2 Events and Probability

Section 2.1: Sets

2.1. Give three examples of impossible events from everyday life.

Solution: Examples may include the sun rising in the West and setting in the East, water flowing upstream, and time going backwards.

2.2. Give three examples of impossible events from fluids engineering.

Solution: Examples may include fluid with no viscosity, higher density fluid rising, and pressure head decreasing with increasing column height.

2.3. Give three examples of impossible events from materials engineering.

Solution: Examples may include temperature dipping below 0 degrees Kelvin, massless material, inextensible metal.

2.4. Give three examples of impossible events from strength of materials.

Solution: Examples may include the proportional limit of a material exceeding the yield stress, fracture being a reversible event, plastic deformation that is recoverable and not permanent, the sum of the forces in a static structure being not zero, and a linear isotropic material with Poisson's ratio greater than 0.5.

2.5. Give three examples of impossible events from mechanical vibration.

Solution: Examples may include a structure that oscillating without any forces or initial conditions on it, for the same structure, a stiffer structure oscillating slower, a structure oscillating with unlimited speeds, a damped oscillator having a non-decaying amplitude in free vibration, a spring with an infinite stiffness, a free pendulum swinging indefinitely without the addition of energy, eliminating low frequency vibration in a baseball bat when hitting a ball outside the sweet zone.

2.6. Give three examples of impossible events from thermal engineering.

Solution: Examples may include decreasing entropy without energy, increasing energy in a closed system, and radiation without temperature.

2.7. Give three examples of certain events from everyday life.

Solution: Examples may include the sun rising tomorrow, having to fill out tax forms every year, the sun causing the evaporation of 1 trillion tons of water daily, being summer time in the northern hemisphere when it is winter time in the southern hemisphere, applying a force on an object results in an equal and opposite force, and all creatures needing energy to survive.

2.8. Give three examples of certain events from fluids engineering.

Solution: Examples include a faster incompressible flow in a smaller diameter pipe, denser fluid sinking, flow exerting a force on impeller.

2.9. Give three examples of certain events from materials engineering.

Solution: All materials have imperfections, a higher pressure results in a higher boiling point, a constrained beam has a higher buckling load.

2.10. Give three examples of certain events from strength of materials.

Solution: An axial force on a column leads to compression of that column; an elastic body will always deform if forces act on it; any beam subjected to an axial force will eventually fail; in polymer composites, nanoparticles do not create large stress concentrations and thus do not compromise the ductility of the polymer.

2.11. Give three examples of certain events from mechanical vibration.

Solution: If a body is oscillating, then there must be inertia; if a body is oscillating, then there is a restoring force; there is an energy exchange mechanism between the kinetic and potential energies; when hit at the sweet zone, the baseball bat will not be excited at its natural frequencies and therefore there will not be energy loss due to these frequencies vibrating, resulting in more energy being transmitted to the ball.

2.12. Give three examples of certain events from thermal engineering.

Solution: Heating an object results in a higher temperature. Heating a metal object results in expansion. Objects expand and contract at different rates.

2.13. Give an example of complementary events from everyday life.

Solution: winning/losing a game; catching/missing a train; night/day;

2.14. Give an example of complementary events from fluids engineering.

Solution: The flow may be considered laminar or turbulent.

2.15. Give an example of complementary events from materials engineering.

Solution: The material extends linearly or nonlinearly. In other words, the material may or may not obey Hooke's law.

2.16. Give an example of complementary events from strength of materials.

Solution: Examples: A material deforming elastically/plastically; a tensile test yielding expected/unexpected results; a part failing/passing a drop test.

2.17. Give an example of complementary events from mechanical vibration.

Solution: An increase in stiffness and lower frequency.

2.18. Give an example of complementary events from thermal engineering.

Solution: The entropy of the system is increasing/decreasing.

2.19. Given the three events:

$$\begin{aligned} X &= \{\text{odd numbers}\} \\ Y &= \{\text{even numbers}\} \\ Z &= \{\text{negative numbers}\}, \end{aligned}$$

obtain the following:

$$\begin{array}{ll} \text{(a) } X \cup Y & \text{(b) } X \cap Y \\ \text{(c) } \overline{X} & \text{(d) } \overline{Y} \\ \text{(e) } \overline{Z} & \text{(f) } Y \cap Z. \end{array}$$

Solution: We are given the following:

$$\begin{aligned} X &= \{\text{odd numbers}\} \\ Y &= \{\text{even numbers}\} \\ Z &= \{\text{negative numbers}\}. \end{aligned}$$

Then,

$$\begin{aligned} X \cup Y &= \{\text{All numbers (odd numbers + even numbers) }\} \\ X \cap Y &= \{\ \} = \phi \\ \bar{X} &= \{\text{even numbers}\} \\ \bar{Y} &= \{\text{odd numbers}\} \\ \bar{Z} &= \{\text{positive numbers}\} \\ Y \cap Z &= \{\text{negative even numbers}\}. \end{aligned}$$

2.20. Extend Example 2.5 to the case where there are three shafts connecting two clutches (instead of two shafts connected to one clutch). The shafts are numbered from left to right as 1, 2, 3. The failure of the drive train is defined as the failure of either of the three shafts, with events defined by E_1, E_2, and E_3, respectively. Assuming the clutches will not fail, find the following events:

(a) failure of the drive train,

(b) no failure of the drive trains, and

(c) show an illustration of de Morgan's rule.

Solution: Define the following events:

$$\begin{aligned} E_1 &= \text{breakage of shaft 1} \\ E_2 &= \text{breakage of shaft 2} \\ E_3 &= \text{breakage of shaft 3.} \end{aligned}$$

(a) The failure of the drive is defined as the failure of any one of the three shafts, or any two of the three shafts, or all the three shafts. Using set notation, this equals

$$\text{failure of drive train} = E_1 \cup E_2 \cup E_3.$$

(b) Therefore, the event of no failure is given by the event

$$\text{no failure of drive train} = \overline{E_1 \cup E_2 \cup E_3}. \tag{1}$$

The event of no failure can also be defined as an event where all three shafts are operational, which implies

$$\text{no failure of drive train} = \bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3. \tag{2}$$

(c) Since Equations 1 and 2 must be equal to each other, we have an illustration of De Morgan's rule,

$$\overline{E_1 \cup E_2 \cup E_3} = \bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3.$$

Section 2.2: Probability

2.21. Consider Figure 2.8 where $d_{av} = 50$ mm and 50 shafts are manufactured. From the measurements we observe that

25 have the diameter d_{av}
10 have the diameter 1.01 d_{av}
6 have the diameter 1.02 d_{av}
5 have the diameter 0.99 d_{av}
4 have the diameter 0.98 d_{av}.

Sketch the frequency diagram showing appropriate numbers along the axes. Using the frequency interpretation for probability, calculate the probability of occurrence for each shaft size and verify that the sum of these probabilities equals 1.

Solution: We are given that $d_{av} = 50$ mm and the number of shafts manufactured is 50. Also, the following measurements are observed:

25 have the diameter d_{av}
10 have the diameter $1.01 \cdot d_{av}$
6 have the diameter $1.02 \cdot d_{av}$
5 have the diameter $0.99 \cdot d_{av}$
4 have the diameter $0.98 \cdot d_{av}$.

From this, we can draw a frequency diagram, as shown in Figure 1. Then,

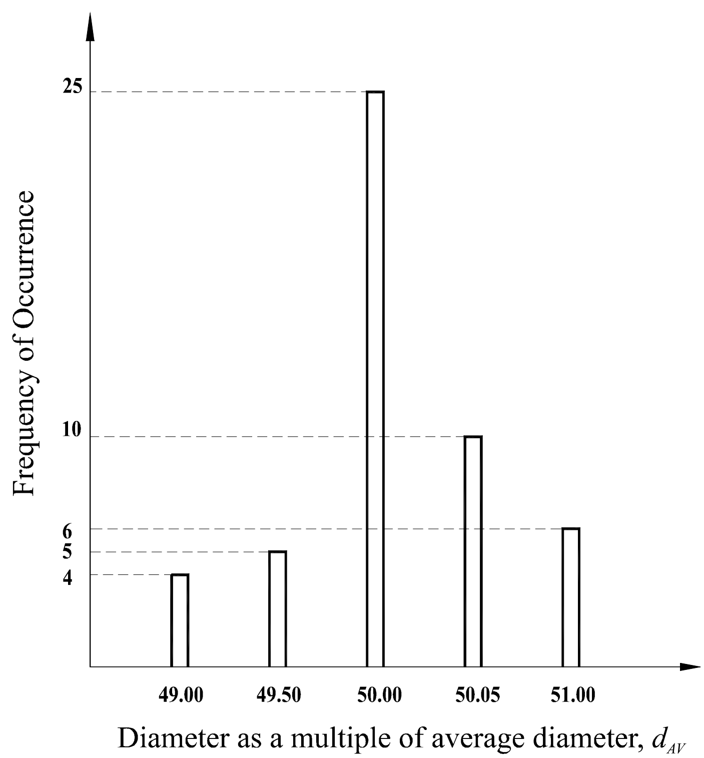

Figure 1: Frequency Diagram of the Number of Shafts

$$
\begin{aligned}
\Pr(d = 0.98 d_{av}) &= \frac{4}{50} = 0.08 \\
\Pr(d = 0.99 d_{av}) &= \frac{5}{50} = 0.1 \\
\Pr(d = d_{av}) &= \frac{25}{50} = 0.5 \\
\Pr(d = 1.01 d_{av}) &= \frac{10}{50} = 0.2 \\
\Pr(d = 1.02 d_{av}) &= \frac{6}{50} = 0.12.
\end{aligned}
$$

$$
\begin{aligned}
\text{Sum of the probabilities} &= \Pr\{d = 0.98 d_{av}\} + \Pr\{d = 0.99 d_{av}\} + \Pr\{d = d_{av}\} \\
&\quad + \Pr\{d = 1.01 d_{av}\} + \Pr\{d = 1.02 d_{av}\} \\
&= 0.08 + 0.1 + 0.5 + 0.2 + 0.12 = 1.0
\end{aligned}
$$

2.22. In your own words, explain the essential ideas of the Theorem of Total Probability, and discuss its importance.

Solution: The total probability theorem provides a way to relate the probability of an event to all other events with which it intersects. The importance in a practical sense is that it is sometimes easier to calculate the intersections than the original probability.

> 2.23. Suppose $\Pr(E_1) = 0.20$, and $\Pr(E_2) = 0.30$.
> (a) If E_1 and E_2 relate to a particular process, are any events not accounted for here? Why?
> (b) If $\Pr(E_1 \cup E_2) = 0.90$ are these processes mutually exclusive? Why?
> (c) If $\Pr(E_1 \cup E_2) = 0.50$, then what is the value of $\Pr(E_1 E_2)$?

Solution: The following data is given

$$\begin{aligned}\Pr\{E_1\} &= 0.20 \\ \Pr\{E_2\} &= 0.30.\end{aligned}$$

(a) We know that the probability of any certain event is 1. If E_1 and E_2 are the only two events occurring, then the probability that any event will definitely occur is given by

$$\Pr\{E_1 \cup E_2\} = \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \cap E_2\}.$$

Even if E_1 and E_2 are assumed to be mutually exclusive, that is $E_1 \cap E_2 = 0$, the probability that E_1 or E_2 or both occur is given by

$$\begin{aligned}\Pr\{E_1 \cup E_2\} &= \Pr\{E_1\} + \Pr\{E_2\} \\ &= 0.20 + 0.30 \\ &= 0.50.\end{aligned}$$

Here even if both events occur, the maximum probability is 0.50, which indicates that some event has been left out.
(b) If $\Pr\{E_1 \cup E_2\} = 0.90$, then there is an inconsistency, as follows:

$$\begin{aligned}\Pr\{E_1 \cup E_2\} &= \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \cap E_2\} \\ \text{So, } \Pr\{E_1 \cap E_2\} &= \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \cup E_2\} \\ &= 0.20 + 0.30 - 0.90 \\ &= -0.40, \text{ which is impossible.}\end{aligned}$$

The maximum probability of the union is $\Pr(E_1 \cup E_2) = 0.50$. Then the two events are mutually exclusive because

$$\begin{aligned}\Pr\{E_1 \cup E_2\} &= \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \cap E_2\} \\ \text{So, } \Pr\{E_1 \cap E_2\} &= \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \cup E_2\} \\ &= 0.20 + 0.30 - 0.50 \\ &= 0, \text{ which indicates that } E_1 \text{ and } E_2 \text{ are mutually exclusive.}\end{aligned}$$

If the union is less than $\Pr = 0.5$, the intersection will have a positive probability, indicating that the two events are not mutually exclusive.
(c) To find $\Pr(E_1 E_2)$ given $\Pr(E_1 \cup E_2) = 0.50$,

$$\begin{aligned}\Pr\{E_1 \cup E_2\} &= \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \cap E_2\} \\ \text{So, } \Pr\{E_1 \cap E_2\} &= \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 \cup E_2\} \\ &= 0.20 + 0.30 - 0.50 \\ &= 0.00.\end{aligned}$$

2.24. Suppose $\Pr(A) = 0.5$, $\Pr(A|B) = 0.3$, and $\Pr(B|A) = 0.1$, calculate $\Pr(B)$.

Solution: We are given the following:

$$\begin{aligned}
\Pr\{A\} &= 0.5 \\
\Pr\{A|B\} &= 0.3 \\
\Pr\{B|A\} &= 0.1.
\end{aligned}$$

To calculate $\Pr\{B\}$, we need to use the definition of conditional probability,

$$\begin{aligned}
\Pr\{A|B\} &= \frac{\Pr\{A \cap B\}}{\Pr\{B\}} \\
\Pr\{B\} &= \frac{\Pr\{A \cap B\}}{\Pr\{A|B\}}.
\end{aligned}$$

$\Pr\{B \cap A\}$is calculated as follows:

$$\begin{aligned}
\Pr\{B|A\} &= \frac{\Pr\{B \cap A\}}{\Pr\{A\}} \\
\Pr\{B \cap A\} &= \Pr\{A\}\Pr\{B|A\} \\
&= (0.5)(0.1) \\
&= 0.05.
\end{aligned}$$

Since $\Pr\{A \cap B\} = \Pr\{B \cap A\}$,

$$\begin{aligned}
\Pr\{B\} &= \frac{\Pr\{B \cap A\}}{\Pr\{A|B\}} \\
&= \frac{0.05}{0.3} = 0.167.
\end{aligned}$$

2.25. Suppose $\Pr(A) = 0.5$, $\Pr(A|B) = 0.5$, and $\Pr(B|A) = 0.1$, calculate $\Pr(B)$. What can be concluded about the statistical relationship, if any, between A and B.

Solution: We are given the following:

$$\begin{aligned} \Pr\{A\} &= 0.5 \\ \Pr\{A|B\} &= 0.5 \\ \Pr\{B|A\} &= 0.1. \end{aligned}$$

Using the relations from the previous problem

$$\begin{aligned} \Pr\{B \cap A\} &= \Pr\{A\}\Pr\{B|A\} \\ &= (0.5)(0.1) = 0.05 \end{aligned}$$

$$\begin{aligned} \Pr\{B\} &= \frac{\Pr\{B \cap A\}}{\Pr\{A|B\}} \\ &= \frac{0.05}{0.5} = 0.10. \end{aligned}$$

Since $\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B\}$, the two events are statistically independent.

2.26. Continuing Example 2.17, let G be the event of rolling a 3 and A_1, A_2, A_3 be the events of selecting die 1, 2, or 3, respectively.
(a) Find the probability of rolling a 3 if one die is selected at random. Use the theorem of total probability.
(b) Determine the probability that die 2 was chosen if a 3 was rolled with the randomly selected die.

Solution: It is given that G is an event of rolling a 3 and A_1, A_2, A_3 equal the event of selecting die 1,2, or 3, respectively.
(a) Since the die is selected at random,

$$\Pr(A_1) = \Pr(A_2) = \Pr(A_3) = 1/3.$$

From the table,

$$\Pr(G|A_1) = 1/12,\ \Pr(G|A_2) = 1/6,\ \Pr(G|A_3) = 1/6.$$

Substituting these into the total probability theorem, we obtain the probability of rolling a 3,

$$\begin{aligned}\Pr(G) &= \Pr(G|A_1)\Pr(A_1) + \Pr(G|A_2)\Pr(A_2) + \Pr(G|A_3)\Pr(A_3) \\ &= \left(\frac{1}{12}\cdot\frac{1}{3}\right) + \left(\frac{1}{6}\cdot\frac{1}{3}\right) + \left(\frac{1}{6}\cdot\frac{1}{3}\right) \\ &= \frac{5}{36} = 0.139\end{aligned}$$

(b) We need to evaluate $\Pr(A_2|G)$. Using Bayes' rule,

$$\Pr(A_2|G) = \frac{\Pr(G|A_2)\Pr(A_2)}{\Pr(G)}.$$

We have then,

$$\Pr(A_2|G) = \frac{\frac{1}{6}\cdot\frac{1}{3}}{\frac{5}{36}} = \frac{2}{5} = 0.4.$$

2.27. Two cables are used to lift load W. Normally, only cable A will be carrying the load; cable B is slightly longer than A, so it does not participate in carrying the load. But if cable A breaks, then B will have to carry the full load until A is replaced. The probability that cable A will break is 0.02. The probability that B will fail if it has to carry the load by itself is 0.30.

$$\begin{aligned} A &= \text{ cable } A \text{ breaks} \\ B &= \text{ cable } B \text{ breaks} \\ \Pr(A) &= 0.02 \\ \Pr(B|A) &= 0.30 \ \ = \text{probability } B \text{ breaks if } A \text{ already broke.} \end{aligned}$$

(a) What is the probability that both cables will fail?
(b) If the load remains lifted, what is the probability that none of the cables have failed?

Solution: Let

$$\begin{aligned} A &= \text{cable } A \text{ breaks} \\ B &= \text{cable } B \text{ breaks.} \end{aligned}$$

Given

$$\begin{aligned} \Pr(A) &= 0.02 \\ \Pr(B|A) &= 0.30 \qquad B \text{ would break only if } A \text{ already broke.} \end{aligned}$$

(a) The probability that both cables will fail is

$$\Pr(AB) = \Pr(BA) = \Pr(B|A)\Pr(A) = 0.3 \cdot 0.02 = 0.006.$$

(b) The condition that the load remains lifted is expressed as $\overline{A} \cup \overline{B}$, but if $\overline{A}$ happened then $\overline{B}$ would definitely occur, that is, $\Pr(\overline{A}\overline{B}) = \Pr(\overline{A})$,

$$\begin{aligned} \Pr(\overline{A}\overline{B}|\overline{A} \cup \overline{B}) &= \Pr(\overline{A}|\overline{A} \cup \overline{B}) = \frac{\Pr[\overline{A} \cap (\overline{A} \cup \overline{B})]}{\Pr(\overline{A} \cup \overline{B})} \\ &= \frac{\Pr(\overline{A})}{1 - \Pr(AB)} \\ &= \frac{1 - \Pr(A)}{1 - \Pr(AB)} = \frac{1 - 0.02}{1 - 0.006} = 0.986. \end{aligned}$$

2.28. Consider the drive train Example 2.5 from a different perspective. The drive train consists of a rotor R and turbine blades B. How well the system operates depends on the precision of the manufactured components. Testing of the individual components by the manufacturer yields the following information:

$$\begin{aligned} &0.1\% \text{ of } R \text{ have imperfections} \\ &0.01\% \text{ of } B \text{ fail.} \end{aligned}$$

Also, it is determined that if R has imperfections, then the blades B are 50% more likely to fail due to the additional vibration forces that result. Determine the probability that the system will pass inspection.

Solution: Let E_R = failure event (imperfection) for R, E_B = failure event for blade. We are given the following:

$$\begin{aligned} \Pr(E_R) &= 0.001 \\ \Pr(E_B) &= 0.0001 \\ \Pr(E_B|E_R) &= 1.5\Pr(E_B|\overline{E_R}). \end{aligned}$$

Then,

$$\begin{aligned} \Pr(\text{failure}) &= \Pr(E_R \cup E_B) \\ &= \Pr(E_R) + \Pr(E_B) - \Pr(E_R \cap E_B) \\ &= \Pr(E_R) + \Pr(E_B) - \Pr(E_B|E_R)\Pr(E_R). \end{aligned}$$

Use Theorem of Total Probability:

$$\begin{aligned} \Pr(E_B) &= \Pr(E_B|E_R)\Pr(E_R) + \Pr(E_B|\overline{E_R})\Pr(\overline{E_R}) \\ 0.0001 &= \Pr(E_B|E_R) \times 0.001 + \frac{1}{1.5}\Pr(E_B|E_R)(1-0.001) \\ \Pr(E_B|E_R) &= 0.00015. \end{aligned}$$

Therefore,

$$\begin{aligned} \Pr(\text{failure}) &= 0.001 + 0.0001 - 0.00015 \times 0.001 \\ &= 0.00109985. \end{aligned}$$

And the probability of passing inspection is $1-\Pr(\text{failure}) = 0.999$