Computational Electromagnetics
Week 2

Instructor:

What will be covered today

- Moment Method
- Finite Element Method and Shape Function

Moment Method

Moment method is one of weighted residual method discussed in the previous class. Let the problem be stated as:

$$Lf = g \qquad \text{. . . . . . (15.65)}$$

where L is an operator, f is the unknown function and g is excitation. The problem becomes; to find f given L and g and the boundary conditions.

Assume

$$f \approx \sum_{n=1}^{N} \alpha_n f_n \qquad \text{. . . (15.66)}$$

where f_n are the basic functions chosen for the problem.

Choose the weight functions w_m for the problem to minimize the residual. Choose the inner product for the problem.

Let us illustrate the process by giving a simple example.

Moment Method

Solve the following problem by moment method.

$$-\frac{d^2 f}{dx^2} = 1 + 4x^2 \qquad \text{. . . . . . (15.67a)}$$

with the domain given as 0 < x < 1 and the boundary conditions are given by

$$f(0) = f(1) = 0 \qquad \text{. . . . . . (15.67b)}$$

The exact answer to the problem is

$$f(x) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3} \qquad \text{. . . . . . (15.67c)}$$

By moment method, substitute (15.66) in (15.65), multiply by w_m and form the inner product.

$$\left\langle w_m, L\sum_{n=1}^{N} \alpha_n f_n \right\rangle = \langle w_m, g \rangle \qquad \text{. . . . . . (15.68)}$$

where ⟨ ⟩ is the symbol for the inner product.

Moment Method

For the problem, the inner product of 2 functions can be expressed as

$$\langle \psi_m, \psi_n \rangle = \int_0^1 \psi_m \psi_n dx \qquad \text{. . . . . . (15.69)}$$

Using (15.69) in (15.68) one obtains

$$\int_0^1 \left[w_m L \sum_{n=1}^{N} \alpha_n f_n \right] dx = \int_0^1 w_m g dx \qquad \text{. . . . . . (15.70)}$$

Now we need to choose the w_m *and* f_n for the problem. This needs some level of skills. Here we define two sub-sectional basic functions.

(1) Pulse function *P(x)*

$$\begin{aligned} Let \quad P(x) &= 1 \qquad |x| < \frac{1}{2(N+1)} \\ &= 0 \qquad otherwise \end{aligned} \qquad \text{. . . (15.71)}$$

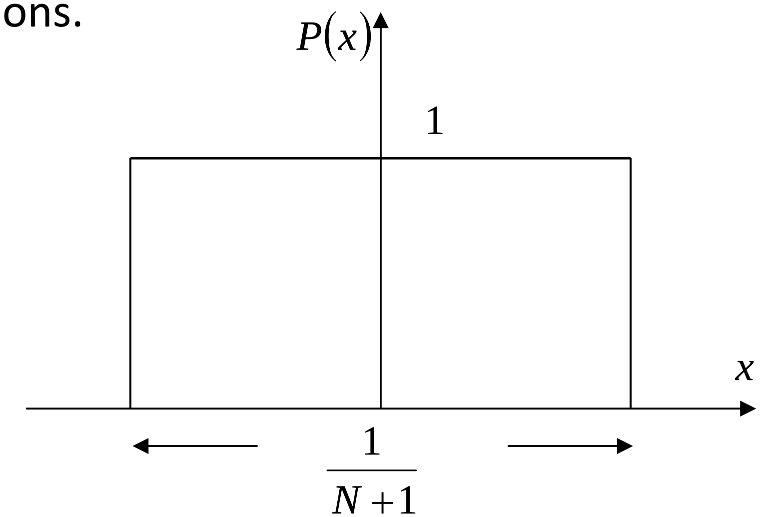


Figure 15.7 sketches this function.

Moment Method

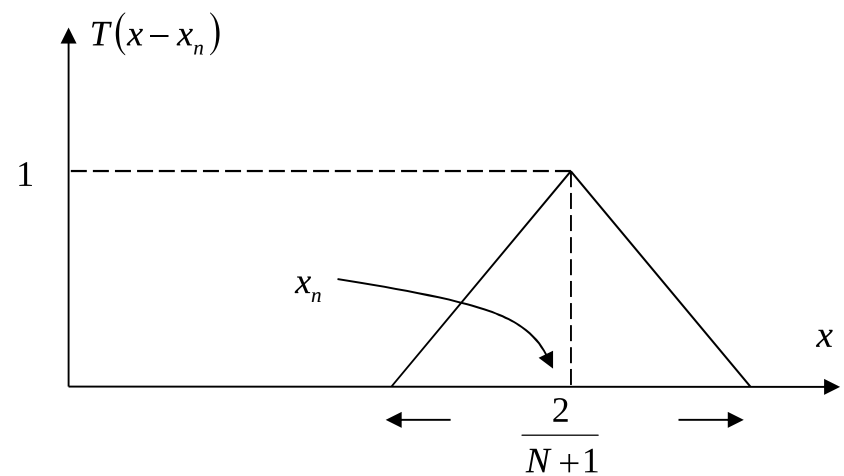
The center of the pulse function can be shifted to $x = x_n$ by defining

$$P(x-x_n)=1 \qquad |x-x_n|<\frac{1}{2(N+1)} \qquad \text{. . . . . . (15.72)}$$
$$=0 \qquad \textit{otherwise}$$

(2) Triangle function *$T(x\text{-}x_n)$*

$$T(x-x_n)=1-|x-x_n|(N+1) \qquad |x-x_n|<\frac{1}{N+1} \qquad \text{. . . . . . (15.73)}$$
$$=0 \qquad |x-x_n|>\frac{1}{N+1}$$

Figure 15.8 sketches this function.



Moment Method

After choosing w_m *and* f_n in (15.70), the result may be written as an algebraic equation

$$\sum_{n=1}^{N} \lambda_{mn}\alpha_n = g_m \qquad \text{. . . . . . (15.74)}$$

where

$$\lambda_{mn} = \int_0^1 w_m L f_n dx \qquad \text{. . . . . . (15.75)}$$

$$g_m = \int_0^1 w_m g dx \qquad \text{. . . . . . (15.76)}$$

Equation (15.74) yields N equations for the N unknowns α_n.

We know that integration, when an impulse function is in the integral, is given by

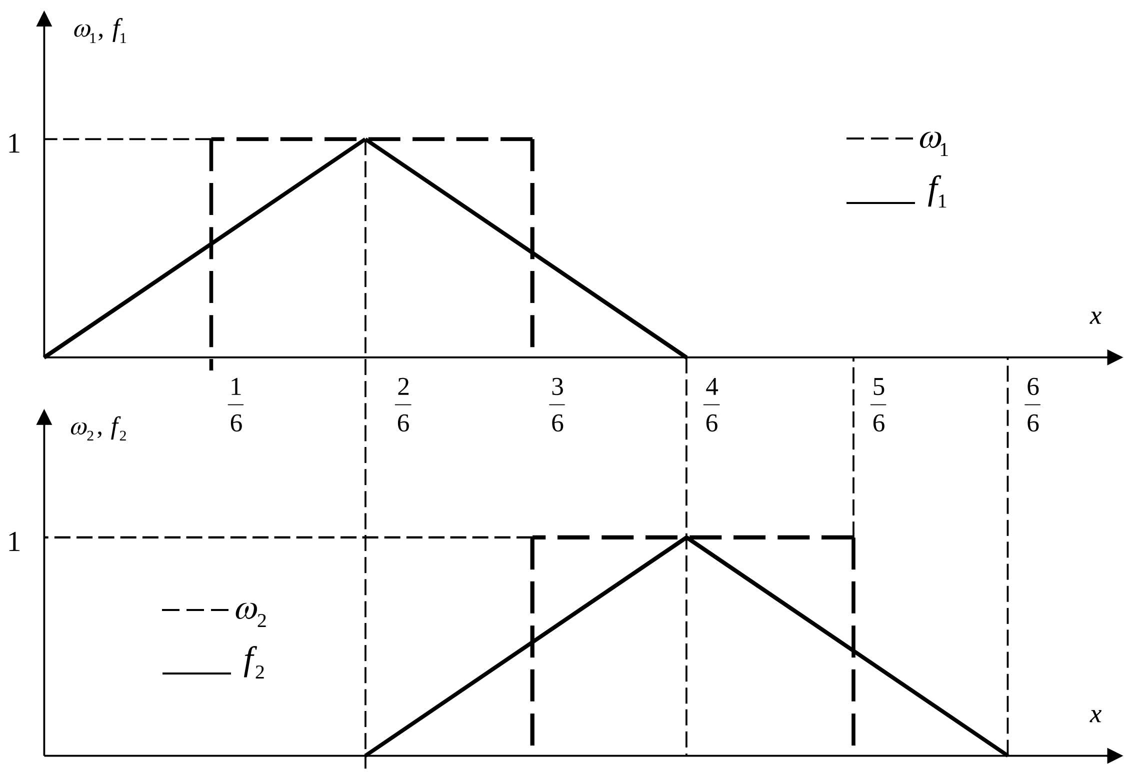
$$\begin{aligned} \int_a^b \delta\left(x - x_j\right) f\left(x\right) dx = f\left(x_j\right) &= f_j \qquad a < x_j < b \\ &= 0 \qquad \textit{Otherwise} \end{aligned} \qquad \text{. . . . . . (15.77)}$$

Thus we can get away by choosing f_n such that $L\, f_n$ is a sum of impulses.

Moment Method

This will be the result if $f_n = T(x-x_n)$ *and* L is a second order derivative shown in Figure 15.9

Let us illustrate the calculations by choosing N =2 in (15.66). The weight functions w_1 and w_2, the basis functions T_1 *and* T_2 are sketched in Figure 15.10



Moment Method

α_1 *and* α_2 can be obtained by two algebraic equations.

$$Lf_1 = -\frac{d^2 f_1}{dx^2} = -3\left[\delta(x) - 2\delta\left(x - \frac{2}{6}\right) + \delta\left(x - \frac{4}{6}\right)\right]$$

$$\lambda_{11} = \int_0^1 w_1 L f_1 dx = \int_{1/6}^{3/6} -3\left[\delta(x) - 2\delta\left(x - \frac{2}{6}\right) + \delta\left(x - \frac{4}{6}\right)\right] dx = 6$$

$$\lambda_{12} = \int_{1/6}^{3/6} -3\left[\delta\left(x - \frac{2}{6}\right) - 2\delta\left(x - \frac{4}{6}\right) + \delta\left(x - \frac{6}{6}\right)\right] dx = -3$$

$$g_1 = \int_0^1 w_1 g dx = \int_{1/6}^{3/6} \left(1 + 4x^2\right) dx = \frac{40}{81}$$

The first equation is

$$\lambda_{11}\alpha_1 + \lambda_{12}\alpha_2 = g_1$$

$$6\alpha_1 - 3\alpha_2 = \frac{40}{81} \qquad \text{. . . . . . (15.78)}$$

Moment Method

Similarly, we can get the second equation

$$\lambda_{21}\alpha_1 + \lambda_{22}\alpha_2 = g_2$$

$$-3\alpha_1 + 6\alpha_2 = \frac{76}{81} \qquad \text{......} (15.79)$$

Solving α_1 *and* α_2, we get

$$\alpha_1 = \frac{52}{243}, \quad \alpha_2 = \frac{64}{243} \qquad \text{......} (15.80)$$

So the approximate solution for moment method is

$$f \approx \frac{52}{243} T_1\left(x - \frac{2}{6}\right) + \frac{64}{243} T_2\left(x - \frac{4}{6}\right) \qquad \text{......} (15.81)$$

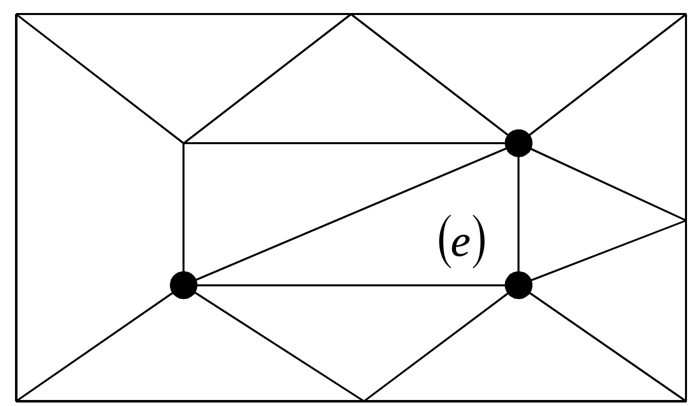
The accuracy can be further improved by choosing a larger value for *N* but the number of equations will increase as well.

Finite Element Method

Two aspects to the finite element method

(i) A continuous domain is broken up into a finite number of elements. Figure 15.11 shows an example. The discrete points are the vertices of the triangle.

The method aims to find the unknown potentials at the finite number of discrete points.



(ii) The second aspect of the finite element method is in the technique of generating the algebraic equations.

Finite Element Method – Variational Principle

Instead of solving the equilibrium equation (15.65) directly, we try to find the function *f* that extremizes its 'functional' *I(f).* For a functional the argument itself is a function. The functional for (15.65), in the language of linear spaces is given by

$$I(f) = \langle Lf, f \rangle - 2\langle f, g \rangle \qquad \text{. . . . . . (15.82)}$$

provided the operator *L* is positive definite, which can be satisfied if

$$\begin{aligned} &\langle LX_1, X_2 \rangle = \langle X_1, LX_2 \rangle \\ &\langle LX, X \rangle \geq 0, \text{ for any } X \end{aligned} \qquad \text{. . . . . . (15.83)}$$

In the above X, X_1, X_2 are arbitrary functions that satisfy the same boundary conditions. Refer to table 15.3 for the common partial differential equations of electromagnetics and their functionals.

Finite Element Method – Variational Principle

Let us use an example to illustrate application of (15.82). Figure 15.12 gives a one dimensional example.

The plate dimensions are assumed to be large compared to *d.* Poission's equation, which can be simplifies to the one dimensional equation

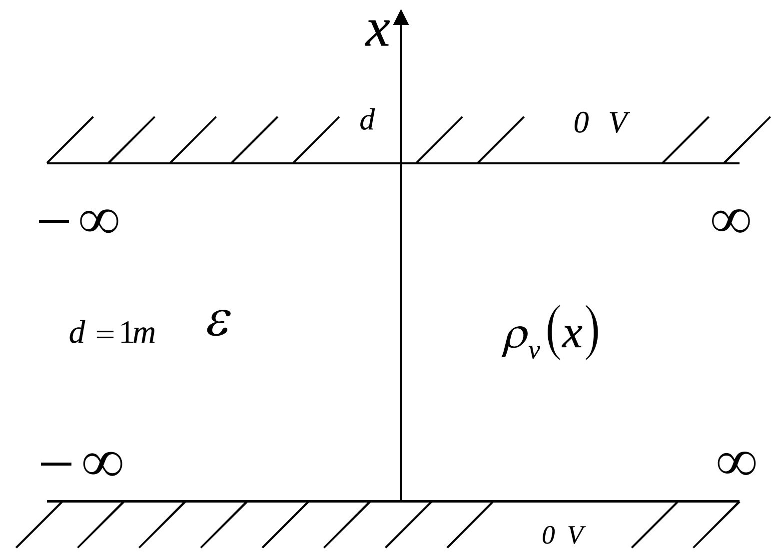
$$-\frac{d^2\Phi}{dx^2} = \frac{\rho_v}{\varepsilon} \qquad \text{. . . . . . (15.85)}$$

with the boundary conditions

$$\Phi(0) = \Phi(1) = 0 \qquad \text{. . . . . . (15.86)}$$

Defining the inner product for the problem as

$$\langle \psi_1, \psi_2 \rangle = \int_0^1 \psi_1 \psi_2 \, dx \qquad \text{. . . . . . (15.87)}$$



Finite Element Method – Variational Principle

The functional for (15.85) can be written as

$$I(\Phi) = \int_0^1 -\frac{d^2\Phi}{dx^2}\Phi dx - 2\int_0^1 \Phi(x)\frac{\rho_v(x)}{\varepsilon}dx \qquad \text{. . . . . . (15.88)}$$

Integrating the first integral on the RHS of (15.88) and using the boundary conditions (15.86), we get

$$I(\Phi) = \int_0^1 \left(\frac{d\Phi}{dx}\right)^2 dx - 2\int_0^1 \Phi(x)\frac{\rho_v(x)}{\varepsilon}dx \qquad \text{. . . . . . (15.89)}$$

The function $\Phi(x)$ that minimized (15.89) is the solution of (15.85).

$$I_E(\Phi) = \frac{1}{2}\varepsilon I(\Phi) = \int_0^1 \frac{1}{2}\varepsilon\left|\frac{d\Phi}{dx}\right|^2 dx - \int_0^1 \Phi(x)\rho_v(x)dx \qquad \text{. . . . . . (15.90)}$$

Notice that the functional (15.90) is the electric potential energy.

Finite Element Method – Variational Principle

The potential function $\Phi(x)$ in each element can be expressed by using linear interpolation:

$$\Phi(x) = \Phi_i + \frac{\Phi_j - \Phi_i}{x_j - x_i}(x - x_i) \quad . . . (15.91)$$

Another way of writing (15.91) is

$$\Phi(x) = N_i(x)\Phi_i + N_j(x)\Phi_j \quad . . . (15.92)$$

where

$$N_i = \frac{x_j - x}{x_j - x_i} \quad . . . . . . (15.93a)$$

$$N_j = \frac{x - x_i}{x_j - x_i} \quad . . . . . . (15.93b)$$

$N_i(x)$ and $N_j(x)$ are called shape functions.

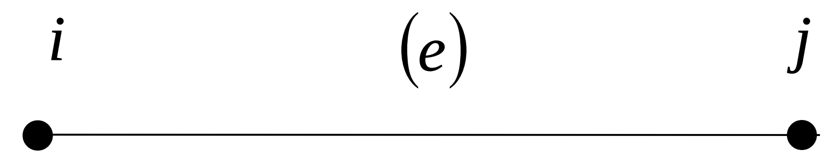


Figure 15.13 Element *(e)* with end points *i and j*

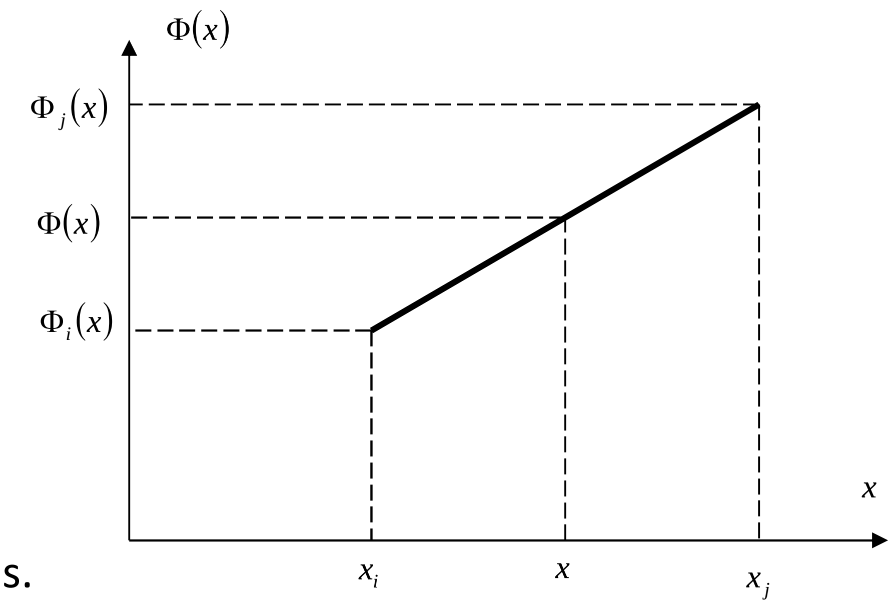


Figure 15.14 Linear interpolation

Finite Element Method – Variational Principle

From (15.92) and (15.93)

$$\frac{d\Phi}{dx} = \frac{1}{x_j - x_i}(-1)\Phi_i + \frac{1}{x_j - x_i}(1)\Phi_j = \frac{1}{x_j - x_i}\left(\Phi_j - \Phi_i\right) \qquad \text{. . . . . . (15.98)}$$

From (15.90)

$$I^{(e)} = \int_{x_i}^{x_j} \frac{1}{2}\varepsilon \frac{1}{\left(x_j - x_i\right)^2}\left(\Phi_j - \Phi_i\right)^2 dx - \int_{x_i}^{x_j}\left[N_i(x)\Phi_i + N_j(x)\Phi_j\right]\rho_v(x)dx \qquad \text{. . . . . . (15.99)}$$

which can be further simplified

$$I^{(e)} = \frac{1}{2}\varepsilon \frac{\left(\Phi_j - \Phi_i\right)^2}{\left(x_j - x_i\right)} - \Phi_i \int_{x_i}^{x_j} N_i(x)\rho_v(x)dx - \Phi_j \int_{x_i}^{x_j} N_j(x)\rho_v(x)dx \qquad \text{. . . . . . (15.100)}$$

The total functional can be obtained by summing up the functional for all the elements

$$I_E(\Phi) = \sum_{elements} I^{(e)} \qquad \text{. . . . . . (15.101)}$$

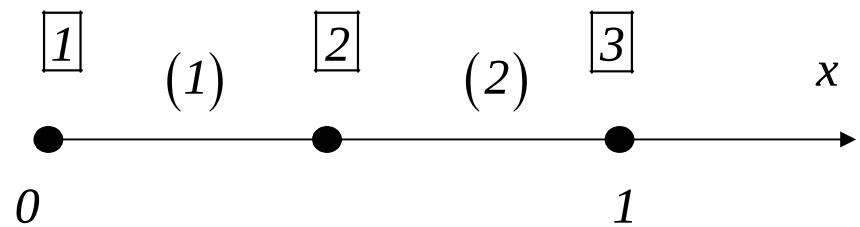
The algebraic equations are then obtained by minimizing (15.101) with respect to each of the unknown potentials:

$$\frac{\partial I_E}{\partial \Phi_k} = 0, \quad \Phi_k \quad \text{(unknown node potential).} \qquad \text{. . . . . . (15.102)}$$

Finite Element Method – Solve the problem

Assume *d = 1, ρ = 1, ε = 1* in Figure 15.11 and let the domain be divided into 2 elements as shown in Figure 15.16

For element (1)



$$x_i = 0, \quad x_j = \frac{1}{2}$$

$$N_i(x) = \frac{\frac{1}{2}-x}{\frac{1}{2}-0} = 1-2x, \quad N_j(x) = \frac{x-0}{\frac{1}{2}-0} = 2x$$

$$\Phi_i = \Phi_{\boxed{1}} = 0; \qquad \Phi_j = \Phi_{\boxed{2}}$$

$$I^{(1)} = \frac{1}{2}\varepsilon\frac{\Phi_{\boxed{2}}^2}{\frac{1}{2}-0} - 0\int_0^{1/2}(1-2x)dx - \Phi_{\boxed{2}}\int_0^{1./2} 2x\,dx$$

$$= \Phi_{\boxed{2}}^2 - 0 - \Phi_{\boxed{2}}\, 2\frac{x^2}{2}\Big|_0^{1/2} = \Phi_{\boxed{2}}^2 - \frac{\Phi_{\boxed{2}}}{4}$$

Finite Element Method – Solve the problem

For element (2)

$$x_i = \frac{1}{2}, \quad x_j = 1$$

$$N_i(x) = \frac{1-x}{1-\frac{1}{2}-} = 2(1-x), \qquad N_i(x) = \frac{x-\frac{1}{2}}{1-\frac{1}{2}} = 2x-1$$

$$\Phi_i = \Phi_{\boxed{2}}; \qquad \Phi_j = \Phi_{\boxed{3}} = 0$$

$$I^{(2)} = \frac{1}{2}\varepsilon\frac{\left(0-\Phi_{\boxed{2}}\right)^2}{1-\frac{1}{2}} - \Phi_{\boxed{2}}\int_{1/2}^{1} 2(1-x)dx - 0\int_{1/2}^{1}(2x-1)dx$$

$$= \Phi_{\boxed{2}}^2 - 2\Phi_{\boxed{2}}\left(x-\frac{x^2}{2}\right)\Bigg|_{\frac{1}{2}}^{1} - 0 = \Phi_{\boxed{2}}^2 - \frac{\Phi_{\boxed{2}}}{4}$$

$$I_E = I^{(1)} + I^{(2)} = 2\Phi_{\boxed{2}}^2 - \frac{\Phi_{\boxed{2}}}{2}$$

Finite Element Method – Solve the problem

For minimum I_E

$$\frac{\partial I_E}{\partial \Phi_2} = 0, \quad 4\Phi_2 - \frac{1}{2} = 0$$

$$\Phi_2 = \frac{1}{8}$$

The exact answer for the problem is:

$$\Phi = \frac{1}{2}x(1-x) \qquad \text{......} \ (15.105)$$

$$\Phi\left(\frac{1}{2}\right) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8} \qquad \text{......} \ (15.106)$$

The exact answer given in (15.105) coincides with the approximate answer by finite element method in (15.103); however it may be noted that this is not true in the entire domain. For example, $\Phi(1/4)$ by exact answer (15.105) is

$$\Phi\left(\frac{1}{4}\right) = \frac{1}{2}\frac{1}{4}\left(1-\frac{1}{4}\right) = \frac{3}{32} \qquad \text{......} \ (15.107)$$

Finite Element Method – Solve the problem

By finite element method we note that *x=1/4* is not an end point of an elementbut it is in the domain of element(1). In this domain

$$N_i(x) = 1 - 2x, \quad N_j(x) = 2x$$

$$\Phi(x) = N_i(x)\Phi_i + N_j(x)\Phi_j$$
$$= (1 - 2x)\Phi_{\boxed{1}} + 2x\Phi_{\boxed{2}}$$

$$\Phi\left(\frac{1}{4}\right) = \left(1 - \frac{2}{4}\right)0 + 2\left(\frac{1}{4}\right)\frac{1}{8} = \frac{2}{32} \qquad \text{. . . . . . (15.108)}$$

The source of the error is obvious, the exact solution shows that the potential varies quadratically where as the finite element method we used assumed a linear interpolation. The shape functions are obtained based on (15.94) and are called First order shape functions. One can define second order shape functions based on quadratic interpolation.

Questions for the week

1. No questions for the week 2
2. Home work: P15.6 – P15.10