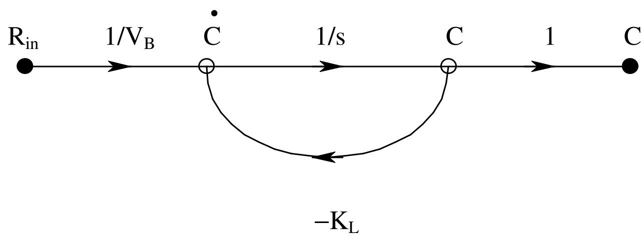
Chapter 2 Home Problem Solutions

2.1 A) SFG for the simple, one-compartment drug infusion system:



$$\frac{C}{R_{in}}(s) = \frac{(1/V_B)}{s + K_L} \quad \text{Transfer function.}$$

B) Let $R_{in}(t) = R_o\delta(t)$, $\therefore R_{in}(s) = R_o$, and by inverse LT tables: $c(t) = (R_o/V_B)e^{-K_L t}$, $t \geq 0$.

C) Let $R_{in}(t) = R_d U(t)$: $\therefore R_{in}(s) = R_d/s$, and by inverse LT tables: $c(t) = (R_d/K_L)(1 - e^{-K_L t})$, $t \geq 0$. The $c_{SS} = R_d/K_L$ as $t \to \infty$.

2.2 LTI system described by 2^{nd} order ODE:

$$\ddot{y} + 7\dot{y} + 10y = x(t)$$

A) $y[p^2 + 7p + 10] = x(t)$; factoring, we find: $y[(p + 2)(p + 5)] = x$. Thus $p_1 = -2$, $p_2 = -5$.

B) Use Laplace transforms, find xfer function:

$$\frac{Y}{X}(s) = \frac{1}{\underset{a}{(s + 2)}\underset{b}{(s + 5)}}$$

Let $x(t) = \delta(t)$, so $X(s) = 1$. The inverse Laplace transform table gives us:

$$y(t) = \frac{1}{(\underset{b}{5} - \underset{a}{2})}[e^{-2t} - e^{-5t}]$$

C) Now let there be a step input, $x(t) = 10U(t)$, so $X(s) = 10/s$: The inverse Laplace transform is:

$$y(t) = \frac{10}{(2)(5)}\left[1 + \frac{1}{(\underset{a}{2} - \underset{b}{5})}(5e^{-2t} - 2e^{-5t})\right] = 1\,[1 - 1/3(5e^{-2t} - 2e^{-5t})]$$

2.3 A) There are two compartments: V_B liters of blood, V_E liters of extracellular fluid.

The ODEs describing the system are:

$$\dot{C}_B V_B = -K_{LB}C_B - K_{DBE}(C_B - C_E) + R_{in} \qquad \mu g/min.$$

$$\dot{C}_E V_E = +K_{DBE}(C_B - C_E) \qquad \mu g/min.$$

These ODEs can be rewritten in terns of concentrations:

$$\dot{C}_B = -C_B(K_{LB} + K_{DBE})/V_B + C_E(K_{DBE}/V_B) + R_{in}/V_B \qquad (\mu g/l)/min.$$

$$\dot{C}_E = +C_B(K_{DBE}/V_E) - C_E(K_{DBE}/V_E) \qquad (\mu g/l)/min.$$

This system and its linear SFG is illustrated on **Page 2B**.

B) Consider the system to be in the DC steady state, so $\dot{C}_B$ & $\dot{C}_E = 0$. Thus we can write:

$$C_B(K_{LB} + K_{DBE})/V_B - C_E(K_{DBE}/V_B) = R_{in}/V_B$$

$$C_B(K_{DBE}/V_E) - C_E(K_{DBE}/_{VE}) = 0$$

The second equation above tells us that $C_B = C_E$ in the SS. The first equation can easily be solved for C_B in the

SS:

$C_{BSS} = C_{ESS} = R_d/K_{LB.}$ (R_d is the IV drip rate in μg/min.)

C) A bolus injection gives the PK system's impulse response. First we use Mason's rule to obtain the system's transfer function, $\frac{C_B}{R_{in}}(s)$, and note that $R_{in}(t) = R_o\delta(t)$, so $R_{in}(s) = R_o$. The C_B transfer function is, after some algebra:

$$\frac{C_B}{R_{in}} = \frac{(1/V_B)(1/s)[1 - (-K_{DBE}/(V_E\, s))]}{1 - \{-(K_{LB} + K_{DBE})/(V_B\, s) - K_{DBE}/(V_E\, s) + K_{DBE}^2/(V_E V_B\, s^2)\} + \{-(K_{LB} + K_{DBE})/(V_B s) \times -K_{DBE}/(V_E\, s)\}}$$

$$C_B(s) = \frac{(R_o/V_B)(s + K_{DBE}/V_E)}{s^2 + s[(K_{LB} + K_{DBE})/V_B + K_{DBE}/V_E] + [(K_{LB} + K_{DBE})/V_B \times K_{DBE}/V_E - K_{DBE}^2/(V_E V_B)]}$$

Subbing numbers:

$$C_B(s) = \frac{R_o\, 0.33333\,(s + 0.06667)}{s^2 + s[0.73333] + 0.02222}, \text{ Factoring the denominator, } C_B(s) = \frac{R_o\, 0.33333\,(s + \overset{a}{0.06667})}{(s + \underset{c}{1.4325})(s + \underset{b}{0.03412})}$$

The inverse Laplace transform yields: $C_B(t) = R_o \dfrac{0.33333}{\underset{c-b}{1.39838}} \{ \overset{a-b}{0.03255}\, e^{-0.03412t} - \overset{a-c}{(-1.3658)} e^{-1.4325t} \}$

2.4 An underdamped, quadratic LPF is described by the second-order ODE:

$$\ddot{y} + (2\xi\omega_n)\dot{y} + \omega_n^2 y = x(t)$$

A) Find the root locations in the complex s-plane: Laplace Xform the ODE:

$Y(s)[s^2 + (2\xi\omega_n) s + \omega_n^2] = X(s)$: The complex-conjugate roots (poles) of the denominator are thus:

$s_1, s_2 = -\xi\omega_n \pm ½\sqrt{(4(\xi\omega_n)^2 - 4\omega_n^2)} < 0 = -\xi\omega_n \pm j\omega_n\sqrt{(1-\xi^2)} = -0.5 \pm j\,0.8660$ (poles lie in the left-half s-plane). The filter's Xfer function is thus:

$$\frac{Y(s)}{X(s)} = \frac{1}{[s^2 + (2\xi\omega_n)s + \omega_n^2]} = \frac{1}{s^2 + 1s + 1}$$

B) The filter's *unit step response* can be found by inverse Laplace Xform tables:

$$Y(s) = \frac{1}{s}\,\frac{1}{[s^2 + (2\xi\omega_n)s + \omega_n^2]}: \quad y(t) = \frac{1}{\omega_n^2}\left\{1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[\omega_n\sqrt{1-\xi^2}\,t + \phi]\right\}, \text{ where } \phi = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

C) The filter's unit impulse is:

$$Y(s) = \frac{1}{[s^2 + 1s + 1]}. \quad \text{Inverse transforming: } y(t) = (1/\omega_n)\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[\omega_n\sqrt{1-\xi^2}\,t]$$

$\omega_n = 1$ r/s, $\xi = 0.5$, so $y(t) = 1.1547\, e^{-0.5t} \sin[0.866\, t]$, so $y(t) = 1.1547\, e^{-0.5t} \sin[0.866\, t]$

2.5 Alternative ODE form for an underdamped, quadratic LPF:

$$\ddot{y} + 2a\,\dot{y} + (b^2 + a^2)y = x(t)$$

Note $\omega_n^2 \equiv (b^2 + a^2)$, & $(2\xi\omega_n) \equiv 2a$, so $a = \omega_n\xi$, and $\omega_n^2 = b^2 + \omega_n^2\xi^2 \rightarrow$ thus $b^2 = \omega_n^2(1-\xi^2) \rightarrow b = \omega_n\sqrt{1-\xi^2}$

A) The filter's Xfer function is:

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + 2a\,s + (b^2 + a^2)}$$

$2a = 1$, $(b^2 + a^2) = 0.7500 + 0.25 = 1$, thus the denom. is: $s^2 + s + 1$

The denominator roots (poles) are at: $s_1, s_2 = -0.5 \pm ½\sqrt{1-4} = -0.5 \pm j\,0.8660.$

B) The input is a unit step: $x(t) = U(t)$, thus $X(s) = 1/s$:

$$Y(s) = \frac{1}{s\,[s^2 + s + 1]}: \text{ By LT tables: } y(t) = \{1 - 1.1547\, e^{-0.5t} \sin[0.8660t + \phi]\},\ \phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = 60^o$$

C) The input is a unit impulse, so $X(s) = 1$. From tables:

$$y(t) = 1.1547\, e^{-0.5t} \sin[0.866\, t]$$

2.6 A hyperbolic capacitor discharge waveform results when a capacitor charged to V_o is allowed to discharge through a nonlinear (square law) conductance beginning at time $t = 0$. Kirchoff's current law for this simple circuit is simply: $C\dot{V}_c + \beta V_c^2 = 0$. This ODE can be linearized and solved as Bernouilli's Equation. Since the hyperbolic waveform for $V_c \to \infty$ at $t = 0$, we delay the start of $V_c(t)$ by τ_o seconds. This yields a hyperbolic waveform section given by $V_c(t) = (C/\beta)/(t + \tau_o)$, for $\tau_o \le t \le \infty$. $\tau_o = C/(\beta V_o)$. Thus $V_c(0) = V_o$.

This simple nonlinear circuit is the core of an analog instantaneous pulse frequency demodulator (IPFD) invented by the author; cf. Northrop, R.B. & H.M. Horowitz. 1966. An instantaneous pulse frequency demodulator for neurophysiological applications. *Proc. Symp. Biomed. Engrg. I,* Milwaukee, Wisc. I Biomedical Instrumentation, Paper 1-1, pp 5-8.

2.7 To create a logarithmic hyperbolic capacitor discharge waveform used for generating a log(IPFD), a nonlinear conductance of the form, $i_{nl} = [C\,k\,\log_{10}e]10^{(Vc/k)}$ is used. $e = 2.71828$, k is a constant, and V_c is the capacitor voltage at time t.

2.8 A) $\det\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = 8 + 3 = 11$ **B)** $\det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 4 & 1 & 3 \end{bmatrix} = 1[9 - 5] = 4$ **C)** $\det\begin{bmatrix} 1 & 0 & 6 \\ 3 & 4 & 15 \\ 5 & 6 & 21 \end{bmatrix} = 1[48 - 90] + 6[18 - 20] = -54.$

2.11 Solve the simultaneous equations using Cramer's Rule:

A) $3x_1 - 5x_2 = 0$

$x_1 + x_2 = 2$ $\quad \Delta = \begin{bmatrix} 3 & -5 \\ 1 & 1 \end{bmatrix} = 3 + 5 = 8$ $\quad x_1 = \begin{bmatrix} 0 & -5 \\ 2 & 1 \end{bmatrix}/\Delta = 10/8,$ $\quad x_2 = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}/\Delta = 6/8$

B) $2x_1 + x_2 + 5x_3 + x_4 = 5$
$x_1 + x_2 - 3x_3 - 4x_4 = -1$
$3x_1 + 6x_2 - 2x_3 + x_4 = 8$
$2x_1 + 2x_2\ 2x_3 - 3x_4 = 2$

$$\Delta = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -3 & -4 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = \ldots$$

2.12 $\ddot{y} + 5\dot{y} + 2y = r(t) \rightarrow \dot{x}_1 = 0x_1 + x_2$

$\dot{x}_2 = -2x_1 - 5x_2 + r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,r$$

2.13 $2\ddot{y} + 3\dot{y} + y = \dot{r} + 2r \rightarrow \ddot{y} + \underset{a_1}{1.5}\dot{y} + \underset{a_2}{0.5}y = \underset{b_1}{0.5}\,\dot{r} + \underset{b_2}{r}$ $\quad b_0 = 0$

The state equations are: $\dot{x}_1 = 0x_1 + x_2 + (0.5)\,r$

$\dot{x}_2 = -0.5x_1 - 1.5x_2 + [1 - 1.5(0.5)]r$ Hence $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix}$ & $\mathbf{B} = \begin{bmatrix} 0.5 \\ -0.25 \end{bmatrix}$

2.14 $\ddot{y} + \underset{a_1}{3}\dot{y} + \underset{a_2}{2}y = \underset{b_0}{3}\ddot{r} + \underset{b_1}{5}\dot{r} + \underset{b_2}{r}$ Where: $y = x_1$, $\dot{x}_1 = x_2$

↓

$\dot{x}_1 = \quad + x_2 + \overset{-4}{(5 - 3\times3)}\,r$ Hence $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ & $\mathbf{B} = \begin{bmatrix} -4 \\ +7 \end{bmatrix}$

$\dot{x}_2 = -2x_1 - 3x_2 + \underset{+7}{[\overset{-5}{(1 - 2\times3)} - \overset{+12}{3(5 - 3\times3)}]}\,r$

2.15 A) $H(s) = \dfrac{s-8}{s^2 + 3s + 2} = \dfrac{s-8}{(s+1)(s+2)}$ Real zero at **s** = +8, two real poles on the LH s-plane at **s** = −1 & **s** = −2.
System is <u>stable</u>.

B) $\dfrac{s+1}{s^2 + 3s + 3}$ Real zero at **s** = −1. Two complex-conjugate poles in the left-hand s-plane at $\mathbf{s_1} = -3/2 - ½\,j\sqrt{3}$ and $\mathbf{s_2} = -3/2 + ½\,j\sqrt{3}$
System is <u>stable.</u>

C) $\dfrac{s+1}{s^2 + 3s - 3}$ Real zero at **s** = −1. Two real poles; one at $\mathbf{s_1}$ = −3.7913, one at $\mathbf{s_2}$ = + 0.7913 in the RH s-plane.
System is <u>unstable</u>.

D) $\dfrac{s+1}{s^2 - 3s - 3}$ Real zero at **s** = −1. Two real poles; one at $\mathbf{s_1}$ = +3.7913 in the RH s-plane, one at $\mathbf{s_2}$ = −0.7913.
System is <u>unstable</u>.

E) $\dfrac{s-1}{s^2 + 3s - 4}$ Real zero at **s** = +1. Real poles at **s** = −4 and **s** = + 1, System is <u>unstable</u>.

2.16 The SISO linear feedback system's loop gain is: $A_L(s) = \dfrac{-K(s+8)}{(s-1)(s+6)}$

A) Considering the FB system's root locus diagram, it is stable for two poles in the LH s-plane this is when the RH plane pole locus branch crosses the origin From **Eq. B.5** in the text, the gain must be greater than:

$$K_c > \frac{6 \times 1}{8} = 0.75$$

B) No, one CL system pole is always in the RH s-plane.

2.17 Using Mason's rule:

$$X/Y = \frac{a\,b\,c}{1 - [-d - e - bf] + [-d \times -e]} = \frac{a\,b\,c}{1 + d + e + bf + de}$$

2.18 The 2I2O LS is described by the ODEs:

$$\dot{x}_1 = -a_{11}x_1 + a_{12}x_2 + b_1u_1$$

$$\dot{x}_2 = a_{21}x1 - a_{22}x_2 + b_2u_2$$

A) The SFG is on **Page 6B.**

B) Find $\mathbf{X_1/U_1}$ using Mason's rule:

$$X_1/U_1 = \frac{(b_1/s)[1 - (-a_{22}/s)]}{1 - [-a_{11}/s - a_{22}/s + a_{21}a_{12}/s^2] + [-a_{11}/s \times -a_{22}/s]} = \frac{b_1(s + a_{22})}{s^2 + s[a_{11} + a_{22}] + a_{11}a_{22} - a_{21}a_{12}}$$

C) $$X_2/U_2 = \frac{(b_2/s)[1 - (-a_{11}/s)]}{1 - [-a_{11}/s - a_{22}/s + a_{21}a_{12}/s^2] + [-a_{11}/s \times -a_{22}/s]} = \frac{b_2(s + a_{11})}{s^2 + s[a_{11} + a_{22}] + a_{11}a_{22} - a_{21}a_{12}}$$

D) $$X_1/U_2 = \frac{(b_2/s)(a_{12}/s)}{1 - [-a_{11}/s - a_{22}/s + a_{21}a_{12}/s^2] + [-a_{11}/s \times -a_{22}/s]} = \frac{b_2a_{12}}{s^2 + s[a_{11} + a_{22}] + a_{11}a_{22} - a_{21}a_{12}}$$

E) $$X_2/U_1 = \frac{(b_1/s)(a_{21}/s)}{1 - [-a_{11}/s - a_{22}/s + a_{21}a_{12}/s^2] + [-a_{11}/s \times -a_{22}/s]} = \frac{b_1a_{21}}{s^2 + s[a_{11} + a_{22}] + a_{11}a_{22} - a_{21}a_{12}}$$

2.19 In SS:

$$\begin{aligned} \dot{x}_1 = 0 &= -a_{11}x_1 + a_{12}x_2 + b_1U_1 \\ \dot{x}_2 = 0 &= a_{21}x_1 - a_{22}x_2 + b_2U_2 \end{aligned} \quad \rightarrow \quad \begin{aligned} b_1U_1 &= a_{11}x_1 - a_{12}x_2 \\ b_2U_2 &= -a_{21}x_1 + a_{22}x_2 \end{aligned}$$

Solve using Cramer's rule:

$$\Delta = \begin{vmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$X_{1SS} = \frac{\begin{vmatrix} b_1U_1 & -a_{12} \\ b_2U_2 & a_{22} \end{vmatrix}}{\Delta} = \frac{b_1 a_{22}U_1 + b_2 a_{12}U_2}{a_{11}a_{22} - a_{21}a_{12}} \qquad X_{2SS} = \frac{\begin{vmatrix} a_{11} & b_1U_1 \\ -a_{12} & b_2U_2 \end{vmatrix}}{\Delta} = \frac{a_{11}b_2U_2 + a_{12}b_1U_1}{a_{11}a_{22} - a_{21}a_{12}}$$

2.20 A) From the SFG of **Fig. P2.20**, show $\mathbf{X_2}(s)/\mathbf{U_1}(s) \to 0$ for an ideally-decoupled system. There are no cofactors. Note that the bottom dashed feedback path has gain $-a_{21}*$, *not* $-a_{22}*$.

$$X_2/U_1 = \frac{c_1 b_1^{-1} b_1 (1/s)a_{21}(1/s) + c_1b_1^{-1}b_1(1/s)(-a_{21}*)b_2^{-1}b_2(1/s)}{\begin{aligned}1 - [&-a_{11}/s - a_{22}/s + a_{12}a_{21}/s^2 - a_{12}* b_1^{-1} b_1(1/s^2) a_{21} - (P_1 - a_{11}*)b_1^{-1} b_1(1/s) - (P_2 - a_{22}*) b_2^{-1} b_2(1/s) \\ &- a_{22}* b_2^{-1} b_2(1/s^2)a_{12}] + \{(-a_{11}/s)(-a_{22}/s) + [-(P_1 - a_{11}*)b_1^{-1} b_1(1/s) \times -(P_2 - a_{22}*) b_2^{-1} b_2(1/s)] + \\ &(-a_{12}* b_1^{-1} b_1(1/s^2) a_{21})(-a_{21}* b_2^{-1} b_2(1/s^2)a_{12})\}\end{aligned}}$$

Reducing the Denominator, $D(s)$: We remove asterisks for ideal decoupling:

$$\begin{aligned}D(s) = 1 - [&-a_{11}/s - a_{22}/s + a_{12}a_{21}/s^2 - a_{12}(1/s^2) a_{21} - (\mathbf{P_1} - a_{11})(1/s) - (\mathbf{P_2} - a_{22})(1/s) - a_{22}(1/s^2)a_{12}] \\ &+ \{+ a_{11}a_{22}/s^2) + (\mathbf{P_1} - a_{11})(\mathbf{P_2} - a_{22})(1/s^2) + a_{12}^2a_{21}^2(1/s^4) \}\end{aligned}$$

Reducing the numerator, $N(s)$:

$N(s) = c_1a_{21}/s^2 - c_1a_{21}/s^2 = 0$, Thus $X_2/U_1 = 0$, x_2 is decoupled from input 1.

2.21 A) See **Page 7B** for NLSFG:

B) See **Page 7B** for Decoupled system NLSFG.

C) ODEs for the perfectly-tuned, decoupled system, where: $K_m* = K_m$, $V_m* = V_m$, $K_{01}* = K_{01}$, $K_{12}* = K_{12}$, $b_1* = b_1$, & $b_2* = b_2$. Thus we have:

$$u_1 = b_1^{-1} [-P_1 x_1 + a_1v_1 + K_{01} x_1^2 - K_{12} x_2 + V_m x_1/(K_m + x_1)]$$

$$u_2 = b_2^{-1} [-P_2 x_2 + a_2 v_2 + K_{12} x_2 - K_{02} - V_m x_1/(K_m + x_1)]$$

Thus:

$$\dot{x}_1 = a_1 v_1 - P_1 x_1 + K_{01} x_1^2 + V_m x_1/(K_m + x_1) - K_{12} x_2 - V_m x_1/(K_m + x_1) - K_{01}x_1^2 + K_{12}x_2 = a_1 v_1 - P_1 x_1$$

$$\dot{x}_2 = a_2 v_2 - K_{02} + K_{12} x_2 - V_m x_1/(K_m + x_1) - P_2 x_2 - K_{12} x_2 + V_m x_1/(K_m + x_1) + K_{02} = a_2v_2 - P_2x_2$$

D) Decoupling controllers for pharmacokinetic drug administration systems are not too effective, because the drug concentrations are non-negative, and 100% effective operation of this kind of controller requires that certain states go negative.