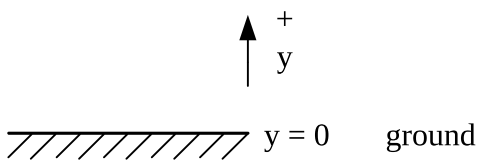
Chapter 2

Problem 2-1

<u>Going Up</u>



$$\sum F_y = ma_y = m\,\frac{dv_y}{dt}$$

$$(1)\ 1\,eq\ 2\,unk\left[\sum F_y, v_y\right]$$

Assuming no air resistance: $\sum F_y = F_g = -mg$ $\quad(2)\ 2\,eq\ 2\,unk$

Substituting (2) into (1) and rearranging,

$$\frac{dv_y}{dt} = -g \quad (3) \quad \text{with } v_y(0) = 35\,m/s$$

Maximum height is when $v_y = 0\frac{m}{s}$

From (3)

$$\int_{35}^{0} dv_y = -g\int_0^t dt \quad =>> \quad v_y|_{35}^{0} = -gt|_0^t \quad =>> \quad 0 - 35 = -gt$$

$t = 3.57\,sec$

We can also obtain an expression for v_y

$$\int_{35}^{v_y} dv_y = -g\int_0^t dt$$

$v_y = 35 - gt = 35 - 9.8t$

For the maximum height,

$$\frac{dy}{dt} = v_y \quad =>> \quad \int_0^y dy = \int_0^t v_y dt = \int_0^t (35 - 9.8t)dt$$

$y = 35t - 4.9t^2$

Then, at $t = 3.57\,s$ $\quad y_{max} = 35(3.57) - 4.9(3.57)^2$

$y_{max} = 62.5\,m$

<u>Going Down</u>

$$\sum F_y = ma_y = m\frac{dv_y}{dt}$$

$$(4)\ 1\ \text{eq.}, 2\ \text{unk}\left[\sum F_y \quad, v_y\right]$$

Assuming no air resistance,

$$\sum F_y = F_g = -mg \qquad (5)\ 2\ \text{eq., } 2\ \text{unk.}$$

$$\frac{dy}{dt} = v_y \qquad (6)\ 3\,eq\ 3\,unk\ [y]$$

Substituting (5) into (4) and rearranging,

$$\frac{d\,v_y}{dt} = -g \quad \Rightarrow> \quad \int_0^{v_y} dv_y = -g \int_0^t dt \quad \Rightarrow> \quad v_y = -gt \quad (7)$$

Using (7) in (6):

$$\frac{dy}{dt} = -gt \quad \Rightarrow> \quad \int_{62.5}^{y} dy = -g \int_0^t t dt \quad \Rightarrow> \quad y = 62.5 - 4.9t^2 \quad (8)$$

For t, $y = 0$

$t = 3.57\ sec$

Problem 2-2

$$\sum F_y = ma_y = m\frac{dv_y}{dt} \qquad (1)$$

$1\ eq., 2\ unk\ [\sum F_y, v_y]$

Considering air resistance (drag force due to air)

$$\sum F_y = F_g + F_d = -mg - 1.0|v_y|v_y \qquad \text{(2) 2 eq., 2 unk.}$$

Eqs (1) and (2) constitute the model for the velocity. For the position,

$$\frac{dy}{dt} = v_y$$
$$(3)\ 3\ eq\ 3\ unk\ [y]$$

Units of 1are units of force (N)/unit of velocity square $(m/s)^2$.

When written as $1.0|v_y|v_y$ the sign of the drag force is equal to the sign of the velocity. When written as $1.0v_y^2$ the sign of the drag force is always positive no matter what is the sign of the velocity.

Problem 2-3

$$\sum F_y = ma_y = m\frac{dv_y}{dt} \qquad (1)$$

$1\ eq., 3\ unk\ [\sum F_y, m, v_y]$

Considering air resistance

$$\sum F_y = F_g + F_d = -mg - 1.0v_y \quad (2) 2\ eq., 3\ unk$$
$$\frac{dy}{dt} = v_y$$
$$(3)\ 3\ eq., 4\ unk\ [y]$$

We need one more equation; at this time there is still one degree of freedom. We don't really have any more equations but, there is a specification that $y(3) = 0\ m,$ that we may use. So:

$$y(3) = 0\ m \qquad (4)\ \ 4\ eq., 4\ unk$$

Substituting (2) into (1):

$$m\frac{dv_y}{dt} = -mg - 1.0v_y = -1(mg + v_y)$$

and using Separation of Variables,

$$\int_0^{v_y} \frac{dv_y}{mg+v_y} = -\frac{1}{m}\int dt \quad =>> \quad ln(mg+v_y)|_0^{v_y} = -\frac{t}{m}$$

$$\ln(mg+v_y) - \ln(mg) = -\frac{t}{m} \quad =>> \quad \ln\left(\frac{mg+v_y}{mg}\right) = -\frac{t}{m}$$

$$v_y = mg\left(e^{-\frac{t}{m}} - 1\right) \qquad (5)$$

Substituting (5) into (3): $\frac{dy}{dt} = mg\left(e^{-\frac{t}{m}} - 1\right)$

$$\int_{30}^{0} dy = mg\int_0^3 \left(e^{-\frac{t}{m}} - 1\right)dt \quad =>> \quad 0 - 30 = \left(-m^2 g e^{-\frac{t}{m}} - mgt\right)|_0^3$$

$$-30 = -m^2 g e^{-\frac{3}{m}} + m^2 g - 3mg$$

By trial and error: $m \approx 2.3\ kg$

Problem 2-4

$$\rho A\frac{dh}{dt} + C_v\sqrt{h} = w_1 + w_2$$

When both inlet flows are shut-off, $w_1 = w_2 = 0.\ So,$

$$\rho A\frac{dh}{dt} + C_v\sqrt{h} = 0 \quad =>> \quad \int_{3.24}^{h}\frac{dh}{\sqrt{h}} = -\frac{C_v}{\rho A}\int_0^t dt \quad =>> \quad 2\sqrt{h}|_{3.24}^{h} = -\frac{C_v}{\rho A}t|_0^t$$

$$h = (1.8 - 0.0285t)^2$$

Drain tank: $h = 0$

$$t = 63.06\ min$$

Problem 2-5

$$\tau\frac{dy(t)}{dt} + y(t) = Kx(t)$$

$$\int_{y(0)}^{y(t)} \frac{dy(t)}{Kx(t)-y(t)} = \frac{1}{\tau}\int_0^t dt \quad =>> \quad \ln[Kx(t) - y(t)]\,|_{y(0)}^{y(t)} = -\frac{t}{\tau}$$

$$\ln[Kx(t) - y(t)] - \ln[Kx(t) - y(0)] = -\frac{t}{\tau}$$

$$\frac{Kx(t) - y(t)}{Kx(t) - y(0)} = e^{-\frac{t}{\tau}}$$

$$y(t) = Kx(t) - [Kx(t) - y(0)]e^{-\frac{t}{\tau}}$$

and using $x(t) = x(0) + D$,

$$y(t) = K(x(0) + D) - [K(x(0) + D) - y(0)]e^{-\frac{t}{\tau}}$$

and with $y(0) = K\ x(0)$,

$$y(t) = y(0) + KD - [y(0) + KD - y(0)]e^{-\frac{t}{\tau}}$$

$$y(t) = y(0) + KD\left(1 - e^{-\frac{t}{\tau}}\right)$$

Problem 2-6

$\boldsymbol{a})\ e^{-2t}\dfrac{dy}{dt} = y^{-1}(1-e^{-2t});\quad y(0)=0$

$$ydy = e^{2t}(1-e^{-2t})dt \ =>> \tfrac{1}{2}y^2|_0^y = \int_0^t (e^{2t}-1)dt$$

$$\tfrac{1}{2}y^2 = \left(\tfrac{1}{2}e^{2t} - t\right)|_0^t$$

$y = \sqrt{e^{2t}-2t-1}$

$\boldsymbol{b})\ \dfrac{dy}{dt} = ycos(t) + y;\qquad y(0)=2$

$$\int_2^y \frac{dy}{y} = \int_0^t (\cos(t)+1)dt \quad =>> \quad \ln(y)|_2^y = (\sin(t)+t)|_0^t$$

$$\ln\left(\frac{y}{2}\right) = \sin(t) + t$$

$$y = 2\exp[\sin(t)+t]$$

$\boldsymbol{c})\ \frac{dy}{dt} = \frac{t+2}{y};\quad y(0)=2$

$$\int_2^y ydy = \int_0^t (t+2)dt \quad =>> \quad \tfrac{1}{2}y^2|_2^y = \tfrac{1}{2}t^2 + 2t|_0^t$$

$$y^2 - 4 = t^2 + 4t$$

$$y = \sqrt{t^2+4t+4}$$

$\boldsymbol{d})\ \dfrac{d}{dt}\left[t\dfrac{dy}{dt}\right] = 2t;\quad y(1)=1\ ;\ y'(1)=3$

Let $u = t\frac{dy}{dt}$ with $u(1)=3$

$$\frac{du}{dt} = 2t \quad =>> \quad \int_3^u du = \int_1^t 2t\,dt \quad =>> \quad u = t^2+2$$

$$u = t^2 + 2 = t\frac{dy}{dt} \quad =>> \quad \int_1^y dy = \int_1^t (t+2t^{-1})\,dt$$

$$y - 1 = \frac{t^2}{2} + 2\ln(t) - \frac{1}{2}$$

$$y = \frac{t^2}{2} + 2\ln(t) + \frac{1}{2}$$

$\boldsymbol{e})\ \frac{d^2y}{dt^2} = 32e^{-4t}\quad y(0)=1\ ;\ y'(0)=0$

$$\frac{d}{dt}\left[\frac{dy}{dt}\right] = 32e^{-4t}$$

Let $u = \frac{dy}{dt}$

Then, $\frac{du}{dt} = 32e^{-4t}; \quad u(0) = 0$

$\int_0^u du = \int_0^t 32e^{-4t}dt$ =>> $u = -\frac{32}{4}e^{-4t}|_0^t = -\frac{32}{4}e^{-4t} + \frac{32}{4}$

$$u = \frac{dy}{dt} = 8 - 8e^{-4t}$$

$\int_1^y dy = \int_0^t (8 - 8e^{-4t})dt$ =>> $y - 1 = 8t + 2e^{-4t}|_0^t = 8t + 2e^{-4t} - 2$

$y = 8t + 2e^{-4t} - 1$

f) $\frac{1}{t^2}\frac{dy}{dt} = y; \quad y(0) = 1$

$\int_1^y \frac{dy}{y} = \int_0^t t^2 dt$ =>> $\ln(y)\,|_1^y = \frac{1}{3}t^3|_0^t$ =>> $\ln(y) = \frac{1}{3}t^3$

$y = e^{\frac{t^3}{3}}$

g) $\frac{dy}{dt} = -y^2e^{2t}; \quad y(0) = 1$

$\int_1^4 \frac{dy}{y^2} = -\int_0^t e^{2t}dt$ =>> $-\frac{1}{y}|_1^y = -\frac{1}{2}e^{2t}|_0^t$ =>>

$-\frac{1}{y} + 1 = -\frac{1}{2}e^{2t} + \frac{1}{2}$

$\frac{1}{y} = \frac{1}{2}(e^{2t} + 1)$ =>> $y = \frac{2}{e^{2t}+1}$

h) $\frac{dy}{dt} - (2t + 1)y = 0; \quad y(0) = 2$

$\frac{dy}{dt} = y(2t + 1)$ =>> $\int_2^y \frac{dy}{y} = \int_0^t (2t + 1)dt$ =>>

$\ln(y)\,|_2^y = (t^2 + t)|_0^t$

$\ln\left(\frac{y}{2}\right) = t^2 + t$ =>> $y = 2e^{(t^2+t)}$

i) $\frac{dy}{dt} + 4ty^2 = 0; \quad y(0) = 1$

$\int_1^y \frac{dy}{y^2} = -\int_0^t 4tdt$ =>> $-\frac{1}{y}|_1^y = -2t^2|_0^t$

$$y = \frac{1}{1+2t^2}$$

j) $\dfrac{d^2y}{dt^2} = \cos\left(\dfrac{t}{2}\right); \quad y(0) = 0;\ y'(0) = 1$

Let $u = \frac{dy}{dt}$, then

$$\frac{du}{dt} = \cos\left(\frac{t}{2}\right); \quad u(0) = 1$$

$$\int_1^u du = \int_0^t \cos\left(\frac{t}{2}\right) dt$$

$$u = 1 + 2\sin\left(\frac{t}{2}\right)$$

and

$$\frac{dy}{dt} = 1 + 2\sin\left(\frac{t}{2}\right); \quad y(0) = 0$$

$$\int_0^y dy = \int_0^t \left(1 + 2\sin\left(\frac{t}{2}\right)\right) dt$$

$$y = \left(t - 4\cos\left(\frac{t}{2}\right)\right)\Big|_0^t = t - 4\cos\left(\frac{t}{2}\right) + 4$$

$$y = 4 + t - 4\cos\left(\frac{t}{2}\right)$$

Problem 2-7

a) $\frac{du}{dy} = y^2 - 1 \ \lll= \ linear$

$$du = (y^2-1)dy$$

$$u = \frac{1}{3}y^3 - y + C$$

b) $\frac{dT}{dt} = -0.0002(T-5)$ <<= linear

$\frac{dT}{T-5} = -0.0002dt$ =>> $\ln(T-5) = -0.0002\text{t} + \text{C}_1$

$$T = 5 + e^{-0.0002t} + C$$

c) $\frac{dy(x)}{dx} = e^{2y}$ <<= nonlinear

$$e^{-2y}dy = dx \quad =>> \quad -\frac{1}{2}e^{-2y} = x + C_1$$

$$y = \ln(x) + C$$

d) $\frac{du}{dt} = u^2\cos(\pi t) \quad u(0) = -\frac{1}{2}$ <<= nonlinear

$$\int_{-\frac{1}{2}}^{u} \frac{du}{u^2} = \int_0^t \cos(\pi t)\, dt \quad =>> \quad \frac{1}{u}\Big|_{-\frac{1}{2}}^{u} = \frac{1}{\pi}\sin(\pi t)\Big|_0^t$$

$$u = \frac{1}{\frac{1}{\pi}\sin(\pi t) - 2}$$

Problem 2-8

$12.5\,\frac{dx_3^{NaOH}}{dt} + x_3^{NaOH} = 0.73\,x_1^{NaOH}$ with $x_3^{NaOH}(0) = 0.55$ and

$x_1^{NaOH} = 0.75 - 0.08\,u(t)$

$$12.5\,\frac{dx_3^{NaOH}}{dt} = 0.73\,x_1^{NaOH} - x_3^{NaOH} = 0.73(0.67) - x_3^{NaOH}$$

or

$$12.5\,\frac{dx_3^{NaOH}}{dt} = [0.4891 - x_3^{NaOH}](1)$$

Using Separation of Variables

$$\int_{0.55}^{x_3^{NaOH}} \frac{dx_3^{NaOH}}{0.4891 - x_3^{NaOH}} = \frac{1}{12.5}\int_0^t dt \quad =>>$$

$$\ln\,(0.4891 - x_3^{NaOH})\Big|_{0.55}^{x_3^{NaOH}} = -\frac{t}{12.5}$$

$$\frac{0.4891 - x_3^{NaOH}}{-0.0609} = e^{-\frac{t}{12.5}} \quad =>> \quad x_3^{NaOH} = 0.4891 + 0.0609e^{-\frac{t}{12.5}}$$

and

$$x_3^{NaOH} = 0.4891 + 0.0609e^{-\frac{t}{12.5}} = 0.55 - 0.0609(1 - e^{-\frac{t}{12.5}})$$

Problem 2-9

$$2\frac{dv}{dt} + 4v = 16u(t) \quad \text{with} \quad v(0) = 0$$

$$\int_0^v \frac{dv}{16-4v} = \int_0^t \frac{1}{2}dt \quad =>> \quad -\frac{1}{4}\ln(16-4v)\,|_0^v = \frac{1}{2}t$$

$$\ln(16-4v) - \ln(16) = -2t \quad =>> \quad \ln\left(\frac{16-4v}{16}\right) = -2t$$

$$v = 4(1 - e^{-2t})$$

Problem 2-10

$$i_1 = 1.43\times10^{-4}v_s + 0.286i_2 \qquad (1)$$

$$2\times10^{-3}\frac{di_2}{dt} + 21428i_2 = 0.286v_s \qquad (2)$$

From (2): $$9.3\times10^{-8}\frac{di_2}{dt} + i_2 = 1.33\times10^{-5}v_s$$

where $$\tau = 9.3\times10^{-8}sec, \quad and \quad K = 1.33\times10^{-5}\frac{V}{A}$$

From Problem 2-5, $$i_2 = i_2(0) + KD\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$i_2 = 2.66 \times 10^{-4} + 0.000267\left(1 - e^{-\frac{t}{9.3\times 10^{-8}}}\right)$$

From (1):

$$i_1 = 57.2 \times 10^{-4} + 0.76 \times 10^{-4} + 7.64 \times 10^{-5}\left(1 - e^{-\frac{t}{9.3\times 10^{-8}}}\right)$$

$$i_1 = 57.96 \times 10^{-4} + 0.76 \times 10^{-4}\left(1 - e^{-\frac{t}{9.3\times 10^{-8}}}\right)$$