. But $\vec{V} = \vec{r} + \frac{1}{2}\vec{B}$, whereas $\vec{A} = 2\vec{r} + \vec{B} = 2\vec{V}$. Since $\vec{A}$ is a multiple of $\vec{V}$, the two vectors are parallel. This implies that $\vartheta = \frac{1}{2}\varphi$.



13. a. $\vec{A} = 4\hat{i} + 4\hat{j} + 4\hat{k}$ b. $\vec{B} = 4\hat{i} + 4\hat{k}$

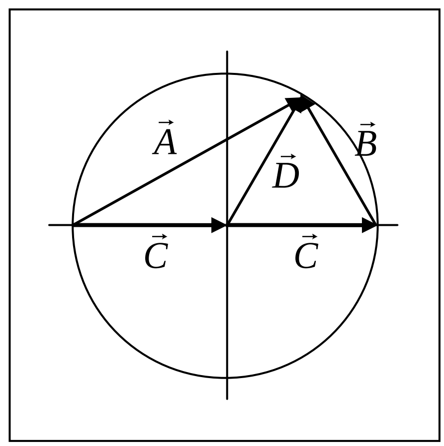
c. $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$, where $A = 4\sqrt{3}$ and $B = 4\sqrt{2}$.

Therefore, $\cos\theta = \dfrac{16+16}{16\sqrt{6}} = \dfrac{2}{\sqrt{6}}$, $\theta = 35.2°$.

14. a. $\vec{A} = \vec{C} + \vec{D}$, $\vec{B} = \vec{D} - \vec{C}$

b. $\vec{A}\cdot\vec{B} = (\vec{C}+\vec{D})\cdot(\vec{D}-\vec{C}) = D^2 - C^2 + \vec{C}\cdot\vec{D} - \vec{D}\cdot\vec{C}$

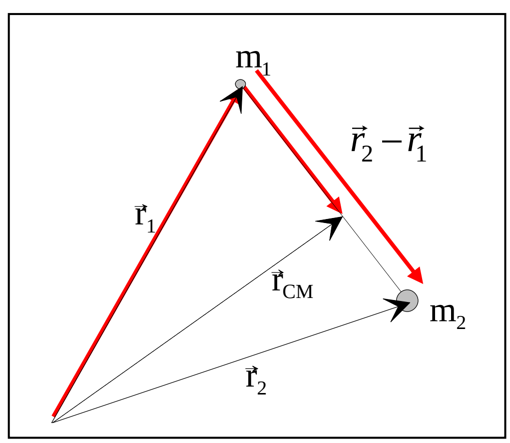
but $D = C$ (they are both radii of the circle), and $\vec{C}\cdot\vec{D} = \vec{D}\cdot\vec{C}$, so $\vec{A}\cdot\vec{B} = 0$ and the two vectors are perpendicular to each other.



15. a. Define the vector $\vec{r}_2 - \vec{r}_1$ as shown to the right. The location of the CM can be written as:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \frac{m_1\vec{r}_1 + (m_2\vec{r}_1 - m_2\vec{r}_1) + m_2\vec{r}_2}{m_1+m_2}$$

$$= \frac{m_1+m_2}{m_1+m_2}\vec{r}_1 + \frac{m_2}{m_1+m_2}(\vec{r}_2 - \vec{r}_1) = \vec{r}_1 + \alpha(\vec{r}_2 - \vec{r}_1)$$



where $\alpha = \dfrac{m_2}{m_1 + m_2}$ and $0 \le \alpha \le 1$.

b. From the figure, $\vec{r}_{cm} = \vec{r}_1 + \alpha(\vec{r}_2 - \vec{r}_1)$ lies along the line joining m_1 and m_2.

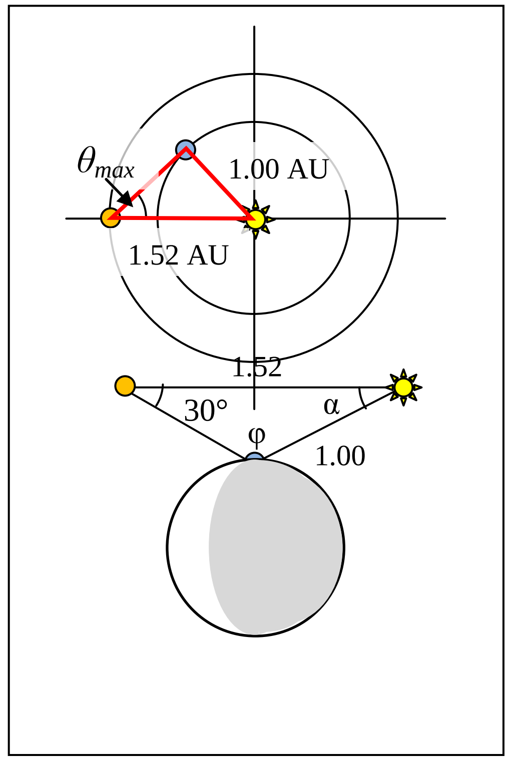
c. Note $\alpha \to 0$ when $m_2 \ll m_1$ and $\alpha \to 1$ when $m_2 \gg m_1$. For $m_2 = 3m_1$, $\alpha = 3/4$, i.e., the CM lies three quarters of the way along the line from m_1 to m_2.

16. a. Draw the figure. $\sin\theta_{\max} = \dfrac{1}{1.52}$, or $\theta_{\max} = 41.1°$

b. When Earth is at position *E′*, it will appear as a crescent planet to an observer on Mars. Draw the triangle with Mars, Earth, and the Sun at its vertices, and solve for the angles using the law of sines:

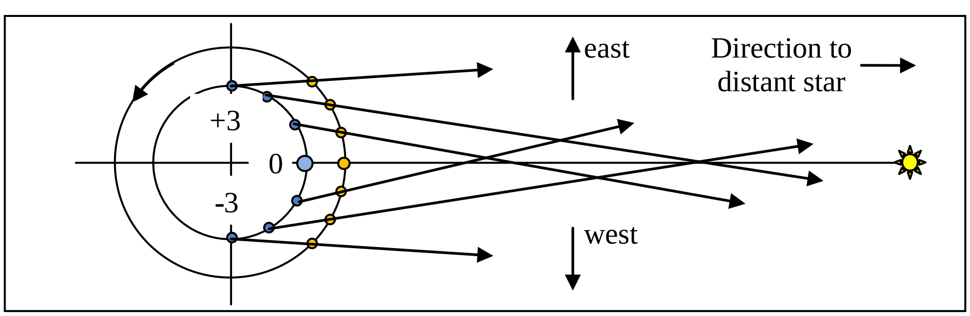
$$\frac{\sin 30°}{1.00} = \frac{\sin\varphi}{1.52}\text{, or } \sin\varphi = 1.52\sin 30° = 0.76,$$

so $\varphi = 49.5°$ or $\varphi = 180° - 49.5° = 130.5°$.
Obviously, the latter choice is the correct one, since φ is an obtuse angle. This makes $\alpha = 19.5°$.

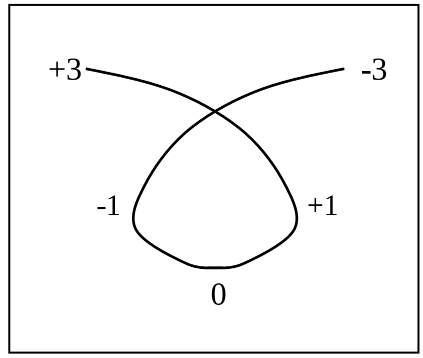


Following the analysis of Aristotle in Chapter 1, the fraction of the Earth that will appear illuminated to someone on Mars is dictated by the sum of angles $\alpha + 30° = 49.5°$. If this sum were 90°, the Earth would appear "half full." So instead, the Earth appears a crescent, a little more than first quarter (see above).

17. Each month, Earth travels about 30° about its Sun-centered orbit, whereas Mars moves only about 15° per month. The positions of the two planets, before and after opposition (the time when Mars and the Sun are on opposite sides of Earth) as shown in the figure below.



Note that the above figure is not drawn to scale: the star is vastly farther from Earth than Mars. No matter where Earth is on its orbit, to see the star an observer must look in the (horizontal) direction. Relative to the starry background, Mars is aligned with the star at opposition (position 0). It appears to move from west to east during the months (-3 to -1), but reverses its apparent motion (retrograde motion) from (-1 to +1), before again traveling west to east from (+1 to +3).



In Fig. 2.9, the earliest images are to the right (west) and the latest images are to the left. Note that at opposition, Mars is closest to Earth, and the disc of the planet appears largest.

d. Earth revolves faster than Mars about the Sun. Mars will once again be in opposition when Earth rotates one full turn more than Mars: in radians, $\theta_E = 2\pi + \theta_M$. But $\theta_E = \dfrac{2\pi}{T_E} t$, and $\theta_M = \dfrac{2\pi}{T_M} t$, where $T_E = 1 \text{ yr}$, $T_M = 1.88 \text{ yr}$, and t is the elapsed time since the initial opposition. Putting these equations together, $\dfrac{2\pi}{T_E} t = \dfrac{2\pi}{T_M} t + 2\pi$, or after rearrangement, $\dfrac{1}{t} = \dfrac{1}{T_E} - \dfrac{1}{T_M} = 1 - \dfrac{1}{1.88}$. Solving, $t = 2.14 \text{ yr} = 25.6 \text{ months}$.

e. All of the superior planets exhibit retrograde motion in the same way as described above.

18. a. This problem is more advanced than the earlier applications of the dot product. In the two coordinate systems, the same vector $\vec{R}$ can be written as $\vec{R} = x\,\hat{i} + y\,\hat{j}$, or $\vec{R} = x'\,\hat{i}' + y'\,\hat{j}'$. To find x', use the dot product: $\vec{R}\cdot\hat{i}' = x'\,\hat{i}'\cdot\hat{i}' + y'\,\hat{j}'\cdot\hat{i}' = x\,\hat{i}\cdot\hat{i}' + y\,\hat{j}\cdot\hat{i}'$. But $\hat{i}'\cdot\hat{i}' = 1$, $\hat{i}'\cdot\hat{j}' = 0$, $\hat{i}'\cdot\hat{i} = \cos\theta$, and $\hat{i}'\cdot\hat{j} = \cos(90-\theta) = \sin\theta$. Therefore, we obtain $\vec{R}\cdot\hat{i}' = x' = x\cos\theta + y\sin\theta$. Likewise, we can prove that $\vec{R}\cdot\hat{j}' = y' = y\cos\theta - x\sin\theta$.

b.
$$\begin{aligned} x'^2 + y'^2 &= (x^2\cos^2\theta + y^2\sin^2\theta + 2xy\cos\theta\sin\theta) + (y^2\cos^2\theta + x^2\sin^2\theta - 2xy\cos\theta\sin\theta) \\ &= x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = x^2 + y^2 \end{aligned}$$

19. a. The latitude φ of Syene must match the axial tilt of the Earth, so $\varphi = 23.5°$ N. Syene is on the Tropic of Cancer.

b. On the summer solstice, the Sun's declination is $\delta = \alpha = +23.5°$. From Eqn. (2.16), $\cos\theta = -\tan\delta\tan\varphi = -\tan^2 23.5° = -.189$, or $\theta = \pm 101°$. The Sun is visible for

$$\frac{2\times 101°}{360°}\times 24\text{ hrs} = 13.5\text{ hrs}.$$

On the winter solstice, $\delta = -\alpha = -23.5°$, and $\cos\theta = +.189$, so $\theta = \pm 79°$. The Sun is visible for just 10.5 hrs.