

Electromagnetic Waves and Materials

Week 2

Instructor:

What will be covered today

- Uniform plane waves
- Good conductor approximation
- Skin effect
- Boundary condition between PEC and dielectric medium
- AC resistance of round wires
- Bounded transmission line

Uniform plane waves

What is a sourceless medium:

$$\rho_v = 0 \text{ and}$$

$$\mathbf{J}_{\text{source}} = 0$$

One-dimensional solution for a simple lossy medium with parameters
in Cartesian coordinates :

$$\epsilon_r \quad \mu_r \quad \sigma$$

$$E_z = H_z = 0 \quad \dots\dots\dots (2.1)$$

$$\eta \mathbf{H} = \hat{\mathbf{z}} \times \mathbf{E} \quad \dots (2.2)$$

$$\mathbf{E} = \mathbf{E}_t \begin{Bmatrix} e^{+jkz} \\ e^{-jkz} \end{Bmatrix} \quad \dots\dots\dots (2.3)$$

where

$$k^2 = \omega^2 \mu \epsilon - j \omega \mu \sigma \quad \dots\dots\dots (2.4)$$

is complex.

Uniform plane waves

The characteristic impedance is also complex

$$\eta = \left(\frac{j\omega\mu}{\sigma + j\omega\epsilon} \right)^{1/2} \dots\dots\dots (2.5)$$

Define: $k = \beta - j\alpha$

where,

α is the attenuation constant (Np/m)

β is the phase constant (rad/m).

Which can be obtained by solving the equation (2.4),

$$\alpha = \omega\sqrt{\mu\epsilon} \left\{ \frac{1}{2} \left[\sqrt{1 + T^2} - 1 \right] \right\}^{1/2} \dots\dots\dots (2.6)$$

$$\beta = \omega\sqrt{\mu\epsilon} \left\{ \frac{1}{2} \left[\sqrt{1 + T^2} + 1 \right] \right\}^{1/2} \dots\dots\dots (2.7)$$

where loss tangent T is defined as

$$T = \frac{\sigma}{\omega\epsilon} \dots\dots\dots (2.8)$$

Uniform plane waves

For a low-loss dielectric, the loss tangent $T \ll 1$ and

$$\alpha \approx \frac{\sigma}{2} \sqrt{\mu/\epsilon}, \quad T \ll 1 \quad \dots\dots (2.9)$$

$$\beta \approx \omega \sqrt{\mu\epsilon}, \quad T \ll 1 \quad \dots\dots (2.10)$$

$$\eta \approx \sqrt{\frac{\mu}{\epsilon}} \angle \tan^{-1} \frac{\sigma}{2\omega\epsilon}, \quad T \ll 1 \quad \dots\dots (2.11)$$

Good conductor approximation

Definition for good conductor

$$T \gg 1$$

We can get

$$\alpha \approx \beta \approx \sqrt{\pi f \mu \sigma} = \frac{1}{\delta}, \quad T \gg 1 \quad \dots\dots (2.12)$$

$$\eta \approx \frac{\sqrt{2}}{\sigma \delta} \angle 45^\circ, \quad T \gg 1 \quad \dots\dots (2.13)$$

δ is called the skin depth. The characteristic impedance Z_s or surface impedance

$$\eta = Z_s = R_s + jX_s = (1 + j) \sqrt{\frac{\omega \mu}{2\sigma}} \quad \dots\dots (2.14)$$

Skin effect

When waves go through the good conductor, they will be attenuated exponentially as

$$E(z,t) = E_0 e^{-z/\delta} \cos(\omega t - z/\delta) \quad \dots\dots (2.15)$$

$$J(z,t) = J_0 e^{-z/\delta} \cos(\omega t - z/\delta) \quad \dots\dots (2.16)$$

where

$$J_0 = \sigma E_0 \quad \dots\dots (2.17)$$

For DC, $f = 0$, current density is uniform inside the good conductor

for the time-harmonic AC case ($f \neq 0$), we define skin depth

$$\delta = 1 / \sqrt{\pi f \mu \sigma}$$

- The amplitude of the current density drops to e^{-1} (38.8%) for a distance of δ .
- It will drop to zero (practically) for a distance of 4δ .
- At high frequencies the current is practically confined to the skin of the conductor and the phenomenon is hence described as **skin effect**.

B.C. between PEC and dielectric medium

For a perfect conductor, $\sigma = \infty \Rightarrow \delta = 0$, we can get $\mathbf{E} = \mathbf{0}$ Inside the conductor

If medium 1 is a perfect conductor and medium 2 is a dielectric,
we can get the boundary conditions as

$$\hat{n}_{12} \cdot \mathbf{D}_2 = \rho_s \quad \dots\dots (2.19)$$

$$\hat{n}_{12} \cdot \mathbf{B}_2 = 0 \quad \dots\dots (2.20)$$

$$\hat{n}_{12} \times \mathbf{E}_2 = 0 \quad \dots\dots (2.21)$$

$$\hat{n}_{12} \times \mathbf{H}_2 = \mathbf{K} \quad \dots\dots (2.22)$$

\hat{n} is **the** normal unit vector on the boundary.

B.C. between PEC and dielectric medium

Since $\mathbf{E} = \mathbf{0}$ Inside the conductor, we get: (2.24) and (2.26) see problem P2.2

$$E_t = 0 \quad \dots\dots (2.13)$$

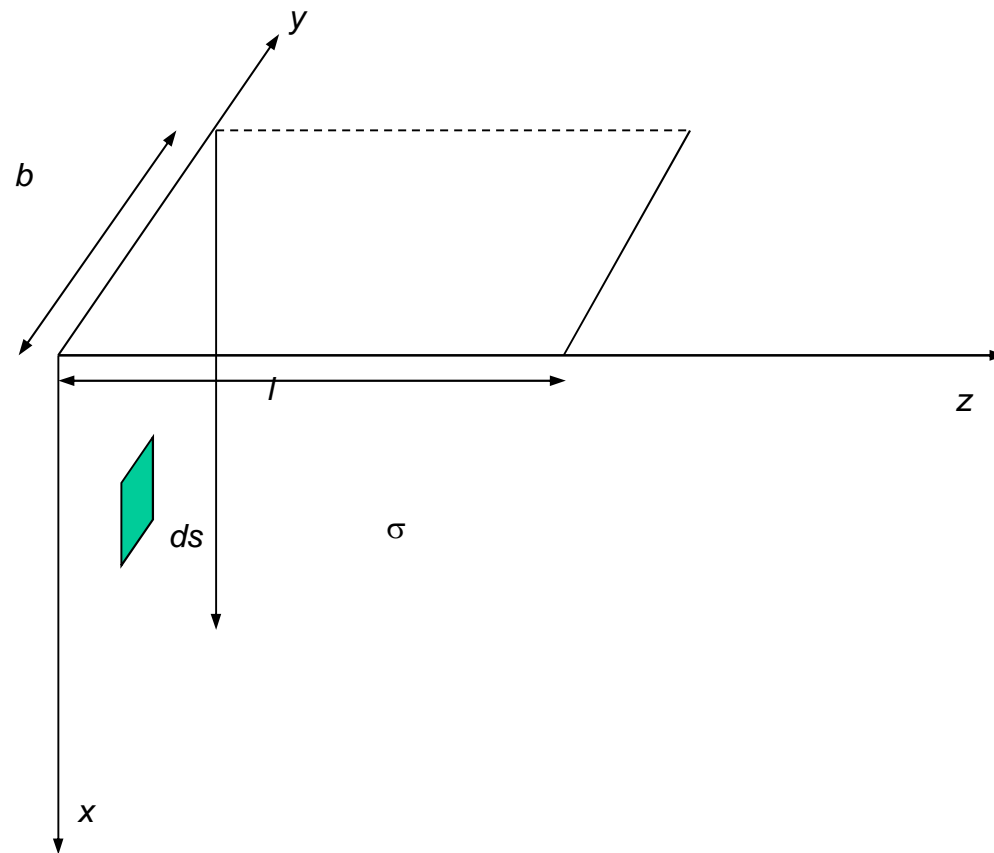
$$\frac{\partial E_n}{\partial n} = 0 \quad \dots\dots (2.24)$$

$$H_n = 0 \quad \dots\dots (2.25)$$

$$\frac{\partial H_t}{\partial n} = 0 \quad \dots\dots (2.26)$$

n is the normal direction and t is the tangential direction.

AC resistance



For an infinitely deep good conductor of conductivity σ defined by the half-space $0 < x < \infty$, shown in the figure, E and H can be expressed as in next slide:

AC resistance

E and ***H*** inside half-space conductor

$$\tilde{\mathbf{E}}(x) = \hat{z}E_0 e^{-x/\delta - jx/\delta} \dots\dots\dots (2.27)$$

$$\tilde{\mathbf{H}}(x) = -\hat{y}E_0 \frac{\sigma\delta}{\sqrt{2}} e^{-x/\delta} e^{-jx/\delta} e^{-j45^\circ} \dots\dots\dots (2.28)$$

Real parts of the fields are

$$\mathbf{E}(x,t) = \hat{z}E_0 e^{-x/\delta} \cos(\omega t - \frac{x}{\delta}) \dots\dots\dots (2.29)$$

$$\mathbf{H}(x,t) = \hat{y}E_0 \frac{\sigma\delta}{\sqrt{2}} e^{-x/\delta} \cos(\omega t - \frac{x}{\delta} - 45^\circ) \dots\dots\dots (2.30)$$

The time averaged power density

$$\langle \mathbf{S} \rangle = \hat{x} \frac{1}{2} E_0^2 \frac{\sigma\delta}{\sqrt{2}} e^{-2x/\delta} \cos 45^\circ = \hat{x} \frac{1}{4} E_0^2 \sigma\delta e^{-2x/\delta} \dots\dots\dots (2.31)$$

AC resistance

Total power entering the conductor of width b and length l is given by

$$P = \langle \mathbf{S} \rangle_{x=0} bl = \frac{1}{4} E_0^2 \sigma \delta bl \quad \dots\dots (2.32)$$

Total phasor current entering the conductor of width b

$$\tilde{I} = \iiint \tilde{\mathbf{J}} \cdot d\mathbf{s} = \iint \sigma \tilde{\mathbf{E}} \cdot d\mathbf{s} = \int_0^b \int_0^\infty \sigma E_0 e^{-(1+j)x/\delta} dx dy \quad \dots\dots (2.33)$$

After integrating, substituting the limits and transferring to time domain, we get

$$I(t) = \frac{\sigma E_0 b \delta}{\sqrt{2}} \cos(\omega t - 45^\circ) \quad \dots\dots (2.35)$$

Define an AC equivalent resistance which consumes the same power as

$$\tilde{I}_{RMS}^2 R_{AC} = \text{Power consumed} = P \quad \dots\dots (2.36)$$

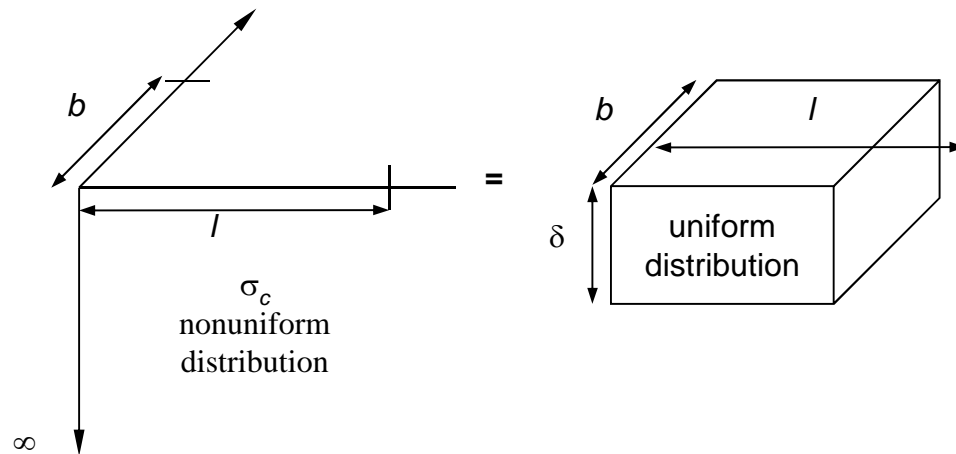
where $\tilde{I}_{RMS} = \frac{\sigma E_0 b \delta}{2}$ is the AC RMS equivalent current.

AC resistance

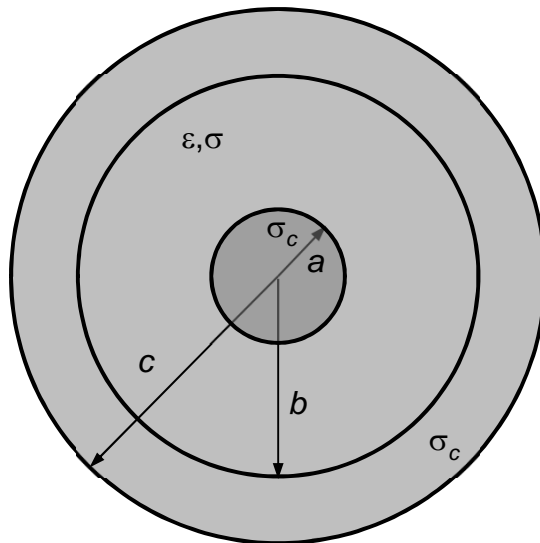
Finally, we get the AC equivalent resistance

$$R_{AC} = \frac{l}{\sigma b \delta} \quad \dots\dots\dots (2.38)$$

When AC waves go through an infinitely deep good conductor of conductivity σ defined by the half-space $0 < x < \infty$, shown in the figure 2.1, the equivalent DC conductor has δ depth.



AC resistance of round wires



The AC resistance of a round wire of radius a and length l .

1. $\delta \gg a$, we get

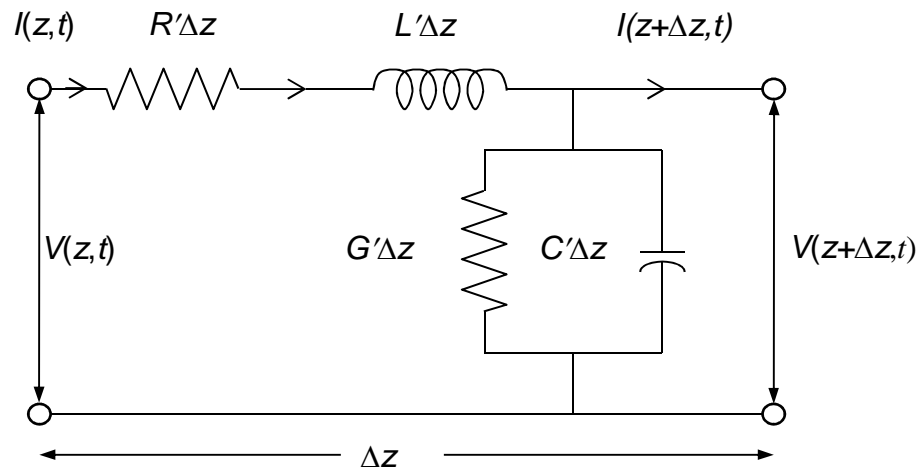
$$R = \frac{l}{\sigma s} = \frac{l}{\sigma \pi a^2}, \delta \gg a \quad \dots\dots\dots (2.39)$$

2. $\delta \ll a$, we get

$$R = \frac{l}{2\pi\sigma a\delta}, \delta \ll a \quad \dots\dots\dots (2.42)$$

Transmission lines

A transmission line is modeled by distributed circuit theory,



where,

R' : The resistance due to imperfect conductors

L' : The inductance due to magnetic flux generated by the currents in the conductors

G' : The conductance due to an imperfect dielectric

C' : The capacitance due to the surface charges on the conductors

Transmission lines

The parameters are computed as though the fields are static. For a coaxial cable shown (slide 14), the parameters are

$$G' = \frac{2\pi\sigma}{\ln(b/a)} \quad \dots\dots\dots (2.42)$$

$$L' = \frac{\mu}{2\pi} \ln(b/a) \quad \dots\dots\dots (2.43)$$

$$C' = \frac{2\pi\epsilon}{\ln(b/a)} \quad \dots\dots\dots (2.44)$$

For high frequencies, $\delta \ll a$

$$R' = \left[\frac{1}{2\pi\sigma_c\delta a} + \frac{1}{2\pi\sigma_c\delta b} \right] \quad \dots\dots\dots (2.45)$$

Transmission lines

L' including the correction due to the internal inductance (inductance due to the magnetic flux in the conductors) is given

$$L' = \frac{\mu}{2\pi} \ln(b/a) + L'' \quad \dots\dots\dots (2.46)$$

where $\omega L'' = R'$

For a two-port network shown in figure 2.4, the following equations can be obtained

$$V(z,t) - V(z + \Delta z, t) = R' \Delta z I(z,t) + L' \Delta z \frac{\partial I(z,t)}{\partial t} \quad \dots\dots\dots (2.48)$$

when $\Delta z \rightarrow 0$

$$-\frac{\partial V(z,t)}{\partial z} = R' I(z,t) + L' \frac{\partial I(z,t)}{\partial t} \quad \dots\dots\dots (2.49)$$

Similarly

$$-\frac{\partial I(z,t)}{\partial z} = G' V(z,t) + C' \frac{\partial V(z,t)}{\partial t} \quad \dots\dots\dots (2.51)$$

Transmission lines

For a the lossless transmission line

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} = 0, \quad R' = G' = 0 \quad \dots\dots (2.52)$$

where $v = \frac{1}{\sqrt{L'C'}}$

If the source is harmonic

$$\frac{\partial^2 \tilde{V}}{\partial z^2} + \beta^2 \tilde{V} = 0 \quad \dots\dots (2.54)$$

where $\beta = \frac{\omega}{v}$

Solution

$$\tilde{V} = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{+j\beta z} \quad \dots\dots (2.57)$$

Transmission lines

The relation between the voltages and the current

$$\tilde{V}^+ = Z_0 \tilde{I}^+ \quad \dots\dots\dots (2.58)$$

$$\tilde{V}^- = -Z_0 \tilde{I}^- \quad \dots\dots\dots (2.59)$$

where $Z_0 = \sqrt{L' / C'}$

For a non-PEC, we get

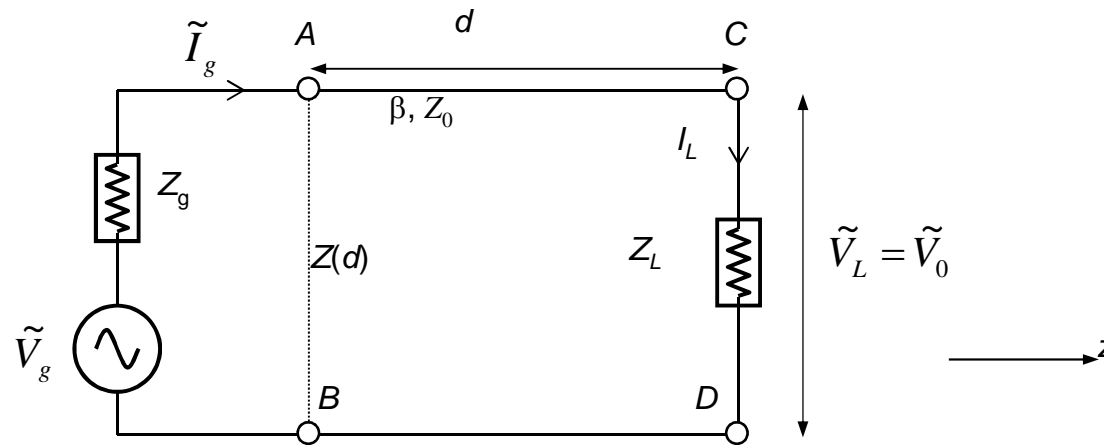
$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad \dots\dots\dots (2.62)$$

Parameters Comparison between TEM wave and transmission

<u>TEM wave</u>	<u>Transmission wave</u>
E	V
H	I
ε	C'
μ	L'
η	Z_0
$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \omega\sqrt{L'C'}$

Bounded Transmission lines

For a bounded transmission line,



where,

d : transmission line of length

V_g : Voltage source

Z_g : internal impedance

Z_L : External load

Bounded Transmission lines

The voltages and the current

$$\tilde{V}(d) = \tilde{V}_0^+ e^{j\beta d} + \tilde{V}_0^- e^{-j\beta d} \quad \dots\dots\dots (2.63)$$

$$\tilde{I}(d) = \frac{1}{Z_0} [\tilde{V}_0^+ e^{j\beta d} - \tilde{V}_0^- e^{-j\beta d}] \quad \dots\dots\dots (2.64)$$

where

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = Z_0 \left[\frac{1 + \Gamma_0 e^{-j2\beta d}}{1 - \Gamma_0 e^{-j2\beta d}} \right] \quad \dots\dots\dots (2.65)$$

Defined Γ_0 as the reflection coefficient at the load

$$\Gamma_0 = \frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \dots\dots\dots (2.66)$$

By substituting (2.66) in (2.65), the input impedance can be obtained

$$Z(d) = Z_0 \frac{Z_L \cos \beta d + jZ_0 \sin \beta d}{Z_0 \cos \beta d + jZ_L \sin \beta d} \quad \dots\dots\dots (2.67)$$

Bounded Transmission lines: see Apendice 2B and 2C for Smith Chart etc.

From 2.67, we get input impedance for special length

$$Z(\lambda/4) = Z_0^2 / Z_L \quad \dots\dots\dots (2.68)$$

$$Z(\lambda/2) = Z_L \quad \dots\dots\dots (2.69)$$

$$Z(d) = Z_L, \quad \frac{d}{\lambda} \ll 1 \quad \dots\dots\dots (2.70)$$

For a matched line, $Z_L = Z_0$, $\Gamma_0 = 0$

$$Z(d) = Z_L. \quad \dots\dots\dots (2.71)$$

Two special cases

$$Z(d) = jZ_0 \tan \beta d, \quad (Z_L = 0) \quad \dots\dots\dots (2.72)$$

$$Z(d) = -jZ_0 \cot \beta d, \quad (Z_L = \infty) \quad \dots\dots\dots (2.73)$$

Questions for the week

A lossfree nonuniform transmission line has

$$L' = L'(z)$$

$$C' = C'(z)$$

where L' and C' are per meter values of the series inductance and parallel capacitance of the transmission line.

Determine the partial differential equation for the instantaneous voltage $V(z,t)$.

For an exponential transmission line

$$L'(z) = L_0 \exp(qz)$$

$$C'(z) = C_0 \exp(-qz),$$

assuming $V(z,t) = V_0 \exp[j(\omega t - kz)]$,
determine the relation between ω and k .