Chapter 2

Trees and Connectivity

2.1 Nonseparable Graphs

1. **Proof.** Suppose first that u and v are two vertices in a nonseparable graph G of order 3 or more. By Theorem 2.3, there is a cycle C of order 3 or more containing u and v. There are two internally disjoint $u-v$ paths on C.

   For the converse, suppose that for every two vertices x and y of G, there are two internally $x-y$ paths. Thus G is connected. Assume, to the contrary, that G contains a cut-vertex v. By Theorem 2.2, G contains two vertices u and w such that v lies on every $u-w$ path of G. Then G does not contain two internally disjoint $u-w$ paths, which is a contradiction. ■

2. **Proof.** Since H is clearly connected, it remains to show that H contains no cut-vertices. Let $x \in V(H)$. If $x = v$, then $H - x = G$, which is connected. If $x \in V(G)$, then $G - x$ is connected since G is nonseparable. Since v is adjacent to at least one vertex of $G - x$, it follows that $H - x$ is connected. Thus H has no cut-vertices. ■

3. **Proof.** Let H be the graph obtained by adding two vertices u and w to G and joining u to the two vertices of U and joining w to the two vertices of W. By Corollary 2.5, H is nonseparable. By Corollary 2.4, H contains two internally disjoint $u-w$ paths. Deleting u and w from these two paths produces two disjoint paths connecting the vertices of U and the vertices of W. ■

4. (a) **Proof.** Suppose first that G contains disjoint k-cycles C and C'. Let $u_1, u_2 \in V(C)$ and $w_1, w_2 \in V(C')$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$. By Corollary 2.6, G contains two disjoint paths P_1 and P_2 connecting the vertices of U and the vertices of W. Let x_i be the last vertex of P_i $(i = 1, 2)$ belonging to C and let y_i be the first vertex of W following x_i on P_i. For $i = 1, 2$, let Q_i be the $x_i - y_i$ subpath of P_i. Suppose that the length of Q_i is $\ell_i \geq 1$. Since there is an $x_1 - x_2$ path on C of length at least $\lceil \frac{k}{2} \rceil$ and a $y_1 - y_2$ path on C' of length at least $\lceil \frac{k}{2} \rceil$, it follows that G contains a cycle of length at least $2\lceil \frac{k}{2} \rceil + \ell_1 + \ell_2 \geq k + 2$, which is a contradiction.

   Next assume that G contains k-cycles C and C' having exactly one vertex v in common. Suppose that $u \in V(C)$ and $w \in V(C')$, where $u, w \neq v$. Since v is not a cut-vertex of G, there is a $u - w$ path P in G not containing v. Let x be the last vertex of P belonging to C and let y be the first vertex of C' following x on P. Let Q be the $x - y$ subpath of P, where Q has length ℓ. Then G contains a cycle of length at least $2\left\lceil \frac{k}{2} \right\rceil + \ell \geq k+1$, again, producing a contradiction. ■

(b) There are two cycles in the graph $K_{2,4}$ that have only two vertices in common.

5. **Proof.** Since v is a cut-vertex of G, it follows that $G - v$ is disconnected. Let $u, w \in V(\overline{G}) - \{v\}$. We show that there is a $u - w$ path in $\overline{G}$ not containing v, which implies that v is not a cut-vertex of $\overline{G}$. We consider two cases.

Case 1. *u and w belong to distinct components of $G - v$.* Then u and w are nonadjacent in G and so uw is an edge in $\overline{G}$. Thus v is not on the $u - w$ path (u, w) in $\overline{G} - v$.

Case 2. *u and w belong to the same component G_1 of $G - v$.* Let G_2 be another component of G and let $x \in V(G_2)$. Then neither u nor w is adjacent to x in G. This implies that ux and wx are edges in $\overline{G}$. Thus (u, x, w) is a $u - w$ path in $\overline{G}$ that does not contain v. ■

6. No. In the graph $G = K_4 - e$, where $V(G) = \{u, v, w, x\}$ with $uv \notin E(G)$, let $P = (u, w, x, v)$. Then there is no $u - v$ path Q such that P and Q are internally disjoint.

7. (a) **Proof.** By Theorem 2.3, every two vertices of a nonseparable graph lie on a common cycle. For the converse, suppose that G is a graph of order 3 or more such that every two vertices u and v of G lie on a common cycle. Then there exist two internally disjoint $u - v$ paths in G. By Corollary 2.4, G is nonseparable.

Next, let G be a nonseparable graph, where $e = uv$ and $f = xy$ are two edges of G. The edges e and f are subdivided by introducing two new vertices s and t, resulting in the four new edges us, sv, xt and ty, producing a nonseparable graph H. Thus s and t lie on a common cycle C'. Since $\deg_H s = \deg_H t = 2$, the edges us, sv, xt and ty lie on the cycle C'. Replacing us and sv by e and replacing xt and ty by f produces a cycle C in G containing e and f. For the converse, suppose that G is a connected graph of order 3 or more with the property that every two edges of G lie on a common cycle. Let u and v be two vertices of G. Let $e = ux$ and $f = vy$ be distinct edges of G incident with u and v, respectively. Then e and f lie on a common cycle C of G. So u and v lie on C. Thus G is nonseparable.

A similar proof can be given of the statement: A graph G of order 3 or more is nonseparable if and only if every vertex and edge of G lie on a common cycle of G. ■

(b) The graph H is complete if and only if G is nonseparable.

Proof. First, if G is nonseparable, then every two vertices of G lie on a common cycle of G. Thus every two vertices of H are adjacent and so H is complete. For the converse, suppose that H is complete. Then every two vertices of H are adjacent. Thus every two vertices of G lie on a common cycle of G and so G is nonseparable. ■

8. **Proof.** Since G is connected (by Exercise 10(a) in Section 1.3), it remains to show that G contains no cut-vertex. Assume, to the contrary, that G contains a cut-vertex

v. Then G contains (necessarily nonadjacent) vertices u and w such that v lies on every $u - w$ path in G. Thus $\deg u + \deg w \geq n$. Let U be the set of vertices adjacent to u and let W be the set of vertices adjacent to w. Since G has order n, the set $U \cap W$ contains at least two vertices. Let $v_1, v_2 \in U \cap W$. At least one of these vertices, say v_1, is different from v. Then (u, v_1, w) is a $u - w$ path in G not containing v, contrary to the hypothesis. ■

9. The statement is false. The graph $G = K_{2,3}$ is itself a block of order 5 in G and there is no cycle in G containing all the vertices of G.

10. $k \geq \ell + 1$. **Proof.** We proceed by induction on ℓ. If $\ell = 0$, then $k = 1$; while if $\ell = 1$, then $k \geq 2$. Assume, for an integer $\ell \geq 2$, that every connected graph with b blocks and c cut-vertices, where $0 \leq c < \ell$, satisfies $b \geq c + 1$. Let G be a connected graph with k blocks and ℓ cut-vertices. By Theorem 2.8, G contains a cut-vertex v with the property that, with at most one exception, all blocks containing v are end-blocks. Since G has at least two cut-vertices, exactly one block B containing v is not an end-block. For each end-block B' containing v, delete $V(B') - \{v\}$ from G. In the resulting graph H, the vertex v is not a cut-vertex. Thus H has j blocks and $\ell - 1$ cut-vertices, where $j \leq k - 1$. By the induction hypothesis, $j \geq (\ell - 1) + 1 = \ell$ and so $k \geq j + 1 \geq \ell + 1$. ■

11. The statement is false. See Figure 2.1.

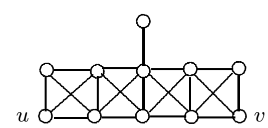


Figure 2.1: The graph in Exercise 11

12. (a) **Proof.** Consider the wheel $W_{2k+2} = C_{2k+2} \vee K_1$, where v is the central vertex of W_{2k+2}. Then $\mathrm{rad}(W_{2k+2}) = 1$ and $\mathrm{rad}(W_{2k+2} - v) = \mathrm{rad}(C_{2k+2}) = k + 1$. Thus $\mathrm{rad}(W_{2k+2} - v) = \mathrm{rad}(W_{2k+2}) + k$. ■

(b) **Proof.** First observe that there are graphs G for which $\mathrm{rad}(G - v) = \mathrm{rad}(G) - 1$ for some vertex v of G that is not a cut-vertex. For example, for an end-vertex v of P_{2k}, $\mathrm{rad}(P_{2k} - v) = \mathrm{rad}(P_{2k-1}) = k - 1 = \mathrm{rad}(P_{2k}) - 1$.

Now, let G be a nontrivial connected graph and let v be a vertex of G that is not a cut-vertex. Let $u \in V(G)$ and let w be a vertex of G that is distinct from u. Suppose that $d_G(u, w) = k$. If there exists a $u - w$ geodesic in G that does not contain v, then $d_{G-v}(u, w) = k$. If, on the other hand, every $u - w$ geodesic in G contains v, then $d_{G-v}(u, w) > k$. This implies that if u is a vertex of G with $e_G(u) = k$ and $d_G(u, v) \neq k$, then $e_{G-v}(u) \geq k$. Suppose that $d_G(u, v) = k$ and $P = (u = u_0, u_1, \ldots, u_k = v)$ is a $u - v$ geodesic in G. Then $d_G(u, u_{k-1}) = d_{G-v}(u, u_{k-1}) = k - 1$. Thus $e_{G-v}(u) \geq k - 1$. Therefore, $\mathrm{rad}(G - v) \geq \mathrm{rad}(G) - 1$. ■

(c) **Proof.** By assumption, H is a connected induced subgraph of G of minimum order having radius r. For a non-cut-vertex v of H, $\mathrm{rad}(H - v) \neq r$. If $\mathrm{rad}(H - v) < r$, then by (b), $\mathrm{rad}(H - v) = r - 1$. If $\mathrm{rad}(H - v) \geq r + 1$, then there are induced subgraphs of $H - v$ whose radius is 1. Since the removal of any non-cut-vertex of a connected graph can reduce the radius by at most 1 (by (b)), it follows that repeated deletion of non-cut-vertices from $H - v$ will result in an induced subgraph of $H - v$ (and therefore of G) with radius r. This is a contradiction. ■

2.2 Trees

1. **Proof.** Since G is a connected graph of order 3 or more, either $\deg u \geq 2$ or $\deg v \geq 2$, say $\deg v \geq 2$. Then there is a vertex w different from u that is adjacent to v. Assume, to the contrary, that v is not a cut-vertex. Thus $G - v$ is connected, and so there is a $u - w$ path P in $G - v$. However then, P together with v and the two edges uv and vw form a cycle containing the bridge uv. This contradicts Theorem 2.10. ■

2. The statement is false. For example, let G be the graph consisting of two components, one of which is P_2 and the other is obtained from K_3 by adding a pendant edge at each vertex of K_3.

3. (a) The only example is a double star where each central vertex has degree 4.
   (b) Let T be a tree of order n where 75% of the vertices have degree 1 and the remaining 25% vertices have degree $x \geq 2$. Then $(3n/4) \cdot 1 + (n/4) \cdot x = 2(n-1)$. Solving for n, we have $n = 8/(5-x)$. The only possible solutions for n are when $x = 3$ or $x = 4$. If $x = 3$, then $T = K_{1,3}$; while if $x = 4$, then T is the tree in (a).

4. Let T be a tree of order n containing x vertices of degree 3. Thus T has $n - x$ end-vertices. Therefore, $3x + 1 \cdot (n - x) = 2(n-1)$. Solving for x, we obtain $x = (n-2)/2$.

5. There are 20 forests of order 6 (see Figure 2.2).

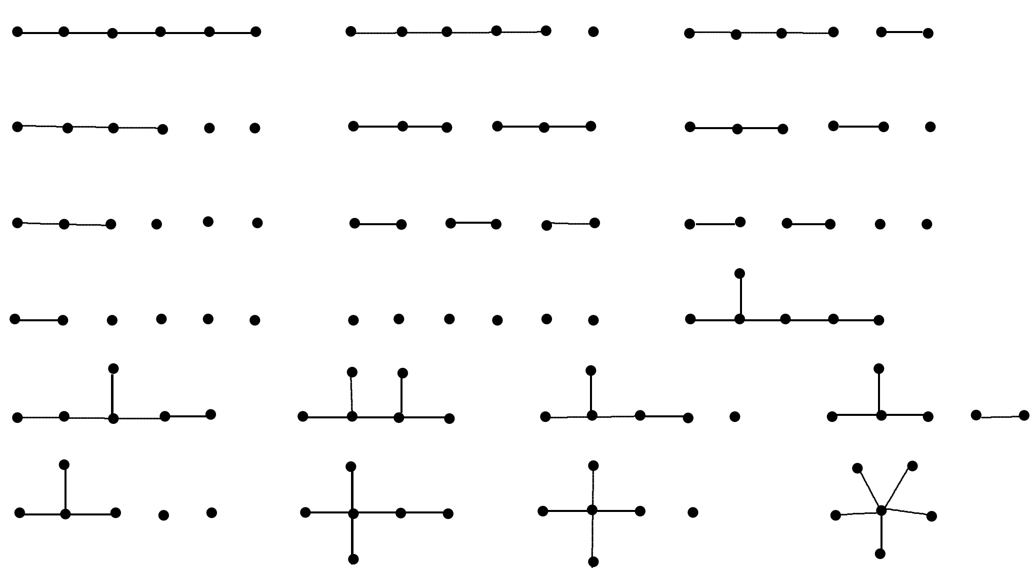


Figure 2.2: The 20 forests of order 6 in Exercise 5

6. **Proof.** Assume first that G is a forest. Then every induced subgraph H of G is a forest. If H contains a nontrivial component (a tree), then H contains at least two end-vertices. Otherwise, H contains at least one isolated vertex. In either case, G contains a vertex of degree at most 1. For the converse, assume that G is not a forest. Then G contains a cycle C. Thus $G[V(C)]$ is an induced subgraph of G that contains the spanning cycle C. This implies that every vertex of $G[V(C)]$ has degree at least 2. ∎

7. **Claim:** A graph G is a forest if and only if every connected subgraph of G is an induced subgraph of G.

   Proof. We first show that every connected subgraph of a forest F is an induced subgraph of F. Assume, to the contrary, that this is not the case. Then there is a connected subgraph H of F that is not an induced subgraph of F. This implies that H contains two nonadjacent vertices u and v that are adjacent in F. Since H is connected, H and therefore F contains $u - v$ path P. However then P together with uv is a cycle in F, which is impossible.

   For the converse, suppose that G is a graph that is not a forest. Then G has a cycle C. Let e be an edge of C. Then $C - e$ is a connected subgraph of G with vertex set $V(C)$. Since $G[V(C)]$ contains e, it follows that $G[V(C)] \neq C - e$. Thus $C - e$ is not an induced subgraph of G. ∎

8. **Proof.** Assume, to the contrary, there exists a connected graph G whose vertices have even degrees but G contains a bridge $e = uv$. Then $G - e$ contains exactly two components, one of which contains u and the other contains v. Then each of the components of $G - e$ has exactly one odd vertex, which is impossible. ∎

9. **Proof.** Assume first that at least one of e_1 and e_2 is not a bridge, say e_1 is not a bridge. Then $G - e_1$ is connected. Thus $G - e_1 - e_2 = (G - e_1) - e_2$ has at most two components.

   For the converse, suppose that e_1 and e_2 are bridges of G. Since e_1 is a bridge, $G - e_1$ has exactly two components, say H and F. We may assume that $e_2 \in E(H)$. Then e_2 is a bridge of H, implying that $H - e_2$ consists of two components. Therefore, $G - e_1 - e_2$ has three components. ∎

10. **Proof.** Let G be a 3-regular graph. If G has a bridge e, then each vertex incident with e is a cut-vertex. For the converse, suppose that G has a cut-vertex u. We may assume that G is connected, for otherwise, consider the component of G containing u. Then $G - u$ is disconnected and so $G - u$ has at least two components. Since $\deg u = 3$, it follows that $G - u$ has at most three components. If $G - u$ has three components, say $G - u = G_1 + G_2 + G_3$, then u is adjacent to exactly one vertex v_i in G_i and each edge uv_i $(1 \leq i \leq 3)$ is a bridge of G. If $G - u$ has two components, say $G - u = G_1 + G_2$, then u is adjacent to exactly one vertex in G_1 or to exactly one vertex in G_2, say u is adjacent to exactly one vertex v_1 in G_1. Then uv_1 is a bridge of G. ∎

11. **Proof.** By Theorem 2.9, the center of every graph lies in a block. Since the only blocks of T are K_2, the only possible centers are K_1 and K_2. ∎

12. (a) **Proof.** Let P be a longest path in T, say P is a $u-v$ path. Thus the length of P is $\text{diam}(T)$. Then u and v are end-vertices of T and $P-u-v$ is a path of length $\text{diam}(T)-2$ in T'. Thus $\text{diam}(T') \geq \text{diam}(T)-2$. Assume, to the contrary, that T' contains a longest path Q of length j, where $j \geq \text{diam}(T)-1$. Suppose that Q is an $x-y$ path. Each of x and y is adjacent to an end-vertex of T, for otherwise, x and y are end-vertices of T and so do not belong to T'. This implies that T contains a path of length $\text{diam}(T)+1$, which is impossible.

    Certainly, no central vertex of T is an end-vertex of T. Thus each central vertex of T belongs to T'. Let $u \in V(T')$ and suppose that $e_T(u) = k$. For each vertex v of T such that $d_T(u,v) = k$, the vertex v is an end-vertex of T. Hence $e_{T'}(u) \leq k-1$. Let w be the neighbor of v on the $u-v$ path in T. Then $d_{T'}(u,w) = k-1$. Thus $e_{T'}(u) = k-1$. Therefore, for every vertex u in T such that u is not an end-vertex of T, it follows that $e_{T'}(u) = e_T(u)-1$. In particular, if u is a central vertex of T, then $e_T(u) = \text{rad}(T) = \text{rad}(T')+1$.

    Let u be a central vertex of T. Then u is not an end-vertex of T. As we saw, $e_{T'}(u) = e_T(u)-1$. Then any central vertex of T is a central vertex of T'. Thus $\text{Cen}(T) = \text{Cen}(T')$. ■

    (b) **Proof.** Let T_1 be the tree obtained from T by deleting the end-vertices of T, let T_2 be the tree obtained by deleting the end-vertices of T_1 and so on until we arrive at a tree T_ℓ which is either K_1 or K_2. Then $\text{diam}(T) = \text{diam}(T_\ell)+2\ell$ and $\text{rad}(T) = \text{rad}(T_\ell)+\ell$. If $T_\ell = K_1$, then $\text{diam}(T_\ell) = 0$ and $\text{rad}(T_\ell) = 0$. So $\text{diam}(T) = 2\ell$ and $\text{rad}(T) = \ell$. Hence $\text{diam}(T) = 2\,\text{rad}(T)$ if $\text{Cen}(T) = K_1$. If $T_\ell = K_2$, then $\text{diam}(T_\ell) = 1$ and $\text{rad}(T_\ell) = 1$. So $\text{diam}(T) = 2\ell+1$ and $\text{rad}(T) = \ell+1$. Thus $\text{diam}(T) = 2(\text{rad}(T)-1)+1 = 2\,\text{rad}(T)-1$ if $\text{Cen}(T) = K_2$. ■

13. (a) **Proof.** Since $b(v_i) \leq \deg v_i$ for $1 \leq i \leq n$, it then follows by the First Theorem of Graph Theory (Theorem 1.4) that

$$\sum_{i=1}^{n} b(v_i) \leq \sum_{i=1}^{n} \deg v_i = 2m,$$

    as desired. ■

    (b) **Proof.** First, assume that $\sum_{i=1}^{n} b(v_i) = 2m$. By (a), $b(v_i) = \deg v_i$ for $1 \leq i \leq n$. This implies that every edge of G is a bridge and so G is a tree.

    For the converse, assume that G is a tree. Then every edge of G is a bridge. This implies that $b(v_i) = \deg v_i$ for $1 \leq i \leq n$. It then follows by the First Theorem of Graph Theory (Theorem 1.4) that $\sum_{i=1}^{n} b(v_i) = 2m$. ■

14. Claim: $T = K_1$ or $T = P_4$.

    Proof. Let T be a tree of order n. Since T and $\overline{T}$ are both trees of order n, it follows that the sizes of T and $\overline{T}$ are $n-1$. Thus $n-1+n-1 = 2(n-1) = \binom{n}{2} = \frac{n(n-1)}{2}$. Hence $4(n-1) = n(n-1)$ and so $(n-1)(n-4) = 0$, implying that $n = 1$ or $n = 4$. If $n = 1$, then $T = K_1$. If $n = 4$, then $T = P_4$ or $T = K_{1,3}$. Since $\overline{P_4} = P_4$ and $\overline{K}_{1,3}$ is not a tree, it follows that $T = P_4$. ■

15. **Proof.** Since each end-block of a tree contains a pendant edge and thus a leaf, the result follows from Theorem 2.8. ■

16. **Proof.** Assume, to the contrary, that $d_k = \deg v_k > \lceil \frac{n-1}{k} \rceil$ for some integer k with $1 \le k \le n$ and so

$$\deg v_k \ge \left\lceil \frac{n-1}{k} \right\rceil + 1 \ge \frac{n-1}{k} + 1 = \frac{n+k-1}{k}.$$

Thus

$$\begin{aligned} 2n-2 &= \sum_{i=1}^{n} \deg v_i = \sum_{i=1}^{k} \deg v_i + \sum_{i=k+1}^{n} \deg v_i \\ &\ge k\left(\frac{n+k-1}{k}\right) + (n-k) = 2n-1, \end{aligned}$$

which is a contradiction. ■

17. **Proof.** Suppose that the order of G is n and the size of G is m. Since $G-u$ and $G-v$ are both trees of order $n-1$, the size of each of $G-u$ and $G-v$ is $n-2$. Therefore, $m - \deg u = n-2$ and $m - \deg v = n-2$. Thus $\deg u = \deg v = m-(n-2)$. ■

18. **Proof.** Assume, to the contrary, that there exists a tree T containing two distinct edges e_1 and e_2 such that the two components of $T - e_1$ are isomorphic and the two components of $T - e_2$ are isomorphic. Let T_1 and T_2 be the two components of $T - e_1$. Since $T_1 \cong T_2$, these two trees have the same size, say k. Thus the size of T is $2k+1$. The edge e_2 belongs either to T_1 or to T_2. Without loss of generality, assume that e_2 belongs to T_2. Since one component of $T - e_2$ contains T_1 and e_1, the size of one component of $T - e_2$ is at least $k+1$ and the size of the other component of $T - e_2$ is therefore at most $k-1$. Thus the two components of $T - e_2$ are not isomorphic, which is a contradiction. ■

19. **Proof.** The graph $\overline{C}_{n+2}$ is $(n-1)$-regular. Since $\delta(\overline{C}_{n+2}) = n-1$, it follows by Theorem 2.20 that T is isomorphic to a subgraph of $\overline{C}_{n+2}$. ■

20. **Proof.** Since T is not a star, the order n of T is at least 4. We proceed by induction on $n \ge 4$. If $n = 4$, then $T = P_4$. Since $\overline{P}_4 = P_4$, the result is true for $n = 4$. Assume that for every tree T of order $n - 1 \ge 4$ that is not a star, $T \subseteq \overline{T}$.

   Let T be a tree of order n that is not a star. We first show that there is an end-vertex $v \in V(T)$ such that $T - v$ is not a star. If $T = P_n$, then let v be an end-vertex of T. So we may assume that $T \ne P_n$ and let $w \in V(T)$ such that $\deg w \ge 3$. If every vertex adjacent to w is not an end-vertex of T, let v be any end-vertex of T; otherwise, let v be an end-vertex of T that is adjacent to w. Thus $T - v$ is not a star.

   By the induction hypothesis, $T - v \subseteq \overline{T-v}$. Hence $T - v$ is isomorphic to a subgraph F of $\overline{T-v}$. Let ϕ be an isomorphism from $T - v$ to F. Let u be the vertex in T that is adjacent to v. We consider two cases.

   Case 1. $\phi(u) \ne u$. Since v is adjacent to $\phi(u)$ in $\overline{T}$, we can extend the isomorphism ϕ from $T - v$ to F to an isomorphism from T to a subgraph F' of $\overline{T}$ by defining $\phi(v) = v$.

   Case 2. $\phi(u) = u$. Since T is not a star, there exists an end-vertex x in T such that x is not adjacent to u in T. Define an isomorphism ϕ^* from T to a subgraph F' of $\overline{T}$ by defining $\phi^*(v) = \phi(x)$, $\phi^*(x) = v$ and $\phi^*(w) = \phi(w)$ for all $w \in V(T - v - x)$. Observe that if x is a adjacent to a vertex y in T, then v is adjacent to $\phi(y)$ in $\overline{T}$. Moreover, $\phi(x)$ is adjacent u in $\overline{T}$. ■

21. **Claim.** The 4-cycle C_4 is the only graph with this property.

Proof. First, observe that if $G = C_4$ and S is a set of three vertices of G, then $G[S] = P_3$. Suppose then that $G \neq C_4$. If G is a tree, then G cannot have three or more end-vertices. Thus $G = P_n = (v_1, v_2, \ldots, v_n)$ for some $n \geq 4$. However, $G[\{v_1, v_2, v_4\}] = K_2 + K_1$, which is not a tree. If G is not a tree, then G contains cycles. Let $C_k = (v_1, v_2, \ldots, v_k = v_1)$ be a smallest cycle in G. If $k = 3$, then $G[\{v_1, v_2, v_3\}] = K_3$, which is not a tree. If $k \geq 5$, then $G[\{v_1, v_2, v_4\}] = K_2 + K_1$, which is not a tree. Thus $k = 4$. Since $G \neq C_4$, there is a vertex $v \in V(G) - V(C_k)$ such that v is adjacent a vertex in C_k, say v is adjacent to v_1. Then v is adjacent to neither v_2 nor v_k as G contains no triangles. So $G[\{v, v_2, v_k\}] = \overline{K}_3$, which is not a tree. ■

2.3 Spanning Trees

1. See Figure 2.3.

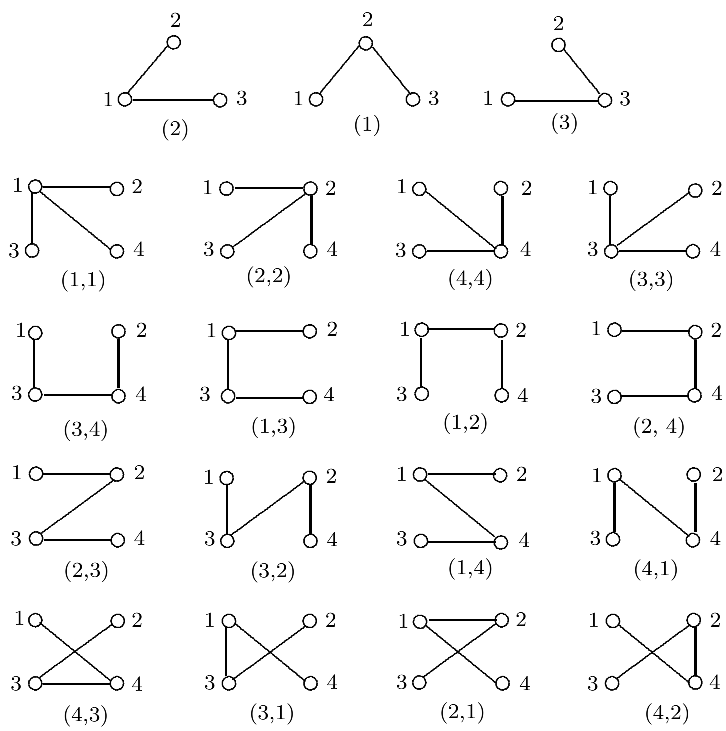


Figure 2.3: The trees in Exercise 1

2. (a) Let T be a tree with vertex set $V(T) = \{1, 2, \ldots, n\}$, $n \geq 3$, that has constant Prüfer code $(a_1, a_2, \ldots, a_n)$ where $a_i = \ell$ for some $\ell \in V(T)$ for $1 \leq$

$i \le n$. Let $V_1 = V(T) - \{\ell\} = \{b_1, b_2, \ldots, b_{n-1}\}$, where $b_1 < b_2 < \ldots < b_{n-1}$. Then $b_1, b_2, \ldots, b_{n-2}$ are end-vertices adjacent to the vertex ℓ. Removing $b_1, b_2, \ldots, b_{n-2}$ from T, we obtain K_2 containing the edge $b_{n-1}\ell$. Thus every vertex in V_1 is adjacent to the vertex ℓ and so $T = K_{1,n-1}$ whose central vertex is ℓ.

(b) Let T be a tree with vertex set $V(T) = \{1, 2, \ldots, n\}$, $n \ge 3$, that has Prüfer code $(a_1, a_2, \ldots, a_n)$ where $a_i \in \{\ell_1, \ell_2\}$ for some $\ell_1, \ell_2 \in V(T)$ for $1 \le i \le n$, $\ell_1 \ne \ell_2$ and each of ℓ_1 and ℓ_2 appears at least once in $(a_1, a_2, \ldots, a_n)$. In this case, $n \ge 4$. Let $V_1 = V(T) - \{\ell_1, \ell_2\} = \{b_1, b_2, \ldots, b_{n-2}\}$, where $b_1 < b_2 < \ldots < b_{n-2}$. We know that ℓ_1 and ℓ_2 are not end-vertices, while $b_1, b_2, \ldots, b_{n-3}$ in V_1 are end-vertices, each of which is adjacent to ℓ_1 or to ℓ_2. Removing all $b_1, b_2, \ldots, b_{n-3}$ from T, we obtain P_3 whose central vertex is $b_{n-2} \in \{\ell_1, \ell_2\}$. Thus every vertex in V_2 is adjacent to ℓ_1 or to ℓ_2, which implies that T is a double star whose central vertices are ℓ_1 and ℓ_2.

(c) Let T be a tree of order $n \ge 3$. Observe that if T contains a vertex ℓ whose degree is 3 or more, then ℓ appears twice in the Prüfer code of T. This implies that $\Delta(T) \le 2$. On the other hand, T is connected and so $T = P_n$.

3. The labeled tree in Figure 2.4 has Prüfer code $(4, 5, 7, 2, 1, 1, 6, 6, 7)$.

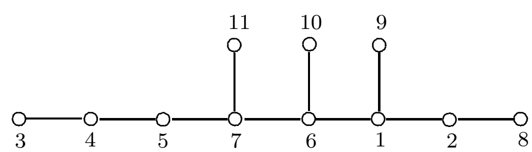


Figure 2.4: The tree in Exercise 3

4. Since seven numbers appear twice each and one number appears three times, the length of the Prüfer code is 17. However, the length of the Prüfer code of a tree of order n is $n-2$, so $n-2 = 17$ and $n = 19$. Thus the number of leaves is $19-7-1 = 11$. Also, by Theorem 2.14, $n_1 = 2 + n_3 + 2n_4 = 2 + 7 + 2 \cdot 1 = 11$.

5. Since every vertex v of a tree T appears $\deg v - 1$ times in its Prüfer code, it follows that the degree sequence of T is $4, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1$.

6. (a) If G contains three or more $u - v$ paths for two distinct vertices u and v, then G contains at least two cycles, which contradicts to the fact that G has cycle rank 1.

(b) The statement in (a) is false if G has cycle rank 2. Let the partite sets of $G = K_{2,3}$ be $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, w_3\}$. Then the graph G has order $n = 5$ and size $m = 6$. Thus $m - n + 1 = 2$ and G has cycle rank 2. There are three $u_1 - u_2$ paths in G.

7. **Proof.** If G is a unicyclic graph of order n, then G has size n and at least three vertices of G lie on a cycle of G. Hence if $s_n : d_1, d_2, \ldots, d_n$ is a degree sequence of G, then $1 \le d_i \le n-1$ for $1 \le i \le n$, at most $n-3$ terms of s_n are 1 and $\sum_{i=1}^{n} d_i = 2n$.

We verify the converse by induction. Suppose that d_1, d_2, d_3 is a sequence of integers with $1 \le d_i \le 2$ for $1 \le i \le 3$ such that $\sum_{i=1}^{3} d_i = 2 \cdot 3 = 6$. Then the sequence is necessarily $2, 2, 2$, which is the degree sequence of the unicyclic graph K_3. The statement is therefore true for $n = 3$. Assume for an integer $k \geq 3$ that if $s_k : d_1, d_2, \ldots, d_k$ is a sequence of integers with $1 \leq d_i \leq k-1$, at most $k-3$ terms of which are 1 which satisfies $\sum_{i=1}^{k} d_i = 2k$, then s_k is a degree sequence of a unicyclic graph of order k. Let

$$s_{k+1} : d_1, d_2, \ldots, d_{k+1}$$

be a sequence of integers with $1 \leq d_i \leq k$, at most $k-2$ terms of which are 1 which satisfies $\sum_{i=1}^{k+1} d_i = 2k+2$. If every term $d_i = 2$, then s_{k+1} is a degree sequence of C_{k+1}, which is a unicyclic graph. Suppose that not all integers d_i are 2. Then some term is 1, say $d_{k+1} = 1$ and some integer d_j is 3 or more. Let d_k be the maximum term in s_{k+1}. Then $d_k \leq k$. If $d_k = k$, then no other term has this value since $\sum_{i=1}^{k+1} d_i = 2k+2$. Let

$$s' : d'_1, d'_2, \ldots, d'_k$$

be the sequence where $d'_i = d_i$ for $1 \leq i \leq k-1$ and $d'_k = d_k - 1$. Then $1 \leq d'_i \leq k-1$ for $1 \leq i \leq k$, $\sum_{i=1}^{k} d'_i = 2k$ and at most $k-3$ terms of s' are 1. By the induction hypothesis, s' is the degree sequence of some unicyclic graph H with $V(H) = \{v_1, v_2, \ldots, v_k\}$ where $\deg v_i = d'_i$. Let G be the graph obtained from H by adding a new vertex v_{k+1} which is adjacent to v_k. Then s_{k+1} is the degree sequence of the unicyclic graph G. ■

8. (a) Let T be any spanning tree of G that is a distance-preserving tree, say from the vertex v. Let $x, y \in V(T)$ such that $d_T(x, y) = \text{diam}(T)$. Now $d_T(x, v) = d_G(x, v) \leq e_G(v)$ and $d_T(y, v) = d_G(y, v) \leq e_G(v)$. Thus

$$\begin{aligned} \text{diam}(T) &= d_T(x, y) \leq d_T(x, v) + d_T(v, y) \\ &\leq 2e_G(v) \leq 2\,\text{diam}(G). \end{aligned}$$

(b) The statement is true. Let k be a positive integer. Consider $G = K_{k+2}$. Then $T = P_{k+2}$ is a spanning tree of G and $\text{diam}(T) = k + 1 > k \cdot 1 = k\,\text{diam}(G)$.

9. See Figure 2.5.

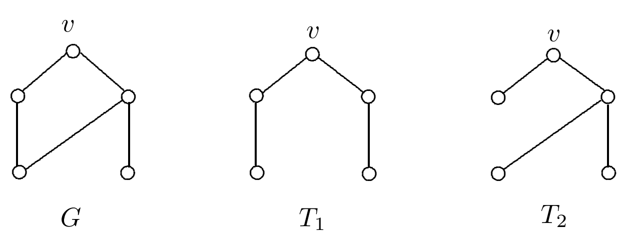


Figure 2.5: The graphs in Exercise 9

10. (a) Suppose that T is a spanning tree of G that is distance-preserving from some vertex of G. The fact that $\text{diam}(G) \leq \text{diam}(T)$ is obvious. For $\text{diam}(T) \leq 2\,\text{diam}(G)$, see the proof in Exercise 8(a).

(b) Let G be the graph of order $2a+1$, where

$$V(G) = \{v_0, v_1, \ldots, v_a\} \cup \{u_1, u_2, \ldots, u_a\}.,$$

such that

$$\begin{aligned} E(G) \quad = \quad & \{v_i v_{i+1} : 0 \le i \le a-1\} \cup \{u_i u_{i+1} : 1 \le i \le a-1\} \cup \{u_i v_i : 1 \le i \le a\} \\ & \cup \{v_i u_{i+1} : 0 \le i \le a-1\}. \end{aligned}$$

(See Figure 2.6 for the graph G when $a = 4$.) Let E denote the edge set of the path

$$P = (v_0, v_1, \cdots, v_a).$$

For $a \le b \le 2a$, we define a spanning tree T_b of G such that T_b is distance-preserving from v_0. Then

$$E(T_a) = E \cup \{v_1 u_1\} \cup \{v_i u_{i+1} : 1 \le i \le a-1\}.$$

For $b = a + j$ with $1 \le j \le a$, let

$$Q_j = (v_0, u_1, \cdots, u_j)$$

and

$$E(T_b) = E \cup E(Q_j) \cup \{u_i v_{i+1} : j \le i \le a-1\}.$$

See Figure 2.6 for the trees T_b for $a \le b \le 2a$.

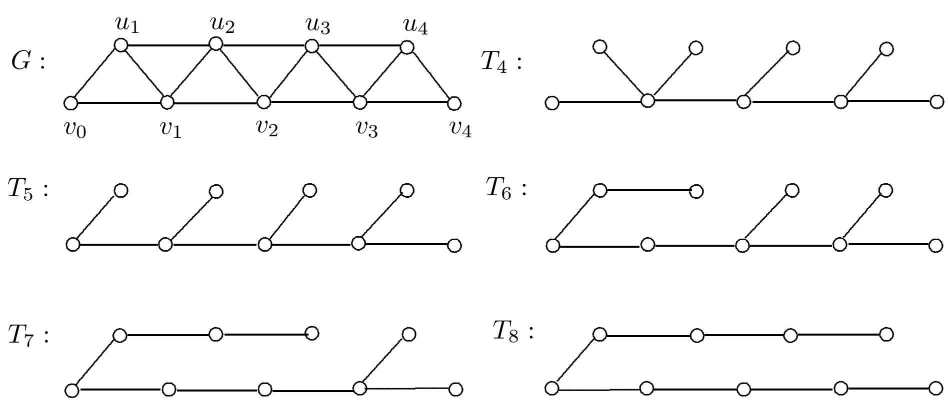


Figure 2.6: The graphs in Exercise 10(b)

11. See Figure 2.7.

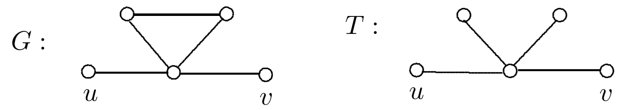


Figure 2.7: The graphs in Exercise 11

12. (a) The matrices D and $D - A$ of G are

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad D - A = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

To calculate a cofactor of $D - A$, we delete the entries in row 5 and column 5 and obtain

$$(-1)^{5+5} \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 11.$$

Consequently, there are 11 distinct spanning trees of the graph G.

(b) The distinct labeled spanning trees of G are shown in Figure 2.8

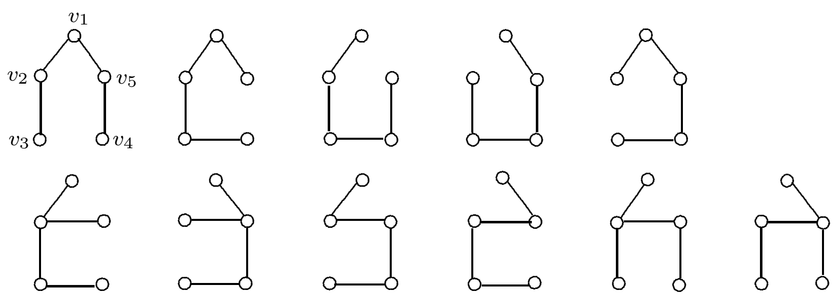


Figure 2.8: The distinct spanning trees of a graph

13. (a) See Figure 2.9.

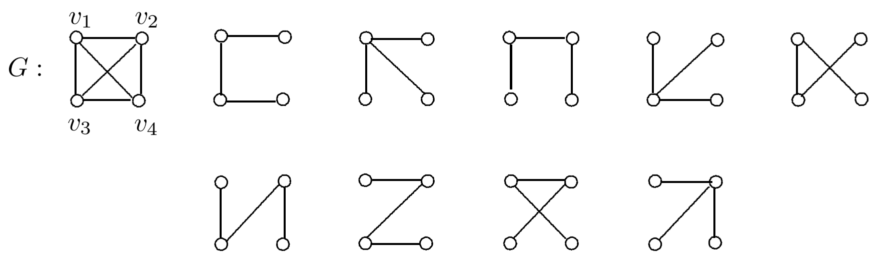


Figure 2.9: The graphs in Exercise 13

(b) **Proof.** A spanning tree T of G having v as an end-vertex is obtained from a spanning tree T' of $G - v$ by joining a vertex u of T' to v. Since $G - v$ has order $n-1$, the number of spanning trees of $T - v$ is $(n-1)^{(n-1)-2} = (n-1)^{n-3}$. Since there are $n - 1$ vertices in $T - v$ to join to v, the number of spanning trees of G having v as an end-vertex is $(n-1)(n-1)^{n-3} = (n-1)^{n-2}$. Note: For $n = 4$, $(n-1)^{n-2} = 3^2 = 9$. ■

14. We use the Matrix-Tree theorem to prove that there are n^{n-2} distinct labeled trees of order n.

Proof. Let t_n be the number of distinct labeled trees of order n. For $n \geq 2$, t_n is the number of spanning trees of K_n and so t_n is any cofactor of the $n \times n$ matrix $D - A = [c_{ij}]$, where $c_{ii} = n - 1$ and $c_{ij} = -1$ for all i, j with $1 \leq i, j \leq n$ and $i \neq j$. Let B be the $(n-1) \times (n-1)$ matrix obtained from $D - A$ by deleting row n and column n. Then

$$t_n = (-1)^{n+n} \det(B) = \det(B).$$

For $2 \leq i \leq n - 1$, replace the ith row R_i in B by $R_i - R_1$, obtaining the matrix B'. Observe that $\det(B') = \det(B)$. Next replace the first column C_1 in B' by $\sum_{i=1}^{n-1} C_i$ where C_i is the ith column for $1 \leq i \leq n-1$, obtaining B''. Observe that $\det(B'') = \det(B') = \det(B)$ and $B'' = [b_{ij}]$ is an upper triangular $(n-1) \times (n-1)$ matrix such that $b_{11} = 1$ and $b_{ii} = n$ for $2 \leq j \leq n - 1$. Since $\det(B'')$ is the product of the entries on the main diagonal, it follows that $\det(B) = n^{n-2}$ and so $t_n = n^{n-2}$. ■

15. **Proof.** Assume first that e is an edge of a connected graph G that is not a bridge. Then $G - e$ is connected. Then every spanning tree of $G - e$ is a spanning tree of G that does not contain e.

For the converse, suppose that e is an edge that does not belong to every spanning tree of G. Let T be a spanning tree of G not containing e. Then $T + e$ has a cycle containing e. Therefore, e is not a bridge. ■

16. By Exercise 15, an edge of F belongs to every spanning tree of G if and only if e is a bridge of G. Thus F is a forest and so F contains no cycles.

17. Since the size of a complete graph of order n is $\binom{n}{2} = \frac{n(n-1)}{2}$ and each spanning tree has $n-1$ edges, the maximum possible number of spanning trees in a connected graph of order $n \geq 4$, no two of which have an edge in common, is $\frac{n(n-1)}{2(n-1)} = \frac{n}{2}$. (We will see in Chapter 10 that this is attainable when n is even.)

18. (a) **Proof.** Let v be an end-vertex of a spanning tree T of G. Then $T - v$ is a tree that is a spanning tree of $G - v$ and so $G - v$ is connected. Thus v is not a cut-vertex of G. ■

(b) **Proof.** Since every tree contains at least two end-vertices, it then follows by (a) that G contains at least two vertices that are not cut-vertices. ■

(c) **Proof.** The edges of G incident with v do not produce a cycle. Let E be the set of edges incident with v. If there is a spanning tree of G with edge set E, then the proof is complete; otherwise, there is an edge e_1 such that $E \cup \{e_1\}$ does not produce a cycle. If there is a spanning tree of G with edge set $E \cup \{e_1\}$, then the proof is complete. Continuing in this manner, we obtain a spanning tree T with $E \subseteq E(T)$. ■

(d) **Proof.** We show that $\Delta(G) \leq 2$. Assume, to the contrary, that $\Delta(G) = k \geq 3$. Let v be a vertex of G with $\deg v = k$. It then follows by (c) that G contains a spanning tree T that contains all edges of G that are incident with v. Thus T contains a vertex whose degree exceeds 2 and so T has more than two end-vertices. By (a), G has more than two vertices that are not cut-vertices, a contradiction. Therefore, as claimed $\Delta(G) \leq 2$. Since G is connected, G is a path or G is a cycle. Since a cycle has no cut-vertices, G is a path. ■

19. For each positive integer k different from 2, we show that there is a connected graph with exactly k spanning trees. First, a connected graph G has exactly one spanning tree if and only if G is a tree. For $k \geq 3$, the graph $G = C_k$ has exactly k spanning trees, as there are k possible edges that can be removed from G to obtain a spanning tree of G. In order for G to have more than one spanning tree, G must contain a cycle and therefore at least three spanning trees.

20. Applying the proof of Theorem 2.24, observe that the tree T can be transformed into T' by a sequence $T = T_0, T_1, \ldots, T_k = T'$ of spanning trees of G such that T_i is transformed into T_{i+1} $(0 \leq i \leq k-1)$ by an edge exchange. In an edge exchange from T_i to T_{i+1}, these two spanning trees have $n - 2$ edges in common for each i.

21. If $r = t$, let $G = K_{1,r}$. Suppose then that $2 \leq r < t$. Let G be the graph obtained by attaching $r - 1$ pendant edges to a vertex v of K_{t-r+2}. Then at least one vertex of K_{t-r+2} different from v is an end-vertex in a spanning tree of G. At the other extreme, all other vertices of K_{t-r+2} different from v can be an end-vertex in a spanning tree of G, for a total of $(t-r+1)+(r-1) = t$ end-vertices. This is illustrated in Figure 2.10 for $r = 4$ and $t = 7$.

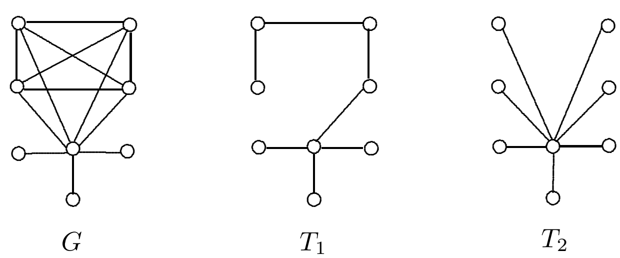


Figure 2.10: A graph G and two spanning trees of G in Exercise 21

22. (a) The adjacency matrix A of G and the degree matrix D of G are

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$D - A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(b) A cofactor of the matrix $D - A$ is

$$(-1)^{1+1} \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{vmatrix}.$$

(c) The matrix C described in the proof of Theorem 2.25 is

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(d) The matrix C_3 described in the proof of Theorem 2.25 is

$$C_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(e) The matrix $C_3 \cdot C_3^t$ is

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant of $C_3 \cdot C_3^t$ is 3.

(f) Note that

$$\begin{aligned} \det(C_3 \cdot C_3^t) &= \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} \\ &+ \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \\ &= 0 + 1^2 + (-1)^2 + 1^2 = 3. \end{aligned}$$

23. For the graph G, there are two nonisomorphic spanning trees, namely $K_{1,3}$ and P_4. For the graph H, there are three nonisomorphic spanning trees namely $K_{1,5}$, T_1 and T_2, where T_1 is the tree obtained from $P_5 = (v_1, v_2, v_3, v_4, v_5)$ by adding a pendant edge at v_3 and T_2 is the tree obtained by subdividing an edge of $K_{1,4}$.

24. See Figure 2.11.

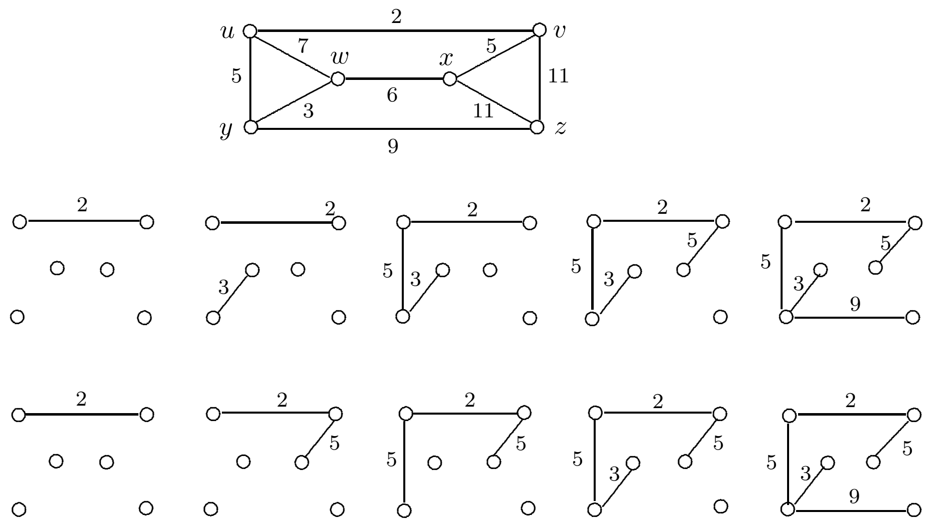


Figure 2.11: The graphs in Exercise 24

25. See Figure 2.12.

26. **Proof.** Suppose that G has order n and assume, to the contrary, that there is a minimum spanning tree T of G that cannot be obtained from Kruskal's algorithm. Let $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$, where $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_{n-1})$. Among all minimum spanning trees of T obtained by Kruskal's algorithm, let T' be one whose edges are selected in the order $f_1, f_2, \ldots, f_{n-1}$ (and so $w(f_1) \leq w(f_2) \leq \ldots \leq w(f_{n-1})$) and such that k $(< n-1)$ is the maximum positive integer for which $\{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_k\}$. Thus f_{k+1} is not in T.

    Let e be an edge of the cycle C in $T + f_{k+1}$ that does not belong to T' and let $T_0 = T + f_{k+1} - e$. Hence T_0 is a spanning tree of G and so $w(T) \leq w(T_0)$. Thus $w(e) \leq w(f_{k+1})$. Furthermore,

$$w(e_{k+1}) \leq w(e) \leq w(f_{k+1}) \leq w(e_{k+1}). \tag{2.1}$$

    Since there is equality throughout (2.1), it follows that $w(e_{k+1}) = w(f_{k+1})$. Because $e_1, e_2, \ldots, e_{k+1}$ do not produce a cycle in G and e_{k+1} is an edge of minimum weight in $E(G) - \{e_1, e_2, \ldots, e_k\}$, there is a minimum spanning tree obtained by Kruskal's algorithm that contains $e_1, e_2, \ldots, e_{k+1}$. This produces a contradiction. ■

27. **Proof.** Let T be a minimum spanning tree of G. First, assume that T is the unique minimum spanning tree of G. Assume, to the contrary, that there exist edges e and f of G such that (1) $e \in E(G) - E(T)$, (2) f is on the cycle of $T + e$ and (3) $w(f) \geq w(e)$. Then $T' = T + e - f$ is a spanning tree of G with

$$w(T') = w(T) + w(e) - w(f) \leq w(T),$$

    which implies that either T' is another minimum spanning tree of G or T is not a minimum spanning tree of G, a contradiction.

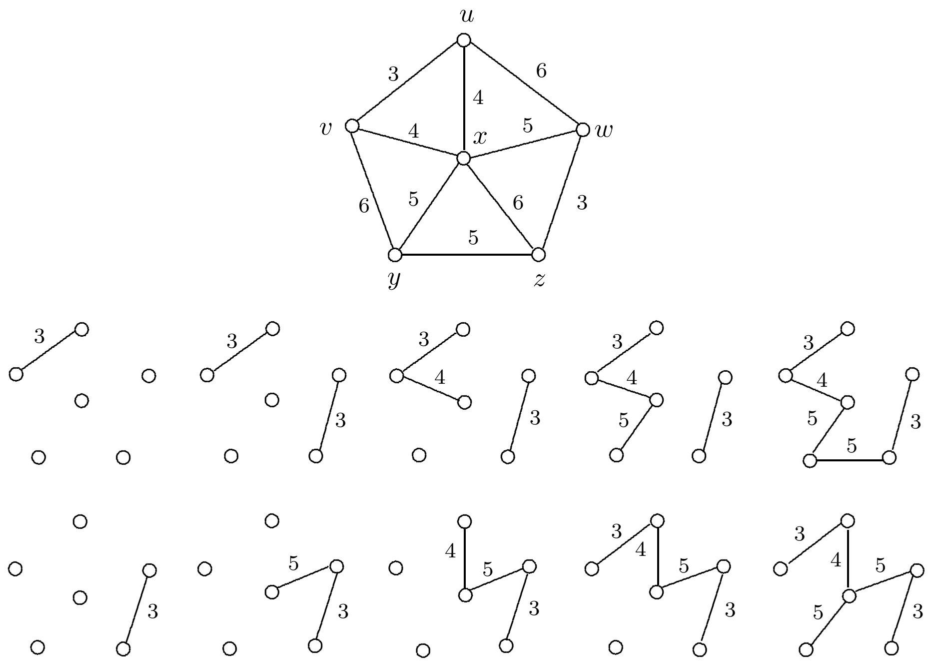


Figure 2.12: The graphs in Exercise 25

For the converse, assume that the weight of each edge $e \in E(G) - E(T)$ exceeds the weight of every edge on the cycle of $T + e$. We claim that T is unique, for otherwise, let $T' \neq T$ be another minimum spanning tree of G. So $w(T') = w(T)$. Let $e' \in E(T') - E(T)$ and let C' be the cycle in $T + e'$. By our assumption, $w(e') > w(f')$ for every edge f' in C'. Thus let $f' \in E(C)$ and so

$$w(T') = w(T) + w(e') - w(f') > w(T) = w(T'),$$

which is a contradiction. ■

2.4 Connectivity and Edge-Connectivity

1. Let $G = K_{n_1,n_2,\cdots,n_k}$ with $n_1 \leq n_2 \leq \cdots \leq n_k$ and $n = \sum_{i=1}^{k} n_i$. Let V_i be a partite set of G with $|V_i| = n_i$ $(1 \leq i \leq k)$.

   Claim: $\kappa(G) = \lambda(G) = \delta(G) = n - n_k$.

   Proof. Certainly, $V(G) - V_k$ is a vertex-cut. So $\kappa(G) \leq \delta(G)$. Let S be a set of vertices with $|S| < \delta(G)$. Then there exist $u, v \in V(G) - S$ such that $u \in V_i$ and $v \in V_j$, where $i \neq j$. Furthermore, $uv \in E(G - S)$ and each vertex of $G - S$ is either adjacent to u or adjacent to v. Thus $G - S$ is connected. ■

   (Also, note that $\operatorname{diam} G = 2$ and so $\lambda(G) = \delta(G)$.)

2. **Proof.** Observe that $\{v_1, v_2, \ldots, v_k\}$ is a vertex-cut of H. Thus $\kappa(H) \leq k$. Let W be a set of $k - 1$ vertices of H. We show that $H - W$ is connected. We consider two cases.

Case 1. $w \in W$. Then $H - W = G - (W - \{w\})$. Since G is k-connected and $|W - \{w\}| = k - 2$, it follows that $G - (W - \{w\})$ is connected and so $H - W$ is connected.

Case 2. $w \notin W$. Since G is k-connected, $G - W$ is connected. Since $|W| = k - 1$, it follows that $H - W$ is obtained from $G - W$ by adding a new vertex w and joining w to a vertex of $G - W$. Therefore, $H - W$ is connected. ■

3. (a) **Proof.** Let $H = G \vee K_1$, where $V(H) - V(G) = \{v\}$. We show that $\kappa(H) \geq k+1$. Let S be a set of vertices of H with $|S| = k$. There are two cases.

   Case 1. $v \notin S$. Since every vertex in G is adjacent to v in H, every vertex in $G - S$ is adjacent to v in $H - S$ and so $H - S$ is connected.

   Case 2. $v \in S$. Then $H - S = G - (S - \{v\})$. Since $\kappa(G) \geq k$ and $|S - \{v\}| = k - 1$, it follows that $G - (S - \{v\})$ is connected.

   In either case, S is not a vertex-cut of H. Thus the removal of k or fewer vertices from H does not disconnect H and so $\kappa(H) \geq k + 1$. Therefore, H is $(k + 1)$-connected. ■

   (b) **Proof.** Let $H = G \vee K_1$, where $V(H) - V(G) = \{v\}$. We show that $\lambda(H) \geq k+1$. Let S be a set of edges of H with $|S| = k$. Suppose that ℓ edges of S belong to G, where $0 \leq \ell \leq k$ and the remaining $k - \ell$ edges are incident with v. Let S' be the set of ℓ edges belonging to G. If $\ell = k$, then v is adjacent to every vertex of G in H and so $H - S$ is connected. If $\ell < k$, then $G - S'$ is connected. Since v is adjacent to at least $k + 1 - \ell > 0$ vertices of G, it follows that $H - S$ is connected. ■

4. Let $H = G \vee K_1$. Observe that H has diameter at most 2. By Theorem 2.34,

$$\lambda(H) = \delta(H) = 1 + d_n.$$

5. Let G be a k-connected graph and let T be a tree of order $k+1$. Then $k \leq \kappa(G) \leq \delta(G)$. By Theorem 2.20, G contains T as a subgraph.

6. (a) **Proof.** Since S is a minimum vertex-cut, $|S| = \kappa(G) = k$. Assume, to the contrary, that $G - S$ has ℓ components, where $\ell \geq 3$. Let G_1 be a component of smallest order n_1 in $G - S$. Then

$$n_1 \leq \frac{n-k}{\ell} \leq \frac{n-k}{3}.$$

   If $u \in V(G_1)$, then

$$\begin{aligned} \deg_G u &\leq \deg_{G_1} u + k \leq n_1 - 1 + k \\ &\leq \frac{n-k}{3} - 1 + k = \frac{n+2k-3}{3}, \end{aligned}$$

   contrary to hypothesis. ■

(b) **Proof.** Assume, to the contrary, that $G-S$ has ℓ components, where $\ell \geq t+1$. Let G_1 be a component of smallest order n_1 in $G-S$. Then

$$n_1 \leq \frac{n-k}{\ell} \leq \frac{n-k}{t+1}.$$

If $u \in V(G_1)$, then

$$\begin{aligned}\deg_G u &\leq \deg_{G_1} u + k \leq n_1 - 1 + k \\ &\leq \frac{n-k}{t+1} - 1 + k = \frac{n+kt-t-1}{t+1},\end{aligned}$$

contrary to hypothesis. ■

7. **Proof.** Assume, to contrary, that there exists a graph G of order n containing at least k pairwise nonadjacent vertices such that

$$\deg_G v \geq \frac{n+(k-1)(\ell-2)}{k}$$

for all $v \in V(G)$ but that G is not (ℓ, k)-connected. Thus $\kappa_k(G) = t \leq \ell - 1$. Since G contains at least k pairwise nonadjacent vertices, there is a set S of t vertices such that $G-S$ contains at least k components. Let G_1 be a component of smallest order n_1 in $G-S$. Thus $n_1 \leq \frac{n-t}{k}$. For $v \in V(G_1)$, it follows that

$$\begin{aligned}\deg_G v &\leq \left(\frac{n-t}{k} - 1\right) + t = \frac{n+kt-t-k}{k} \\ &= \frac{n+(k-1)(t-1)-1}{k} \leq \frac{n+(k-1)(\ell-2)-1}{k},\end{aligned}$$

producing a contradiction. ■

8. **Proof.** Let S be a minimum edge-cut of G. Since $\mathrm{diam}(G) = 2$, it follows by Theorem 2.34 that

$$|S| = \lambda(G) = \delta(G) = \delta.$$

Furthermore, $G-S$ has exactly two components. Suppose that G_1 and G_2 are the two components of $G-S$ with $V(G_1) = A$ and $V(G_2) = B$. Since $\mathrm{diam}(G) = 2$, at most one of A and B contains a vertex that is not incident with any edges in S; for otherwise, there are vertices $x \in A$ and $y \in B$ such that x and y are not incident with any edges in S, implying that $d(x, y) \geq 3$, which is impossible. Assume, without loss of generality, that every vertex in A is incident with some edge in S; that is, every vertex in A is adjacent to some vertex of B in G.

Let $v \in A$ such that v is incident with a maximum number k of edges in S, say v is incident with the edges $e_1, e_2, \cdots, e_k$ in S. Since there are $\delta - k$ edges in $S - \{e_1, e_2, \cdots, e_k\}$ and every vertex in A is incident with some edge in S, there are at most $\delta - k$ vertices in $A - \{v\}$ and so $|A| \leq \delta - k + 1$. On the other hand, v must also be adjacent to $\deg v - k \geq \delta - k$ vertices in A and so there are at least $\delta - k$ vertices in $A - \{v\}$. Therefore, $|A| \geq \delta - k + 1$ and so $|A| = \delta - k + 1$. Let u be any other vertex of A, where u is incident, say, with ℓ edges, where then $\ell \le k$. Thus u is incident with $\deg u - \ell$ vertices in $A - \{u\}$ and so

$$\delta - k + 1 = |A| \ge \deg u - \ell + 1 \ge \delta - k + 1,$$

which implies that $\deg u = \delta$ and $\ell = k$. Hence

$$(1)\ G_1 = K_{\delta-k+1} \text{ and } (2)\ |S| = k(\delta - k + 1) = \delta.$$

Solving (2) for δ, we obtain $\delta = k$ or $\delta = 1$. Thus $G_1 = K_1$ or $G_1 = K_\delta$. ∎

9. **Proof.** Let F_1 and F_2 be two copies of K_{c+1} with $V(F_1) = \{u_1, u_2, \ldots, u_{c+1}\}$ and $V(F_2) = \{v_1, v_2, \ldots, v_{c+1}\}$. Let G be the graph obtained from F_1 and F_2 by adding the a edges u_iv_i $(1 \le i \le a)$ and $b-a$ edges joining u_1 and $b-a$ vertices of F_2 different from v_1. Then $\kappa(G) = a$, where $\{u_1, u_2, \ldots, u_a\}$ is a minimum vertex-cut, $\lambda(G) = b$, where the b edges between F_1 and F_2 form a minimum edge-cut and $\delta(G) = c$. ∎

10. Choose k and n such that $n \ge k + 1$ and let $N = (n - k + 1)/2$ (so k and n are of opposite parity). Let $G = 2K_N \vee K_{k-1}$. Then $U = V(K_{k-1})$ is a vertex-cut and so $\kappa(G) \le k - 1$. Let $v \in V(2K_N)$. Then $\deg v = (N - 1) + (k - 1) = (n + k - 3)/2$. In fact, $\delta(G) = (n + k - 3)/2$.

11. Let G be the graph obtained from two copies F_1 and F_2 of K_k $(k \ge 3)$ by adding an edge joining a vertex of F_1 and a vertex of F_2. Then $\lambda(G) = 1$ and $\delta(G) = k - 1 \ge 2$.

12. **Claim:** If G is a connected graph of order $n \ge 2$, then

$$\kappa(G) + c(G) = n + 1.$$

Proof. Let S be a minimum vertex-cut of G. Then $|S| = \kappa(G)$. Since $G - S$ is an induced disconnected subgraph of order $n - \kappa(G)$ in G, it follows that $c(G) \ge n - \kappa(G) + 1$ or $\kappa(G) + c(G) \ge n + 1$.

Let H be an induced disconnected subgraph of order $c(G) - 1$ and let $S = V(G) - V(H)$. Since $H = G - S$ is disconnected, S is a vertex-cut of G and so

$$\kappa(G) \le |S| = n - (c(G) - 1) = n - c(G) + 1,$$

or $\kappa(G) + c(G) \le n + 1$. Combining the two inequalities gives us the desired result. ∎

13. **Proof.** Let $k = 2\ell$, where $\ell \ge 1$. Any k-connected graph of order n has minimum degree at least k and so has size at least $kn/2$. We show that there exists a k-connected graph G of order n and size $kn/2$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ where v_i is adjacent to the vertices $v_{i\pm1}, v_{i\pm2}, \ldots, v_{i\pm\ell}$. Thus G is a k-regular graph of order n and size $kn/2$. It remains to show that $\kappa(G) = k$. Assume, to the contrary, that $\kappa(G) < k$. Then there exists a set S of $k - 1$ vertices of G such that $G - S$ is disconnected. Let v_1 and v_r be vertices in two components of $G - S$. Let $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}\}$, where $1 < i_1 < i_2 < \cdots < i_{k-1} \le n$. Either at most $\ell - 1$ of the integers $i_1, i_2, \ldots, i_{k-1}$ are less than r or at most $\ell - 1$ of these integers exceed r, say the former. Let $j_1 = 1$. There is a vertex $v_{j_2} \notin S$ such that $2 \le j_2 \le \ell + 1$. Also, there is a vertex $v_{j_3} \notin S$ such that $j_2 + 1 \le j_3 \le j_2 + \ell$. Continuing in this manner, there are integers $j_1, j_2, \ldots, j_s$ with $1 = j_1 < j_2 < \ldots < j_s = r$ such that $j_{i+1} - j_i \le \ell$ and $v_{j_i} \in V(G - S)$ for $1 \le i \le s - 1$. Since $v_{j_i}v_{j_{i+1}} \in E(G - S)$, the path $(v_1 = v_{j_1}, v_{j_2}, \ldots, v_{j_s} = v_r)$ is a $v_1 - v_r$ path in $G - S$, producing a contradiction. ∎

14. The statement is false. For example, let $G = K_1 \vee (K_1 + K_2)$, where $v \in V(G)$ with $\deg v = 1$. Then $\kappa(G) = 1$ and $\kappa(G - v) = 2$.

15. (a) **Proof.** Let $W = \{w_1, w_2, \ldots, w_{k-2}\}$ be a set of $k-2$ vertices of $G-e$. We show that $(G-e)-W$ is connected. First, assume that e is incident with some vertex of W. Then $(G-e)-W = G-W$. Since G is k-connected and $|W| = k-2$, it follows that $G-W$ is connected. Thus, we may assume that e is not incident with any vertex of W. Let $x, y \in V(G)-W$. We show that x and y are connected in $(G-e)-W$. There are two cases.

    Case 1. $e = xy$. Since G contains at least $k+1$ vertices, there exists $z \in V(G) - (W \cup \{x, y\})$. Since G is k-connected, $G - (W \cup \{y\})$ and $G - (W \cup \{x\})$ are both connected. Thus there is an $x-z$ path P in $G - (W \cup \{y\})$ and a $z-y$ path Q in $G - (W \cup \{x\})$. Note that P and Q do not contain e and so are paths in $G-e-W$. Then the walk obtained by following P by Q is an $x-y$ walk in $G-e-W$ and so $G-e-W$ contains an $x-y$ path by Theorem 1.16.

    Case 2. $e \neq xy$. So e is either not incident with x or not incident with y, say e is not incident with x. Suppose that e is incident with a vertex $u \neq x$. Since G is k-connected, $G - (W \cup \{u\})$ is connected and so there is an $x-y$ path P in $G - (W \cup \{u\})$. This implies that P is also an $x-y$ path in $G-W$. Since P does not contain u, it follows that P does not contain e. Therefore, P is an $x-y$ path in $G-e-W$. ■

    (b) **Proof.** Let $X = \{e_1, e_2, \ldots, e_{k-2}\}$ be a set of $k-2$ edges of $G-e$. Since $|X \cup \{e\}| = k-1$, the graph $G-e-X$ is connected. ■

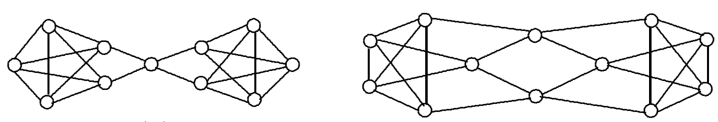
16. (a) If G is a 0-regular graph, then $G = \overline{K_n}$ for some positive integer n. Then $\kappa(G) = \lambda(G) = 0$.

    (b) Let G be a 1-regular graph. If G is disconnected, then $\kappa(G) = \lambda(G) = 0$. If G is connected, then $G = K_2$ and so $\kappa(G) = \lambda(G) = 1$.

    (c) If G is disconnected, then $\kappa(G) = \lambda(G) = 0$. If G is connected, then $G = C_n$, where $n \geq 3$. Thus $\kappa(G) = \lambda(G) = 2$.

    (d) The graph G shown in Figure 2.13(a) is 4-regular with $\kappa(G) = 1$ and $\lambda(G) = 2$. Thus $r = 4$.

    (e) The graph G shown in Figure 2.13(b) is 4-regular with $\kappa(G) = 2$ and $\lambda(G) = 4$. Thus $r = 4$.



(a) (b)

Figure 2.13: The graphs in Exercises 16(d) and 16(e)

17. Claim: (1) $\overline{\kappa}(G) \geq \kappa(G)$, (2) $\overline{\lambda}(G) \geq \lambda(G)$, and (3) $\overline{\lambda}(G) \geq \overline{\kappa}(G)$.

    Proof. Observe that (1) and (2) are immediate consequences of the definitions. For (3), let H be a subgraph of G such that $\kappa(H) = \overline{\kappa}(G)$. Then $\overline{\lambda}(G) \geq \lambda(H) \geq \kappa(H) = \overline{\kappa}(G)$. ■

18. Claim: $\max\{\kappa(G) : G \in \mathcal{G}\} = k$.

    Proof. Let $G \in \mathcal{G}$ and let $E = [V(G_1), V(G_2)]$, so $|E| = k$. Let W be the set of vertices of G_1 incident with at least one edge of E. Then $G - W$ is disconnected and $|W| \le k$. Thus $\max\{\kappa(G) : G \in \mathcal{G}\} \le k$.

    Since G_1 and G_2 are k-connected, each has order at least $k + 1$. Let $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$. Suppose that $E = \{u_1v_1,\ u_2v_2,\ \cdots,\ u_kv_k\}$ is the set of k edges added to G_1 and G_2, producing the graph H. Let $U = \{u_1,\ u_2,\ \cdots,\ u_k\}$ and $V = \{v_1, v_2, \cdots, v_k\}$. Since $H - U$ is disconnected, it follows that $\kappa(H) \le k$. We show that $\kappa(H) = k$ in this case. Let S be a subset of $V(H)$ of cardinality $k-1$ and let $S_i = S \cap V(G_i)$ for $i = 1, 2$. Since G_i $(i = 1, 2)$ is k-connected and $|S_i| < k$ for $i = 1, 2$, it follows that $G_i - S_i$ is connected. Furthermore, there exists j $(1 \le j \le k)$ such that $u_j, v_j \notin S$. Thus u_j, v_j, and u_jv_j are present in $H - S$ and so $H - S$ is connected. Hence S is not a vertex-cut and so $\kappa(H) > k - 1$, implying that $\kappa(H) = k$. ■

2.5 Menger's Theorem

1. For $n \ge k \ge 3$, every k vertices of $G = C_n$ lie on a common cycle but G is not k-connected.

2. **Proof.** Let $S = \{v_1, v_2, \cdots, v_k\}$ and $T = \{v_1, v_2\}$. Since G is k-connected, there are k internally disjoint $v_1 - v_2$ paths (by Theorem 2.37). Since $v_3,\ v_4,\ \cdots,\ v_k$ can belong to at most $k - 2$ of these paths, there are two of these paths that contain no vertices of $S - T$. These two paths produce a cycle containing no vertices of $S - T$.

   For the converse, let G be a graph having the property that for each set S of k distinct vertices of G and each two-element subset T of S, there is a cycle containing the vertices of T and avoiding the vertices of $S - T$. Assume, to the contrary, that $\kappa(G) < k$. Then G contains a vertex-cut $W = \{v_1, v_2, \cdots, v_{k-1}\}$. Since $n \ge k + 1$, it follows that $G - W$ is not trivial. Let u and v be two vertices belonging to different components of $G - W$. Let $S = \{v_1, v_2, \cdots, v_{k-2}, u, v\}$ and $T = \{u, v\}$. By assumption, there is a cycle C containing the vertices of T, but avoiding the vertices of $S - T$. This is impossible, however, since C must contain two vertices of W. ■

3. **Proof.** Let W be a set of $k - 1$ vertices of H. We show that $H - W$ is connected. We consider two cases.

   Case 1. $w \in W$. Then $H - W = G - (W - \{w\})$. Since G is k-connected and $|W - \{w\}| = k - 2$, it follows that $G - (W - \{w\})$ is connected and so $H - W$ is connected.

   Case 2. $w \notin W$. Since G is k-connected, $G - W$ is connected. Since $|W| = k - 1$, it follows that $S - W \neq \emptyset$. Thus $H - W$ is obtained from $G - W$ by adding a new vertex w and joining w to the vertices of $S - W$. Therefore, $H - W$ is connected. ■

4. **Proof.** Construct a graph H from G by adding a new vertex v and joining v to each of $v_1, v_2, \ldots, v_t$. It is a consequence of Corollary 2.38 that H is t-connected. By Theorem 2.37, H contains t internally disjoint $u - v$ paths $Q_1, Q_2, \ldots, Q_t$. Then $Q_i - v$ is a $u - v_i$ path for each i $(1 \le i \le t)$, every two paths of which have only u in common. ■

5. **Proof.** Let G be a k-connected graph order $n \geq 2k$ and let the graph H be obtained from G by adding two vertices v_1 and v_2 and joining v_i to the vertices of V_i for $i = 1, 2$. By applying Corollary 2.38 twice, it follows that H is k-connected. Thus H contains k internally disjoint $v_1 - v_2$ paths. This produces k disjoint paths connecting V_1 and V_2.

   For the converse, let G be a graph of order $n \geq 2k$ that is not k-connected. Then G contains a vertex-cut S with $|S| = k - 1$. Let G_1 be a component of $G - S$ of minimum order n_1. If $n_1 \geq k$, let V_1 be a set of k vertices of G_1, and let V_2 be a set of k vertices of $G - S$, none of which belong to G_1. Then there do not exist k disjoint paths connecting V_1 and V_2 since all such paths must pass through S. If $n_1 < k$, then define V_1 to be the union of $V(G_1)$ and $k - n_1$ vertices of S. Let V_2 be the union of the remaining $n_1 - 1$ vertices of S and $k - n_1 + 1$ vertices of $G - S$ not belonging to G_1. Here too there do not exist k disjoint paths connecting V_1 and V_2. ■

6. **Proof.** Since G is k-connected, $\delta(G) \geq k$. By t applications of Corollary 2.38, it follows that G_t is k-connected. ■

7. **Proof.** Let $u \in S_1$ and $v \in S_2$. Then there exist k internally disjoint $u - v$ paths $P'_1, P'_2, \ldots, P'_k$ in G. Let u_i be the last vertex of P'_i belonging to S_1 and let v_i be the first vertex following u_i on P'_i that belongs to S_2. For $1 \leq i \leq k$, let P_i be the $u_i - v_i$ subpath of P'_i. Then the paths $P_1, P_2, \ldots, P_k$ have the desired properties. ■

8. **Proof.** Since G is k-connected, $G - v$ is $(k-1)$-connected, where $k - 1 \geq 2$. Therefore, by Theorem 2.41 the vertices $v_1, v_2, \cdots, v_{k-1}$ lie on a common cycle C of $G - v$. Since G is k-connected, G contains $k - 1$ internally disjoint $v - v_i$ paths Q_i $(1 \leq i \leq k - 1)$. Let u_i be the first vertex of Q_i on C and denote the $v - u_i$ subpath of Q_i by P_i. ■

9. **Proof.** Assume first that G is k-edge-connected. Thus $\lambda(G) \geq k$ and so the removal of any fewer than k edges of G results in a connected graph. Hence for every two vertices u and v, the minimum number of edges that separate u and v is at least k. By Theorem 2.42, G contains at least k pairwise edge-disjoint $u - v$ paths.

   For the converse, suppose that G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G. Hence for every pair u, v of distinct vertices of G, the maximum number of pairwise edge-disjoint $u - v$ paths in G is at least k. By Theorem 2.42, the minimum number of edges that separate u and v is at least k. Thus $\lambda(G) \geq k$ and so G is k-edge-connected. ■

10. The statement is false. Let $G = K_1 \vee 2K_2$. Then G is 2-edge-connected. Let v_1 be the cut-vertex of G, and let v and v_2 be two nonadjacent vertices of G.

11. The statement is true. **Proof.** Let $u \in S_1$ and $v \in S_2$. Then there exist k edge-disjoint $u - v$ paths $Q_1, Q_2, \cdots, Q_k$. For each $i = 1, 2, \cdots, k$, let u_i be the last vertex of Q_i that belongs to S_1 and let v_i be the first vertex of S_2 that belongs to Q_i after u_i is encountered. Let P_i be the $u_i - v_i$ subpath of Q_i. ■

12. **Proof.** Since Q_n is n-regular, $\delta(Q_n) = n$ and so $\kappa(Q_n) \leq \lambda(Q_n) \leq n$. Thus it suffices to show that $\kappa(Q_n) \geq n$. We show that for every two vertices u and v of Q_n, there are n internally disjoint $u - v$ paths. We verify this by induction. This is clearly true for $n = 1$ and $n = 2$.

Assume that every two vertices of Q_{n-1} are connected by $n-1$ internally disjoint paths, where $n-1 \geq 2$. Suppose that Q_n is constructed from two copies H and H' of Q_{n-1}, where $V(H) = \{v_1, v_2, \ldots, v_{2^{n-1}}\}$ and $V(H') = \{v'_1, v'_2, \ldots, v'_{2^{n-1}}\}$, such that $v_i v'_i \in E(Q_n)$. Let u and v be two vertices of Q_n. We consider two cases.

Case 1. *$u, v \in V(H)$ or $u, v \in V(H')$, say the former.* By the induction hypothesis, H contains $n-1$ internally disjoint $u-v$ paths $P_1, P_2, \ldots, P_{n-1}$. Let P' be a $u'-v'$ path in H'. Define $P_n = (u, P', v)$. Then $\{P_1, P_2, \ldots, P_n\}$ is a collection of n internally disjoint $u-v$ paths in Q_n.

Case 2. *$u \in V(H)$ and $v \in V(H')$.* Then $v = w'$ for some $w \in V(H)$. We consider two subcases.

Subcase 2.1. $w \neq u$. By the induction hypothesis, H contains $n-1$ internally disjoint $u-w$ paths $P'_1, P'_2, \ldots, P'_{n-1}$. Each of these paths necessarily contains at least one neighbor of w in H. Suppose that the next-to-last vertex of P'_i $(1 \leq i \leq n-1)$ is x_i. For $1 \leq i \leq n-2$, let $P_i = (P'_i - w, x'_i, w')$. Let $P_{n-1} = (P'_{n-1}, w' = v)$. Let P''_{n-1} be the $u'-w'$ path in H' corresponding to P'_{n-1} and let $P_n = (u, P''_{n-1})$. Then $\{P_1, P_2, \ldots, P_n\}$ is a collection of n internally disjoint $u-v$ paths in Q_n.

Subcase 2.2. $w = u$. Then we show that Q_n contains n internally disjoint $u-u'$ paths. Let $x \in V(H)$ such that $x \neq u$. Then H contains $n-1$ internally disjoint $u-x$ paths $P'_1, P'_2, \ldots, P'_{n-1}$. For each i with $1 \leq i \leq n-1$, let $x_i \in V(H)$ such that x_i is adjacent to x on P'_i and let Q_i be the $u-x_i$ subpath of P'_i. Now for each i with $1 \leq i \leq n-1$, let Q^*_i be the $x'_i - u$ path in H' corresponding to Q_i (by reversing the order of the vertices on Q_i). For $1 \leq i \leq n-1$, define $P_i = (Q_i, Q^*_i)$ and $P_n = (u, v)$. Then $\{P_1, P_2, \ldots, P_n\}$ is a collection of n internally disjoint $u-u'$ paths in Q_n. ■

13. In the proof of Theorem 2.42, there is a step where each edge of S is subdivided, introducing k new vertices $w_1, w_2, \ldots, w_k$, which are then identified producing a new vertex w. The resulting structure may be a multigraph (not a graph) and it need not occur that there are k pairwise edge-disjoint $u-w$ paths in the underlying graph of H and k pairwise edge-disjoint $w-v$ paths in the underlying graph of H. See Figure 2.14.

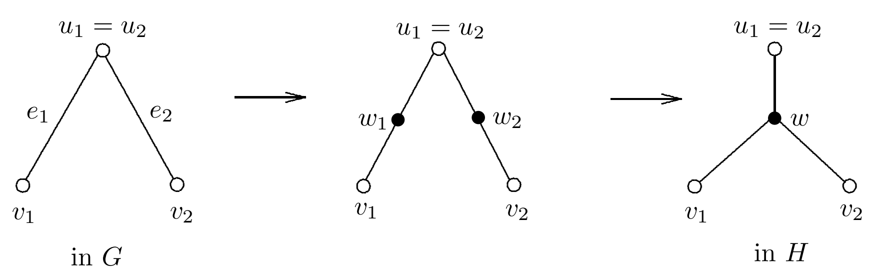


Figure 2.14: The graphs in Exercise 13

14. **Proof.** Suppose that $\kappa(G) = k$. Let u and v be two vertices of G such that $d(u, v) = \text{diam}(G)$. Since G has connectivity k, it follows that G contains k internally disjoint $u-v$ paths $P_1, P_2, \ldots, P_k$. Since $d(u, v) = \text{diam}(G)$, each path P_i $(1 \leq i \leq k)$ contains at least $\text{diam}(G) - 1$ vertices different from u and v. Thus $n \geq k(\text{diam}(G) - 1) + 2$. ■

15. **Proof.** Let u and v be two adjacent vertices of a 3-connected graph G. Then there are three internally disjoint $u-v$ paths P_1, P_2, P_3, where say the lengths of P_1 and P_2 are at least 2. Then P_1 and P_2 together with the edge uv form a chorded cycle in G. This need not be the case for a 2-connected graph. For example, the graph $G = C_n$ for $n \geq 4$ contains no chorded cycle. ■

16. (a) **Proof.** Suppose that the statement is false. Then there is a 3-connected graph G and two vertices u and v such that all paths in every set of three internally disjoint $u-v$ paths have the same length. Let P_1, P_2 and P_3 be three internally disjoint $u-v$ paths of length a, where then $a \geq 2$. Since $G - \{u, v\}$ is connected, $G - \{u, v\}$ contains an $x - y$ path P connecting two of the three paths P_1, P_2 and P_3 such that x, say, belongs to P_1, y belongs to P_2 and no other vertex of P belongs to P_1, P_2 or P_3. Suppose that the length of P is $b \geq 1$, the length of the $u - x$ subpath of P_1 is k and the length of the $u - y$ subpath of P_2 is ℓ. Then there are $u - v$ paths of lengths $k + b + a - \ell$ and $\ell + b + a - k$ internally disjoint with P_3. Hence $k + b + a - \ell = \ell + b + a - k = a$, which implies that $b = 0$, a contradiction. ■

    (b) Let u and v be any two antipodal vertices of C_n, where $n \geq 4$ is even.

17. (a) **Proof.** By Whitney's theorem (Theorem 2.37), $\kappa(G) = k$. Let S be a vertex-cut of G with $|S| = k$. Since $n \geq k + 3$, it follows that $G - S$ has at least three vertices. Since $G - S$ is disconnected, there are vertices u and v belonging to different components of $G - S$. Let $w \in V(G) - (S \cup \{u, v\})$. Then u and v belong to different components of $G - (S \cup \{w\})$ and so $S \cup \{w\}$ is a vertex-cut with $|S \cup \{w\}| = k + 1$. ■

    (b) **Proof.** Let u and v be two vertices of G with $d(u, v) = \operatorname{diam} G = k$. Since G is k-connected, there exist k internally disjoint $v - u$ paths Q_i $(i = 1, 2, \cdots, k)$. Since $d(u, v) = k$, each path Q_i has length k or more. For each i with $1 \leq i \leq k$, let v_i be a vertex on Q_i such that the $v - v_i$ subpath P_i has length i. This gives the desired result. ■