MTE, by RBH & JI

Solution to Problems

Chapter 2 :

Problem 2.1

To show (a) we write LHS of (a) in components & obtain

$$[(\underline{a} \times \underline{b}) \times \underline{c}]_i =$$

$$= \epsilon_{ijk} (\underline{a} \times \underline{b})_j c_k = \epsilon_{ijk} \epsilon_{jpq} a_p b_q c_k$$

$$= \epsilon_{jki} \epsilon_{jpq} a_p b_q c_k \qquad (1)$$

Now, by using the $\epsilon - \delta$ identity [see Eq. (2.1.10)] we obtain

$$\epsilon_{jki} \epsilon_{jpq} = \delta_{kp} \delta_{iq} - \delta_{kq} \delta_{ip} \qquad (2)$$

Therefore substituting (2) into (1) we receive

$$[(\underline{a} \times \underline{b}) \times \underline{c}]_i = a_k c_k b_i - b_k c_k a_i \qquad (3)$$

or

$$[(\underline{a} \times \underline{b}) \times \underline{c}]_i = (\underline{a} \cdot \underline{c}) b_i - (\underline{b} \cdot \underline{c}) a_i \qquad (4)$$

Eq. (4) is equivalent to (a), & this proves (a). To show (b) we replace the vector $\underline{c}$ in (a) by $\underline{c} \times \underline{d}$, & arrive at (b).

Problem 2.2

The relation (a) in components reads

$$u_i = (\underline{u}^{\perp})_i + (\underline{u}^{\parallel})_i \qquad (1)$$

where

$$(\underline{u}^{\perp})_i = (u_k n_k) n_i \, , \quad (\underline{u}^{\parallel})_i = \epsilon_{ijk} n_j \epsilon_{kpq} u_p n_q \qquad (2)$$

Since by the ϵ-δ identity

$$\epsilon_{ijk} \epsilon_{kpq} = \epsilon_{kij} \epsilon_{kpq} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{pj} \qquad (3)$$

therefore

$$(\underline{u}^{\parallel})_i = (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{pj}) n_j n_q u_p$$

$$= n_q n_q u_i - n_p u_p n_i \qquad (4)$$

Now, if we substitute $(\underline{u}^{\perp})_i$ & $(\underline{u}^{\parallel})_i$ from (2) & (4), respectively, into RHS of (1), & take into account that $n_q n_q = 1$, we arrive at LHS of (1) which proves (1).

The relations (c) hold true, since by (2) & (4)

$$\underline{u}^{\perp} \cdot \underline{u}^{\parallel} = (\underline{u}^{\perp})_i (\underline{u}^{\parallel})_i =$$

$$= (u_k n_k) n_i (u_i - n_i u_p n_p) = (u_k n_k)^2 - (u_k n_k)^2 = 0 \quad (5)$$

$$\underline{u}^{\perp} \cdot \underline{n} = [(u_k n_k) n_i] n_i = u_k n_k = \underline{u} \cdot \underline{n} \quad (6).$$

and

$$\underline{u}^{\parallel} \cdot \underline{n} = (\underline{u}^{\parallel})_i n_i = (u_i - n_i u_p n_p) n_i = 0 \quad (7),$$

Problem 2.3

The tensor product of vectors $\underline{a}$ & $\underline{b}$ is defined as a second order tensor $\underline{P}$ with the components

$$P_{ij} = a_i b_j \quad (1)$$

or in direct notation

$$\underline{P} = \underline{a} \otimes \underline{b} \quad (2)$$

Therefore, Eqs. (b) in components read

$$(\underline{u}^{\perp})_i = n_i u_j n_j \; ; (\underline{u}^{\parallel})_i = (\delta_{ij} - n_i n_j) u_j \quad (4)$$

Substituting (4) into RHS of (9) written in components, we arrive at LHS of (9) written in components. This proves (9).

Problem 2.4

Eqs. (b) & (c), respectively, in components, take the form

$$(\underline{T}^{\perp})_{ij} = n_i T_{jk} n_k + n_j T_{ik} n_k - n_k T_{kp} n_p n_i n_j \quad (1)$$

and

$$(\underline{T}^{\parallel})_{ij} = (\delta_{ip} - n_i n_p) T_{pq} (\delta_{qj} - n_q n_j)$$

$$= T_{ij} - n_j T_{iq} n_q - n_i T_{pj} n_p + n_i n_j n_p n_q T_{pq} \quad (2)$$

Since $\underline{T}$ is a symmetric tensor, therefore

$$n_i T_{jk} n_k = n_i T_{pj} n_p \quad (3)$$

Writing (a) in components, and substituting (1) & (2) into RHS of (a) we arrive at LHS of (a) which proves (a).

To prove $(d)_1$ note that

$$\underline{T}^{\perp} \cdot \underline{T}^{\parallel} = (\underline{T}^{\perp})_{ij} (\underline{T}^{\parallel})_{ij} \quad (4)$$

If we note that

$$(\delta_{ip} - n_i n_p) n_i' = 0 \qquad (\delta_{qj} - n_q n_j) n_j' = 0 \quad (5)$$

then substituting (1) & (2) into (4) & taking into account (5) we obtain $(d)_1$.

To show $(d)_2$ we write RHS of $(d)_2$ in components to obtain

$$(\underline{T}^{\perp} \underline{n})_i = (n_i T_{jk} n_k + n_j T_{ik} n_k$$
$$- n_k T_{kp} n_p n_i n_j) n_j$$

$$= T_{ik} n_k + n_i n_j n_k T_{jk} - n_i n_p n_k T_{kp}$$

$$= T_{ik} n_k. \qquad (6)$$

Hence RHS of $(d)_2$ = LHS of $(d)_2$ and this proves $(d)_2$.

Finally, writing LHS of $(d)_3$ in components we obtain

$$(\underline{T}^{\parallel} \underline{n})_i = (\underline{T}^{\parallel})_{ij} n_j$$
$$= (\delta_{ip} - n_i n_p) T_{pq} (\delta_{qj} - n_q n_j) n_j \qquad (7)$$

If Eq. $(5)_2$ is substituted into (7) we obtain $(d)_3$. This completes solution to Problem 2.4.

Problem 2.5

Substituting $n_1 = 0$, $n_2 = 0$, $n_3 = -1$, into Eqs. (1) and (2), respectively, in the solution of Problem 2.4, we obtain

$$\underline{T}^{\perp} = \begin{bmatrix} 0 & 0 & T_{13} \\ 0 & 0 & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

and

$$\underline{T}^{\parallel} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence

$$\underline{T} = \underline{T}^{\perp} + \underline{T}^{\parallel}$$

which proves the decomposition formula of Problem 2.5.

Problem 2.6

To show (a) we use the result (a) of Example 2.1.1.

$$\epsilon_{ijk} a_i b_j c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (1)$$

where $\underline{a}$, $\underline{b}$, & $\underline{c}$ are arbitrary vectors.

By letting

$$a_i = T_{i1}, \quad b_i = T_{i2}, \quad c_i = T_{i3} \quad (2)$$

in (1) we obtain

$$\epsilon_{ijk} T_{i1} T_{j2} T_{k3} = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{vmatrix} \quad (3)$$

Since

$$\det(\underline{T}) = \det(\underline{T}^T) \quad (4)$$

Eq. (3) is equivalent to (a), and this proves (a).

To show (b) we let

$$a_i = T_{ip}, \quad b_i = T_{iq}, \quad c_i = T_{ir} \quad (5)$$

in (1), where $p, q,$ & r are fixed numbers from the set $\{1, 2, 3\}$, and obtain

$$\epsilon_{ijk} T_{ip} T_{jq} T_{kr} = \begin{vmatrix} T_{1p} & T_{2p} & T_{3p} \\ T_{1q} & T_{2q} & T_{3q} \\ T_{1r} & T_{2r} & T_{3r} \end{vmatrix} \qquad (6)$$

Next, multiplying (a) by ϵ_{pqr}, we get

$$\epsilon_{pqr} (\det T) = \epsilon_{pqr} \, \epsilon_{ijk} T_{i1} T_{j2} T_{k3} \qquad (7)$$

Since by (2.1.12)

$$\epsilon_{pqr} \, \epsilon_{ijk} = \begin{vmatrix} \delta_{pi} & \delta_{pj} & \delta_{pk} \\ \delta_{qi} & \delta_{qj} & \delta_{qk} \\ \delta_{ri} & \delta_{rj} & \delta_{rk} \end{vmatrix} \qquad (8)$$

therefore, substituting (8) into (7) and multiplying T_{i1}, T_{j2}, and T_{k3}, respectively, by the first, second, and third column of the determinant on RHS of (8), we obtain

$$\epsilon_{pqr} (\det T) = \begin{vmatrix} T_{p1} & T_{p2} & T_{p3} \\ T_{q1} & T_{q2} & T_{q3} \\ T_{r1} & T_{r3} & T_{r3} \end{vmatrix} \qquad (9)$$

Since $\det(T) = \det(T^T)$, RHS of (6)

is identical to RHS of (9), and this proves (b).

Finally, to show (c) we multiply (b) by ϵ_{ijk} and obtain

$$\epsilon_{ijk}\,\epsilon_{pqr}\,(\det T) =$$

$$= \epsilon_{ijk}\,\epsilon_{abc}\,T_{ap}\,T_{bq}\,T_{cr} \tag{10}$$

or by (2.1.12)

$$\epsilon_{ijk}\,\epsilon_{pqr}\,(\det T) = \begin{vmatrix} \delta_{ia} & \delta_{ib} & \delta_{ic} \\ \delta_{ja} & \delta_{jb} & \delta_{jc} \\ \delta_{ka} & \delta_{kb} & \delta_{kc} \end{vmatrix} T_{ap}\,T_{bq}\,T_{cr} \tag{11}$$

Now, multiplying the first, second, & third column of the determinant on RHS of (11) by T_{ap}, T_{bq}, and T_{cr}, respectively, we get

$$\epsilon_{ijk}\,\epsilon_{pqr}\,(\det T) = \begin{vmatrix} T_{ip} & T_{iq} & T_{ir} \\ T_{jp} & T_{jq} & T_{jr} \\ T_{kp} & T_{kq} & T_{kr} \end{vmatrix} \tag{12}$$

This proves (c), and a solution to Problem 2.6 is complete.

Problem 2.7

The relation (b) in components takes the form

$$T_{ik}\hat{T}^T_{kj} = \hat{T}^T_{ik} T_{kj} = (\det T)\,\delta_{ij} \qquad (1)$$

Using (a) we obtain

$$T_{ik}\hat{T}^T_{kj} = T_{ik}\hat{T}_{jk} = \frac{1}{2}\epsilon_{jab}\,\epsilon_{kcd}\,T_{ac}T_{bd}T_{ik} \qquad (2)$$

Since, by virtue of (b) in Problem (2.6)

$$\epsilon_{pqr}(\det T^T) = \epsilon_{ijk}T_{pi}T_{qj}T_{rk} \qquad (3)$$

and

$$\det(\underline{T}^T) = \det(\underline{T}) \qquad (4)$$

therefore, Eq. (2) can be written in the form

$$T_{ik}\hat{T}^T_{kj} = \frac{1}{2}\epsilon_{jab}\,\epsilon_{iab}(\det T) \qquad (5)$$

Now, using the $\epsilon - \delta$ identity (2.1.10)

$$\epsilon_{mis}\,\epsilon_{jks} = \delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij} \qquad (6)$$

which is equivalent to

$$\epsilon_{jkb}\epsilon_{imb} = \delta_{mk}\delta_{ij} - \delta_{mj}\delta_{ik} \qquad (7)$$

and letting $k = m = a$ in (7) we obtain

$$\epsilon_{jab}\,\epsilon_{iab} = 2\,\delta_{ij} \qquad (8)$$

Therefore, by virtue of (5) & (8) we obtain

$$T_{ik}\hat{T}^T_{kj} = (\det T)\,\delta_{ij} \qquad (9)$$

which proves the first part of (1).

To prove that

$$T_{ik}\hat{T}^T_{kj} = \hat{T}^T_{ik}T_{kj} \qquad (10)$$

we note that

$$\hat{T}^T_{ik}T_{kj} = \hat{T}_{ki}T_{kj} = \frac{1}{2}\epsilon_{kab}\epsilon_{icd}T_{ac}T_{bd}T_{kj}$$
$$= \frac{1}{2}\epsilon_{icd}\epsilon_{jcd}(\det T) = (\det T)\delta_{ij} \qquad (11)$$

and this completes the proof of (b).

To show (c) we note that

$$\underline{T}\,\underline{T}^{-1} = \underline{T}^{-1}\underline{T} = \underline{1} \qquad (12)$$

and by virtue of (b)

$$\underline{T}\,\underline{\hat{T}}^T = (\det T)\,\underline{1} \qquad (13)$$

Multiplying (13) by $\underline{T}^{-1}$, taking into account (12) as well as the relation

$$\underline{A}\,\underline{B}\,\underline{C} = (\underline{A}\,\underline{B})\underline{C} = \underline{A}(\underline{B}\,\underline{C}) \qquad (14)$$

where $\underline{A}$, $\underline{B}$, & $\underline{C}$ are arbitrary matrices, we obtain

$$\hat{T}^T = (\det T)\,\underline{T}^{-1} \qquad (15)$$

and this proves (c).

Problem 2.8

The relations (a)–(c) follow from the formulae

$$\underline{T}' = \underline{Q}^T\,\underline{T}\,\underline{Q} \qquad (1)$$

where $\underline{Q}^T$ is the matrix

$$\underline{Q}^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2)$$

To show that (b) and (c) are equivalent to (d) use the identities

$$\exp(ik\theta) = \cos k\theta + i\sin k\theta,\quad k=1,2 \qquad (3)$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad (4)$$

Finally, using the correspondence

$$x_1' = r \,, \quad x_2' = \theta \,, \quad x_3' = x_3 \qquad (5)$$

and

$$\begin{aligned} T_{11}' &= T_{rr} \,, \quad T_{22}' = T_{\theta\theta} \,, \quad T_{12}' = T_{r\theta} \\ T_{13}' &= T_{r3} \,, \quad T_{23}' = T_{\theta 3} \,, \quad T_{33}' = T_{33} \end{aligned} \qquad (6)$$

we transform (d) into (e), and this completes a solution to Problem 2.8.

Problem 2.9

We are to show that $\underline{T}\,\underline{T}^T$ and $\underline{T}^T\underline{T}$ satisfy the inequalities

$$\underline{u} \cdot (\underline{T}\,\underline{T}^T)\underline{u} > 0 \quad \text{for every } \underline{u} \neq \underline{0} \quad (1)$$

and

$$\underline{u} \cdot (\underline{T}^T\underline{T})\underline{u} > 0 \quad \text{for every } \underline{u} \neq \underline{0} \quad (2)$$

To prove (1) & (2) note that (1) & (2) are equivalent to

$$(\underline{T}^T\underline{u}) \cdot (\underline{T}^T\underline{u}) > 0 \quad \text{for every } \underline{u} \neq \underline{0} \quad (3)$$

and

$$(\underline{T}\underline{u}) \cdot (\underline{T}\underline{u}) > 0 \quad \text{for every } \underline{u} \neq \underline{0} \quad (4)$$

The equivalency is implied by the identities

$$\underline{u} \cdot (\underline{T}\,\underline{T}^T)\underline{u} = (\underline{T}^T\underline{u})\cdot(\underline{T}^T\underline{u}) \qquad (5)$$

and

$$\underline{u} \cdot (\underline{T}^T\underline{T})\underline{u} = (\underline{T}\underline{u})\cdot(\underline{T}\underline{u}) \qquad (6)$$

Now, since $\underline{T}$ is invertible, $\underline{T}^T$ is invertible. Hence

$$\underline{T}\,\underline{u} \neq \underline{0} \quad \text{for every } \underline{u} \neq \underline{0} \qquad (7)$$

and

$$\underline{T}^T\underline{u} \neq \underline{0} \quad \text{for every } \underline{u} \neq \underline{0} \qquad (8)$$

As a result, the inequalities (3), (4), (7), and (8) imply that $\underline{T}\,\underline{T}^T$ & $\underline{T}^T\underline{T}$ are positive definite, and this completes solution of Problem 2.9.

Problem 2.10

By taking steps similar to those of Example 2.1.11, we find that λ_i and $\underline{n}^{(i)}$ $(i=1,2,3)$, respectively, given by (b) and (c)–(d) are the eigenvalues & eigenvectors for the matrix $\underline{T}$ given by eq.(a). This completes a solution to Problem 2.10.

Problem 2.11

The orthonormal basis $\{e_i^*\}$ obtained from Eqs. (c)–(e) in Problem 2.10 is defined by

$$\underset{\sim}{e}_1^* = \left[\frac{1}{\sqrt{2}} \, \frac{1}{1-\sqrt{2}} \, \frac{1}{\sqrt{2+\sqrt{2}}} \,, \; 0 \,, \; \frac{1}{\sqrt{2}} \, \frac{1}{\sqrt{2+\sqrt{2}}} \right] \tag{1}$$

$$\underset{\sim}{e}_2^* = [\, 0 \,, \; 1 \,, \; 0 \,]$$

$$\underset{\sim}{e}_3^* = \frac{1}{\sqrt{2}} \, \frac{1}{1+\sqrt{2}} \, \frac{1}{\sqrt{2-\sqrt{2}}} \,, \; 0 \,, \; \frac{1}{\sqrt{2}} \, \frac{1}{\sqrt{2-\sqrt{2}}} \,]$$

Since

$$\underset{\sim}{e}_1 = [1, 0, 0] \,, \quad e_2 = [0, 1, 0] \,, \quad e_3 = [0, 0, 1] \; (e_i$$

Eq. (a) implies that

$$Q_{11}^T = \frac{1}{\sqrt{2}} \, \frac{1}{1-\sqrt{2}} \, \frac{1}{\sqrt{2+\sqrt{2}}} \,, \quad Q_{12}^T = 0$$

$$Q_{13}^T = \frac{1}{\sqrt{2}} \, \frac{1}{\sqrt{2+\sqrt{2}}} \,; \quad Q_{21}^T = 0$$

$$Q_{22}^T = 1 \,, \quad Q_{23}^T = 0 \,,$$

$$Q_{31}^T = \frac{1}{\sqrt{2}} \, \frac{1}{1+\sqrt{2}} \, \frac{1}{\sqrt{2-\sqrt{2}}} \,, \quad Q_{32}^T = 0 \,,$$

$$Q_{33}^{T} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2-\sqrt{2}}} \tag{3}$$

Hence, we obtain

$$\begin{aligned} &Q_{11} = Q_{11}^{T}, \quad Q_{12} = 0, \quad Q_{13} = Q_{31}^{T} \\ &Q_{21} = 0, \quad Q_{22} = Q_{22}^{T}, \quad Q_{23} = 0, \\ &Q_{31} = Q_{13}^{T}, \quad Q_{32} = Q_{23}^{T}, \quad Q_{33} = Q_{33}^{T} \end{aligned} \tag{4}$$

Since, by Eq. (b),

$$T_{ij}^{*} = Q_{ik}^{T} T_{ka} Q_{aj} \tag{5}$$

therefore substituting T_{ka}, Q_{ik}^{T}, and Q_{aj} from Eqs. (a), (3), & (4), respectively, into RHS of (5), we obtain

$$\begin{aligned} &T_{11}^{*} = \frac{2(3-2\sqrt{2})}{2-\sqrt{2}}, \quad T_{12}^{*} = T_{13}^{*} = 0 \\ &T_{21}^{*} = 0, \quad T_{22}^{*} = 2, \quad T_{23}^{*} = 0 \\ &T_{31}^{*} = 0, \quad T_{32}^{*} = 0, \quad T_{33}^{*} = \frac{2}{2-\sqrt{2}}. \end{aligned} \tag{6}$$

Eqs.(6) imply that $\underline{T}^*$ is represented by a diagonal matrix. In addition, it follows from (6) that (c) holds true, and this completes a solution to Problem 2.11.

Problem 2.12

To show (a) we recall that for any vector field $\underline{\varphi} = \underline{\varphi}(\underline{x})$ the curl operator is defined by [see Eq. (2.2.11)]

$$(\mathrm{curl}\, \underline{\varphi})_i = \epsilon_{ijk}\, \varphi_{k,j} \tag{1}$$

By letting $\underline{\varphi} = \underline{\nabla}\phi$ in (1), we obtain

$$(\mathrm{curl}\, \underline{\nabla}\phi)_i = \epsilon_{ijk}\, \phi_{,kj} \tag{2}$$

Since $\phi_{,kj}$ is a second order tensor that is symmetric with respect to the indexes k & j, while ϵ_{ijk} is asymmetric with respect to k, j, then by Eq.(a) in Example 2.1.5, RHS of (2) vanishes, and this proves (a).

To show (b) we let $\underline{\varphi} = \underline{u}$ in (1) and obtain

$$(\text{curl}\, \underline{u})_i = \epsilon_{ijk} u_{k,j} \qquad (3)$$

By taking the operator div on (3) we get

$$[(\text{curl}\, \underline{u})_i]_{,i} = \epsilon_{ijk} u_{k,ij} \qquad (4)$$

Since $u_{k,ij}$ is a third order tensor that is symmetric with respect to the indices i & j, while ϵ_{ijk} is asymmetric with respect to those indices, by Eq. (a) in Example 2.1.5, RHS of (4) vanishes, and this proves (b).

To show (c) we write (c) in components

$$\epsilon_{ida} \epsilon_{abc} u_{c,bd} = u_{a,ai} - u_{i,aa} \qquad (5)$$

Since, by the $\epsilon-\delta$ relation [see Eq. (2.1.10)]

$$\epsilon_{ida} \epsilon_{abc} = \epsilon_{ida} \epsilon_{bca} =$$

$$= \delta_{ib} \delta_{dc} - \delta_{ic} \delta_{bd} \qquad (6)$$

therefore

$$\epsilon_{ida} \epsilon_{abc} u_{c,bd} = (\delta_{ib} \delta_{dc} - \delta_{ic} \delta_{bd}) u_{c,bd} \qquad (7)$$

and using the filtrating property of the Kronecker's delta.

$$\delta_{ab}\, a_b = a_a \quad (8)$$

we obtain

$$\epsilon_{ida}\, \epsilon_{abc}\, u_{c,bd} = u_{c,ci} - u_{i,bb} \quad (9)$$

This proves (c).

To show (d) we note that

$$(\underline{\nabla}\, \underline{u})_{ij} = u_{i,j} \quad (10)$$

& by the definition of curl of a second-order tensor field $\underline{T} = \underline{T}(\underline{x})$ [see Eq. (2.2.18)]

$$(\text{Curl}\, \underline{T})_{ij} = \epsilon_{ipq}\, T_{jq,p} \quad (11)$$

Substituting $\underline{T} = \underline{\nabla}\, \underline{u}$ into (11) we get

$$(\text{curl}\, \underline{\nabla T})_{ij} = \epsilon_{ipq}\, u_{j,qp} \quad (12)$$

Eq. (12) together with Eq. (9) in Example 2.1.5 implies (d), and this completes a proof of (d).

To show (e) we replace $\underline{T}$ by $\underline{T}^T$ in (11) & obtain

$$(\text{curl}\ \underline{T}^T)_{ij} = \epsilon_{ipq} T^T_{jq,p}$$

$$= \epsilon_{ipq} T_{qj,p} \qquad (13)$$

Next, by letting $\underline{T} = \underline{\nabla}\underline{u}$ in Eq. (13) we obtain

$$(\text{curl}\ \underline{\nabla}\underline{u}^T)_{ij} = \epsilon_{ipq} u_{q,jp}$$

$$= [\epsilon_{ipq} u_{q,p}]_{,j} = (\underline{\nabla}\ \text{curl}\ \underline{u})_{ij} \qquad (14)$$

This proves that (e) holds true.

To show (f) we need to prove that

$$u_{i,j} + u_{j,i} = 0 \Rightarrow u_{i,jk} = 0 \qquad (15)$$

To this end we note that Eq. $u_{i,j}+u_{j,i}=0$ implies

$$u_{i,jk} + u_{j,ki} = 0. \qquad (16)$$

By replacing j by k & k by j in (16) we get

$$u_{i,jk} + u_{k,ji} = 0 \qquad (17)$$

Now, if Eqs. (16) & (17) are added side by side we obtain

$$2u_{i,jk} + (u_{j,k} + u_{k,j})_{,i} = 0 \qquad (18)$$

Since the second term on LHS of (18) vanishes by the hypothesis, Eq. (18) implies (15).

To show (g), we write (f) in components and obtain [see Eq. (11)].

$$\epsilon_{ipq} T_{jq,pj} = \epsilon_{ipq} T^{T}_{qj,jp} \qquad (19)$$

Since

$$T_{jq} = T^{T}_{qj} \qquad (20)$$

Eq. (19) is an identity, & this proves (g).

The relation (h) in components takes the form

$$\epsilon_{jpq} T_{iq,pj} = 0 \qquad (21)$$

Eq. (21) represents an identity as ϵ_{jpq} is asymmetric with respect to indices p & j, and $T_{iq,pj}$ is symmetric with respect to p & j, and Eq. (a) in Example 2.1.5 holds true. This proves (h).

To show (i) introduce the notations

$$\operatorname{curl} \underline{S} = \underline{A} \quad , \quad \operatorname{curl} \underline{S}^T = \underline{B} \qquad (22)$$

Then Eq. (i) is equivalent to

$$(\operatorname{curl} \underline{A})^T = \operatorname{curl} \underline{B} \qquad (23)$$

Eq. (23) in components takes the form

$$\epsilon_{jpq} A_{iq,p} = \epsilon_{ipq} B_{jq,p} \qquad (24)$$

where

$$A_{iq} = \epsilon_{iab} S_{qb,a} , \qquad (25)$$

$$B_{jq} = \epsilon_{jab} S^T_{qb,a} = \epsilon_{jab} S_{bq,a} \qquad (26)$$

Substituting (25) and (26) into (24) we obtain

$$\epsilon_{jpq} \epsilon_{iab} S_{qb,ap} = \epsilon_{ipq} \epsilon_{jab} S_{bq,ap} \qquad (27)$$

By letting $a=p$, $b=q$ in RHS of (27) we arrive at an identity, and this proves (i).

To show (j), note that Eq. (j) in components takes the form

$$\epsilon_{ipq}(\phi\,\delta_{jq})_{,p} = -\,\epsilon_{jpq}(\phi\,\delta_{iq})_{,p} \qquad (28)$$

or equivalently

$$\epsilon_{ipj}\,\phi_{,p} = -\,\epsilon_{jpi}\,\phi_{,p} \qquad (29)$$

Since

$$-\,\epsilon_{jpi} = +\,\epsilon_{jip} = -\,\epsilon_{ijp} = \epsilon_{ipj} \qquad (30)$$

therefore Eq. (29) is an identity, and this proves (j).

To show (k) we note that Eq. (k) in component reads

$$(S^T_{ij}u_j)_{,i} = u_k S_{kj,j} + S_{ij}u_{i,j} \qquad (31)$$

Since

$$(S^T_{ij}u_j)_{,i} = (S_{ji}u_j)_{,i} =$$
$$= S_{ji,i}u_j + S_{ji}u_{j,i} \qquad (32)$$

therefore (31) is an identity, and this proves (k).

To prove (l) we note that Eq. (l) in components takes the form

$$\epsilon_{ipq} S_{iq,p} = 0 \qquad (32)$$

Eq. (32) is an identity since $S_{iq} = S_{qi}$ & $\epsilon_{ipq} = -\epsilon_{qpi}$, and this completes proof of (l).

To show (m) note that LHS of (m), written in components, takes the form

$$L_{ij} \equiv \epsilon_{ipq} \epsilon_{jap} S_{qb,ap} = \qquad (33)$$
$$= \epsilon_{iqp} \epsilon_{jba} S_{qb,pa}$$

Using the identity [see Eq. (2.1.12)]

$$\epsilon_{iqp} \epsilon_{jba} = \begin{vmatrix} \delta_{ij} & \delta_{ib} & \delta_{ia} \\ \delta_{qj} & \delta_{qb} & \delta_{qa} \\ \delta_{pj} & \delta_{pb} & \delta_{pa} \end{vmatrix} =$$

$$\begin{aligned} &= \delta_{ij} (\delta_{qb} \delta_{pa} - \delta_{qa} \delta_{pb}) \\ &\quad - \delta_{ib} (\delta_{qj} \delta_{pa} - \delta_{pj} \delta_{qa}) \qquad (34) \\ &\quad + \delta_{ia} (\delta_{qj} \delta_{pb} - \delta_{pj} \delta_{qb}) \end{aligned}$$

as well as the filtrating property of the Kronecker symbol

$$\delta_{ab}\, a_b = a_a \tag{35}$$

, where a_a is an arbitrary vector, we reduce Eq. (33) to the form

$$\begin{aligned} L_{ij} &= \delta_{ij}\,(S_{qq,aa} - S_{ab,ab}) \\ &- \delta_{ib}\,(S_{jb,aa} - S_{ab,ja}) \\ &+ \delta_{ia}\,(S_{jb,ab} - S_{bb,ja}) \end{aligned} \tag{36}$$

or

$$\begin{aligned} L_{ij} &= -S_{ij,aa} + S_{ia,aj} + S_{jb,bi} \\ &- S_{bb,ij} + \delta_{ij}\,(S_{qq,aa} - S_{ab,ab}) \end{aligned} \tag{37}$$

Therefore

$$L_{ij} = R_{ij} \tag{38}$$

, where R_{ij} is RHS of (m) written in components, and this completes a proof of Eq. (ii). Note that the symmetry of $\underline{S}$ was used to obtain (37) from (36).

To show (n) we substitute

$$S_{ij} = G_{ij} - \delta_{ij} G_{kk} \qquad (39)$$

into RHS of (37) and obtain

$$R_{ij}' = -(G_{ij} - \delta_{ij} G_{kk})_{,aa}$$
$$+ G_{ia,aj} + G_{jb,bi} - 2G_{kk,ij}$$
$$+ 2G_{aa,ij} - \delta_{ij}(G_{aa,bb} + G_{ab,ab})$$

$$= -G_{ij,aa} + 2G_{(ia,aj)} - \delta_{ij} G_{ab,ab} \qquad (40).$$

and this proves that (n) holds true.

Finally, to show (o) we recall the definition of an axial vector $\underline{\omega}$ corresponding to a tensor $\underline{P}$. [see Eq. (2.1.33)]

$$\omega_i = -\frac{1}{2} \epsilon_{ijk} P_{jk} \qquad (41)$$

An equivalent form of (41) reads

$$P_{ij} = -\epsilon_{ijk} \omega_k \qquad (42)$$

By letting $P_{ij} = S_{ij}$ in (42) we obtain

$$S_{ij} = -\epsilon_{ijk} \omega_k \qquad (43)$$

Taking the curl of (43) we get

$$\epsilon_{ipq} S_{jq,p} = -\epsilon_{ipq}\epsilon_{jqa}\omega_{a,p} \qquad (44)$$

Since

$$-\epsilon_{ipq}\epsilon_{jqa} = \epsilon_{ipq}\epsilon_{jaq} = \delta_{ij}\delta_{pa} - \delta_{ia}\delta_{jp} \qquad (45)$$

Eq. (44) implies

$$\epsilon_{ipq} S_{jq,p} = \delta_{ij}\omega_{p,p} - \omega_{i,j} \qquad (46)$$

and this proves (c).

As a result the solution to Problem 2.12 is complete.

Problem 2.13

To show (a) we use the formula [see Eq.(2.3.3)]

$$\int_R \hat{u}_{k,k}\, dv = \int_{\partial R} \hat{u}_k n_k \, da \qquad (1)$$

where $\hat{u}_k = \hat{u}_k(\underline{x})$ is an arbitrary vector field. By letting $\hat{u}_k = \delta_{ki} f$ into (1), where i is a fixed number from the set $\{1, 2, 3\}$, we obtain (a); and this completes proof of (a).

To show (b) we note that Eq. (b) in components reads

$$\int_R \epsilon_{ijk} u_{k,j}\, dv = \int_{\partial R} \epsilon_{ijk} n_j u_k\, da \qquad (2)$$

By letting $\hat{u}_k = \epsilon_{ika} u_a$ in (1), where i is a fixed number from the set $\{1,2,3\}$, we get

$$\int_R \epsilon_{ika} u_{a,k}\, dv = \int_{\partial R} \epsilon_{ika} n_k u_a\, da \qquad (3)$$

Eq. (3) is equivalent to Eq. (b), and this completes proof of (b).

To show (c), note that (c) written in components, takes the form

$$\int_R u_{i,j}\, dv = \int_{\partial R} u_i n_j\, da \qquad (4)$$

By letting $\hat{u}_k = \delta_{jk} u_i$ in (1), where i & j are fixed numbers from the set $\{1,2,3\}$ we get

$$\int_R \delta_{jk} u_{i,k}\, dv = \int_{\partial R} \delta_{jk} u_i n_k\, da \qquad (5)$$

or

$$\int_R u_{i,j}\, dv = \int_{\partial R} u_i n_j\, da \qquad (6)$$

Eq. (6) is equivalent to (c), and this completes proof of (c).

To show (d) we note that Eq. (d) in components takes the form

$$\int_R (u_i T_{jk,k} + u_{i,a} T^T_{aj})\, dv = \int_{\partial R} u_i T_{jk} n_k\, da \quad (7)$$

Since

$$u_i T_{jk,k} + u_{i,a} T^T_{aj} =$$

$$= u_i T_{jk,k} + u_{i,k} T_{jk} = (u_i T_{jk})_{,k} \qquad (8)$$

therefore, by letting $\hat{u}_k = u_i T_{jk}$ in (1), where i and j are fixed indices from the set $\{1, 2, 3\}$, we obtain (7); and this completes proof of (d).

Problem 2.14

Define a scalar field f, and a vector field $\underline{g}$ by

$$f = \underline{u}\cdot(\mathrm{curl}\,\underline{g}), \quad \underline{g} = \mathrm{curl}\,\underline{u} \qquad (1)$$

or in components

$$f = u_k \epsilon_{kab}\, g_{b,a}, \quad g_b = \epsilon_{bcd}\, u_{d,c} \qquad (2)$$

Also, note that

$$f = (u_k \epsilon_{kab}\, g_b)_{,a} - u_{k,a}\epsilon_{kab}\, g_b$$

$$= (u_k \epsilon_{kab}\, g_b)_{,a}$$

$$+ \epsilon_{bak}\, u_{k,a}\, \epsilon_{bcd}\, u_{d,c}$$

$$= (u_k \epsilon_{kab}\, g_b)_{,a} + (\mathrm{curl}\,\underline{u})^2 \qquad (3)$$

By letting $\hat{u}_k = u_c \epsilon_{ckb}\, g_b$ in Eq. (1) of Problem 2.13, we obtain

$$\int_R [u_c \epsilon_{ckb}\, g_b]_{,k}\, dv = \int_{\partial R} u_c \epsilon_{ckb}\, g_b\, n_k\, da \qquad (4)$$

Therefore, if either (a) or (b) holds true, RHS of (4) vanishes. Hence, integrating

Eq. (3) over R and using (4), we find that Eq. (c) holds true provided either Eq. (a) or Eq. (b) is satisfied. This completes solution to Problem 2.14.

Problem 2.15

To show that (a) and (b) imply (c) we integrate Eq. (a) twice with respect to time over the interval $[0, t]$, and take into account the initial conditions (b).

To show that (c) implies (a) and (b), we take the two steps:

(A) We let $t = 0$ in (c) to obtain $(b)_1$,

We differentiate Eq. (c) with respect to time and take the result at $t = 0$ to obtain $(b)_2$.

(B) We differentiate Eq. (c) twice with respect to time, take into account the formula

$$\frac{\partial^2}{\partial t^2}(t * f) = f \qquad (1)$$

valid for an arbitrary function $f = f(x, t)$, and arrive at Eq. (a). This completes solution to Problem 2.15.