$$= ½\left[\frac{2m}{2^{k+1}}a + (2 - \frac{2m}{2^{k+1}})b\right]$$

$$= \frac{m}{2^{k+1}}a + (1 - \frac{m}{2^{k+1}})b$$

Hence, $d_r \equiv CD_r < ra + (1 - r)b$ and the theorem follows for order $k + 1$.

16. By the triangle inequality, $|d - d_n| = |CD - CD_n| \le DD_n = |AD - AD_n|$, as shown in the figure above, preceding page. Dividing by AB,

$$\frac{DD_n}{AB} = \left|\frac{AD_n}{AB} - \frac{AD}{AB}\right| = |\, p_n - p\,|$$

Hence as $n \to \infty$ and $p_n \to p$ as limit, then $DD_n \to 0$ and $|d - d_n| \to 0 \Rightarrow d = \lim d_n = \lim (p_n a + q_n b) = pa + qb$.

Section 2.1, pages 56–59

1. 90. In general, $m\angle AOB$ could be taken as the number of degrees in either of the two arcs of the circle cut off by A and B, resulting in two measures, θ and $360 - \theta$.

2. If ray OC is the ray opposite ray OA, lines OA and OB are perpendicular iff $m\angle AOB = m\angle AOC$. Proof: Assuming that $m\angle AOB + m\angle AOC = 180$, then with $m\angle AOB = m\angle AOC$, $2m\angle AOB = 180 \Rightarrow m\angle AOB = 90$.

3. 165.

4. $hk = m\angle AOB = 105$, $hu = m\angle AOC = 90$, $ku = m\angle BOC = 165 \Rightarrow$ neither (hku), (khu), nor (huk) hold.

5. If $hu = 10$.
NOTE: Answer in text incorrectly given as 70.

6. $hu = -10$ (not allowed by **Axiom 7**).

7. 105.

8. **(a)** 105
(b) 67.5.

9. Assuming the scale on the protractor is perfectly uniform, each number on the protractor is $180/179 = 1.0056$ too large and an angle-sum of 180° is measured as $180^\circ\times 1.0056 = 181.008^\circ \Rightarrow$ error $= 1.008^\circ$

10.* 179.7°: Each number on the protarctor is measured as $180^\circ/x$ too large and an angle-sum of 180.3° is measured as $180^\circ\times 180/x = 180.3^\circ \Rightarrow 32{,}400/x = 180.3 \Rightarrow x = 179.7^\circ$.

11. **(a)** 90, 210
(b) $90 + (330 - 120) = 300$; $330 - 30 = 300$ (in agreement)
(c) $m\angle AOB = 90$, $m\angle BOC = 150$, $m\angle AOC = 60$.

12. $m\angle AOB = |30 - 120| = 90$; $|120 - 330| = 210 > 180 \Rightarrow m\angle BOC = 360 - 210 = 150$; $m\angle AOC = 360 - |30 - 300| = 6$.

13. $m\angle AOB + m\angle BOC' = 90 + (210 - 120) = 90 + 90 = 180 = m\angle AOC'$.

14. If both (huk) and (hku) then $hu + uk = hk = hu - uk \Rightarrow uk = -uk \Rightarrow uk = 0 \Rightarrow u = k$, a contradiction. If ($huk$) and ($khu$), similar reasoning produces $k = h$ in contradiction.

15. The remaining betweenness relation is (ukv); $uv = hv - hu = (hk + kv) - (hk - uk) = kv + uk \Rightarrow$ (ukv).

16. If $h[0]$ and $k[x]$ $(0 < x < 180)$ are the two sides of $\angle hk$, and $h'[y]$ and $k'[z]$ are the rays opposite (by definition, the angles hh' and kk' are straight angles), then by the theorem in this section, $180 = hh' = |0 - y|$ (or $180 = 360 - |0 - y|$) $\Rightarrow$ $|y| = 180$ $\Rightarrow$ $y = 180$. Similarly, $180 = kk' = |x - z|$ (or $180 = 360 - |x - z|$) $\Rightarrow |x - z| = 180$ $\Rightarrow$ $x - z = \pm 180$ $\Rightarrow$ $z = x \mp 180$. Since all coordinates lie in the range $[0, 360)$, and $x - 180 < 0$, we conclude $z = x + 180$. Since $|y - z| = |180 - x - 180| = x < 180$, the measure of angle $h'k'$ is $|y - z| = x = hk$.

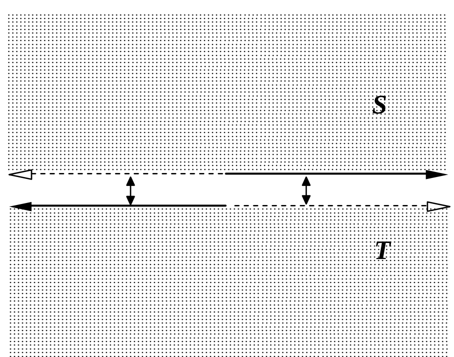
Section 2.3, pages 67–68

1. (a) Convex
(b) Not convex.

2. (a) Not convex
(b) Convex.

3. (a) Not convex
(b) Convex.

4. Yes: one possibility is if $\boldsymbol{S}$ is an open half-plane (the set of points (x, y) for $y > 0$ in the coordinate plane, for example) and $\boldsymbol{T}$ is the complementary closed half-plane. Another possibility is illustrated in the figure.



5. If $\angle ABC$ is convex and (ADC) then $D \in \overline{AC} \subseteq \angle ABC \equiv \overrightarrow{BA} \cup \overrightarrow{BC} \Rightarrow D$ is an interior point of $\overrightarrow{BA}$ or $\overrightarrow{BC}$, or both. If both, then by Theorem **3**, Section **1.8**, $\overrightarrow{BA} = \overrightarrow{BC}$ and $\angle ABC$ is degenerate. Otherwise, with A and D on line AB, then C and D lie on line AB (or BC), and D must lie on the opposite ray $\overrightarrow{BC'} \Rightarrow$ $\overrightarrow{BA} = \overrightarrow{BD} = \overrightarrow{BC'}$ and $\angle ABC$ is a straight angle.

6. *Corollary*. If A lies on line l, B lies in $\boldsymbol{H}_1$, and (ABC), where $AC < \alpha$, then C lies in $\boldsymbol{H}_1$. *Proof:* Suppose C lies on l or in $\boldsymbol{H}_2$, to obtain a contradiction. First, $C \in l \Rightarrow \overleftrightarrow{AC} = l \Rightarrow B \in l$, a contradiction. Suppose $C \in \boldsymbol{H}_2$; since B is an interior point of segment AC, by Theorem **1**, $B \in \boldsymbol{H}_2$, a contradiction. Therefore, $C \in \boldsymbol{H}_1$.

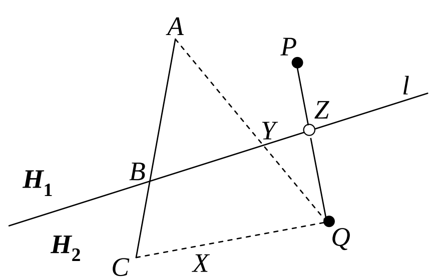
7. If A and C lie on opposite sides of line l, there exists X on l such that (AXC); since l cannot meet segment BC at an interior point (since B and C lie on the same side of l), by the Postulate of Pasch, l meets segment AB at an interior point, a contradiction since A and B lie on the same side of l, hence lie in the same half plane $\boldsymbol{H}$ determined by l, with $\overline{AB} \subset \boldsymbol{H}$. Note that this argument assumes that A and C do not lie on l. This follows implicitly by definition: if A and C lie on opposite sides of line l, they lie in opposite half planes determined by l, hence do not lie on l.

8. **NOTE:** It should have been stated in the problem that A and C do not lie on line l.
Parts (b) and (c) of the postulates are to be established; for convenience, we prove (a) and (c) first, then (b).

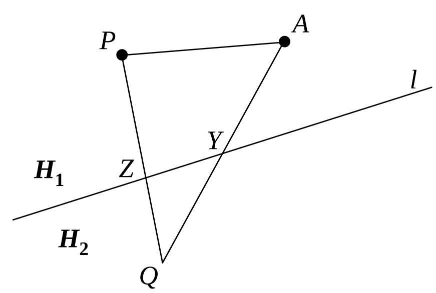
(1) Any point in the plane either lies on l or in $\boldsymbol{H}_1$ or in $\boldsymbol{H}_2$: Let P be a point not on l; then it is to be proven that $P \in \boldsymbol{H}_1$ or $P \in \boldsymbol{H}_2$. Since (ABC) and $B \in l$, by the Postulate of Pasch l passes through an interior point of segment AP or CP, not both. If the former then there exists no X on l such that $(CXP) \Rightarrow P \in \boldsymbol{H}_2$. If the latter, then no point X exists such that (AXP) and $P \in \boldsymbol{H}_1$.

(2) Let $P \in \boldsymbol{H}_1$ and $Q \in \boldsymbol{H}_2$ (see figure). Then there exists no X on l such that (CXQ); hence, since (ABC) and $B \in l$, by the Postulate of Pasch, l passes through a point Y such that (AYQ). Similarly, l passes through a point Z such that (PZQ), as was to be proven.

(3) (Convexity of $\boldsymbol{H}_1$ (and $\boldsymbol{H}_2$). It will be proven first that if $P \in \boldsymbol{H}_1$ and (PZQ) for some $Z \in l$, then $Q \in \boldsymbol{H}_2$. By definition, there exists no point $X \in l$ such that (PXA); since l passes through an interior point of PQ, it must pass through an interior point of either QA or PA, hence QA, and (QYA) for some $Y \in l \Rightarrow Q \notin \boldsymbol{H}_1$. Now consider P, Q points in $\boldsymbol{H}_1$, and R such that (PRQ). If $R \in l$ then $Q \in \boldsymbol{H}_2$ by the above; if $R \in \boldsymbol{H}_2$ then by (2) there exists $W \in l$ such that $(PWR) \Rightarrow (PWRQ) \Rightarrow (PWQ)$ and again $Q \in \boldsymbol{H}_2$, a contradiction. Therefore, $R \in \boldsymbol{H}_1$ and $\boldsymbol{H}_1$ is convex.



(2)



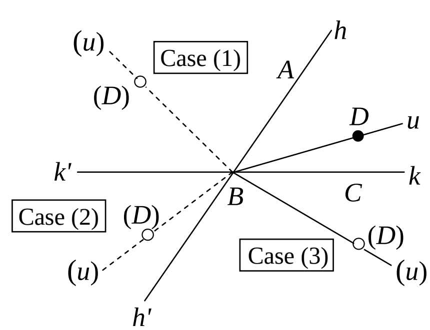
(3)

9. The reflexive and symmetry laws are often, by nature, automatically true, as they are here, by examining the definition of $\approx$. To prove the transitive law, suppose $A \approx B$ and $B \approx C$. If $A \not\approx C$ then l contains X such that $(AXC) \Rightarrow l$ passes through Y such that (AYB) (Postulate of Pasch) contradicting $A \approx B$, or through Z such that (BZC) contradicting $B \approx C$. Therefore, $A \approx C$ and $\approx$ is an equivalence relation. Now suppose there are at least three equivalence classes: That is, there exist points A, B, and C such that $A \not\approx B$, $B \not\approx C$, and $B \not\approx C$. Then l passes through an interior point of all three segments AB, BC, and AC, in denial of the Postulate of Pasch. **Axiom 0** implies there are two equivalence classes since there exists $P \notin l$, and for any $Q \in l$

(PQR) exists $\Rightarrow P \not\approx Q$. Now let $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ be the two equivalence classes. To prove the three parts (a), (b), and (c) of the plane separation postulate, we notice that (a) is trivial by letting $A \in \boldsymbol{H}_1$; with P a given point not on l, then either $P \approx A \Rightarrow P \in \boldsymbol{H}_1$, or $P \not\approx A \Rightarrow P \in \boldsymbol{H}_2$. For convexity (part (b)), let P and Q belong to the same equivalence class $\Rightarrow P \approx Q$, and suppose that (PRQ). If $R \in l$ then $P \not\approx Q$, a contradiction, so that either $R \approx P$ or $R \approx Q \Rightarrow R \in \boldsymbol{H}_1$. Finally for (c) let $P \in \boldsymbol{H}_1$ and $Q \in \boldsymbol{H}_2 \Rightarrow P \not\approx Q \Rightarrow$ there exists $X \in l$ such that (PXQ).

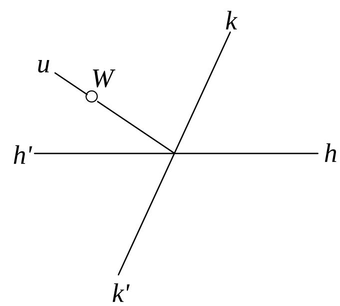
Section 2.4, pages 78–82

1. (Drawing experiment.)
2. (Drawing experiment.)
3. (Drawing experiment.)
4. Half-plane, interior of a parallel strip, the interior of a triangle, or the interior of a truncated parallel strip.
5. **(a)** 144°
(b) 141°
(c) 141°, 147°.
6. 60.
7. 44.
8. (a) 51 or 87
(b) 93 or 129
9. **(a)** 126
(b) 128
10. **NOTE:** Arc shown in figure P.10 should extend to ray AD making $m\angle CAD = 175$. With this correction, $y = 30$.
11. $(h'uk) \Rightarrow h'u + uk = h'k$; by the linear pair theorem, $h'u = 180 - hu$ and $h'k = 180 - hk \Rightarrow$ (by substitution) $(80 - hu) + uk = 180 - hk \Rightarrow -hu + uk = -hk \Rightarrow$ either $u = h$ or (ukh).
12. It must be proven that if $m\angle ABD + m\angle DBC = \angle ABC$ for rays $\overrightarrow{BA}$, $\overrightarrow{BD}$, and $\overrightarrow{BC}$, then D lies in the interior of $\angle ABC$ or is an interior point of one of its sides. In the trivial case $m\angle ABC = 0$, then both angles ABD and DBC are degenerate, and D lies interior to either ray BA or ray BC. Another trivial case is $m\angle ABC = 180 \Rightarrow \angle ABC$ is a straight angle and every point in the plane, including D, lies in the interior of $\angle ABC$. Hence let $0 < m\angle ABC < 180$; with the



notation $h = \overrightarrow{BA}$, $u = \overrightarrow{BD}$, and $k = \overrightarrow{BC}$, the hypothesis is $hu + uk = hk$. Suppose D does not lie in the interior of $\angle hk$ or on either side, to gain a contradiction. Since D is a point in the plane such that $0 < AD < \alpha$, then, as shown in the figure above (preceding page), either (1) $D \in \text{Int}\angle hk'$ or $D \in k'$, (2) $D \in \text{Int}\angle k'h'$, or $D \in h'$, or (3) $D \in \text{Int}\angle h'k$. By **Axiom 10**, case (1) implies $hu + uk' = hk'$ (if $D \in k'$ then $k' = u$) $\Rightarrow hu + (180 - uk) = 180 - hk \Rightarrow hu + hk = uk = hk - hu \Rightarrow 2hu = 0 \Rightarrow h = u$, a contradiction. Similarly, case (2) implies $k'u + uh' = h'k' \Rightarrow (180 - ku) + (180 - uh) = 180 - hk \Rightarrow 180 = ku + uh - hk = 0$, a contradiction. Finally, in case (3), $180 - hu + uk = 180 - hk \Rightarrow uk = hu - hk = -uk \Rightarrow 2uk = 0$ or $u = k$, a contradiction.

13. If either of the angles hk, hu, or uk is a straight angle, we already have either (huk), (hku), or (hku). Otherwise, $W \notin h, k, h'$, or k' since the given rays are distinct (see figure). As in Problem 12, W belongs to either $\text{Int}\angle hk$, $\text{Int}\angle h'k$, $\text{Int}\angle h'k'$, or $\text{Int}\angle k'h$. By **Axiom 10** either (huk), $(h'uk)$, $(h'uk')$, or $(k'uh)$. By the results of Problem 11, $(h'uk) \Rightarrow (ukh)$, and $(huk') \Rightarrow (khu)$. Finally, if $(h'uk')$, then by the vertical pair theorem, $h'u + uk' = h'k' = hk \Rightarrow hu' + u'k = hk$ and $(hu'k)$ holds $(hu' = 180 - hu = h'u)$.



14. By Problem 13, either (huk), (hku), (khu), or $(hu'k)$. If (huk) then $hu + uk \geq hk$. If (hku) then $hu > hk \Rightarrow hu + uk > hk$. If (khu) then $ku > hk \Rightarrow hu + uk > hk$. Finally, if $(hu'k)$ then $(180 - hu) + (180 - uk) = hk \Rightarrow hu + uk = 360 - hk \geq 180 \geq hk$.

15. The betweenness relations (huk) and (uvk) must be established (since the other two needed for $(huvk)$ are given).

(1) By the triangle inequality for rays (Problem 14),

$$hk \leq hu + uk$$

But also by the triangle inequality

$$uk \leq uv + vk$$

Add hu to both sides and use (huv) and (hvk):

$$uk + hu \leq (hu + uv) + vk = hv + vk = hk$$

proving the reverse inequality $hk \geq hu + uk$. Therefore, equality holds and (huk).

(2) Using the betweenness relations already established,

$$uv + vk = (hv - hu) + (hk - hv) = -hu + hk = uk$$

or (uvk).

16. D lies on the A-side of line BC, and on the C-side of line AB (Theorem **1**, Section **2.3**).

Therefore, $D \in \text{Int}\angle ABC$.

17. It is given that $A + B = 180$, $A + C = 90$, and $B + C = 180 \Rightarrow$ (subtracting in the first two equations) $B - C = 180 - 90 = 90$. Use this with $B + C = 180$ to obtain $B = 135$, $C = 45 \Rightarrow$ $A = 45$.

18. It is given that $A + B = 180 = A + C$ (hence $B = C$) and $B + C = 90 \Rightarrow 2B = 90 \Rightarrow B = 45 = C$ and $A = 135$.

19. It is given that $A + B = 180$, $A + C = 90$, and $B = 4C \Rightarrow A + 4C = 180$, which, with $A + C =$ 90, implies $3C = 90$ (by subtraction) $\Rightarrow C = 30$, $B = 120 \Rightarrow A = 60$.

20. If lines BD and CE are perpendicular at A, with (BAD) and (CAE), then $(\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD})$ and $(\overrightarrow{AC}\,\overrightarrow{AD}\,\overrightarrow{AE})$. But if $(\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\,\overrightarrow{AE})$, then $(\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AE}) \Rightarrow m\angle BAC + m\angle CAE = 270 =$ $m\angle BAE$, contradicting **Axiom 8**.

21. Simply let $\angle ABC$ be a straight angle. Then if $(\overrightarrow{BA}\,\overrightarrow{BC}\,\overrightarrow{BD})$ holds for some ray $\overrightarrow{BD}$, $m\angle ABD =$ $m\angle ABC + m\angle CBD > 180$, not allowed by our axioms.

22. $\angle WMT$ is a right angle.

23. By Theorem 6, $(\overrightarrow{AB}\,\overrightarrow{AD}\,\overrightarrow{AC}) \Rightarrow D \in \text{Int}\angle BAC$ (converse of **Axiom 10**) $\Rightarrow \overrightarrow{AD}$ meets $\overline{BE}$ at a point F such that (BFE) (crossbar theorem). It remains to prove (AFD). The same argument proves ray BE meets segment AD at F' such that $(AF'D)$, and we now have lines AD and BE meeting at points F' and $F \Rightarrow F' = F$ or $FF' = \alpha$. If (AFD) does not hold, then since A, B, and C are noncollinear $F \neq A, D \Rightarrow (ADF)$ since $F \in \overrightarrow{AD}$. This then implies $(AF'DF) \Rightarrow F'F < \alpha$ $\Rightarrow F' = F \Rightarrow (AFD)$.

24. By Theorem 6, (CEA) implies $(\overrightarrow{DB}\,\overrightarrow{DE}\,\overrightarrow{DA}) \Rightarrow$ (by the crossbar theorem) $\overrightarrow{AB}$ meets segment BA at F such that (AFB). It remains to prove (FED). But by Theorem 6 $(BCD) \Rightarrow$ $(\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}) \Rightarrow (FED)$, again by Theorem 6 (since F, E, and D are collinear).

25. Since angle measure is the only property that has been changed, all the axioms dealing with distance and collinearity hold (**Axioms 0–6**). **Axioms 7–8** are evidently true since every nondegenerate angle has positive measure ≤ 180. The plane separation axiom is independent of angle measure, so **Axiom 9** holds. But if $D \in \text{Int}\angle ABC$ and $\angle ABC$ is not a straight angle then since D does not lie on the sides of $\angle ABC$, $m\angle ABD < m\angle ABC$, contradicting $m\angle ABD =$ 90.

Section 2.6, pages 89–93

1. Since $m\angle GEF = m\angle ABC = 40$ and $(\overrightarrow{ED}\,\overrightarrow{EG}\,\overrightarrow{EF})$, $m\angle DEG = m\angle DEF - m\angle GEF = 135 -$ $40 = 95$.

2. If $\angle GEF$ has been constructed congruent to $\angle ABC$ and $(\overrightarrow{ED}\,\overrightarrow{EG}\,\overrightarrow{EF})$, then $m\angle DEG =$ $m\angle DEF - \theta$.

3. **(a)** 360 – (150 + 110) = 100

(b) One assumes W is not interior to $\angle RST$ from the figure, hence ray SW is not between rays SR and ST. Also, ray SR is not between rays SW and ST (since 110 + 150 > 180), and ray ST is not between rays SW and SR (since 150 > 110). By Theorem **7**, Section **2.4**, ray ST' lies between rays SR and $SW \Rightarrow m\angle WST = 180 - m\angle WST' = 180 - (m\angle WSR - m\angle T'SR) = 180 -$ $110 + (180 - m\angle TSR) = 70 + (180 - 150) = 100$.

4. (a) $115 - 65 = 50$

(b) $115 - 90 = 25$

(c) $90 - 65 = 25$

(d) $m\angle KPW = m\angle KPT + m\angle TPW \Rightarrow (\overrightarrow{PK}\,\overrightarrow{PT}\,\overrightarrow{PW})$; $m\angle KPT = m\angle TPW$, so by definition, ray PT is the bisector of $\angle KPW$.

5. (a) $135 - 64 = 71$

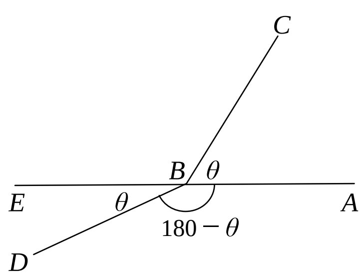
(b) $180 - 92 = 88$

(c) Let x be that coordinate. If the bisector is ray OX, then $64 < x < 135$, and $\angle DOX \cong \angle XDB \Rightarrow 135 - x = x - 64$ or $x = 99.5$. (To actually prove $64 < x < 135$, it can be shown from the coordinatization theorem for rays that if $x < 0$ then ray OX lies on the opposite side of line OA as OB, contradicting $(\overrightarrow{OB}\,\overrightarrow{OX}\,\overrightarrow{OD})$ from the definition of angle bisector $\Rightarrow$ $x > 0$; if $x < 64$ then $64 - x = m\angle BOX = m\angle XOD = 135 - x$ (impossible), and if $x > 135$ then $x - 135 = x - 64$ (impossible) $\Rightarrow 64 < x < 135$.)

6. $hk = |-179 + 61| = 118$, $hu = 360 - |-179 - 5| = 176$, $hv = 360 - |-179 - 120| = 61$, $ku = |-61 - 5| = 66$, $kv = 360 - |-61 - 120| = 179$, and $uv = |-5 - 120| = 115$.

7. Construct point C such that $(C'BC)$; thus C' lies on the D-side of line $AB \Rightarrow m\angle ABC' = 180 - m\angle ABC = m\angle ABD$ (linear pair theorem). According to **Axiom 11** there can be only one such ray BD on the D-side of line AB and rays BD and BC' coincide.

8. (See figure.) By the linear pair theorem, $m\angle ABD = 180 - m\angle DBE = 180 - m\angle ABC$ so that $\angle ABD$ and $\angle ABC$ are supplementary. By Problem 7, ray BD coincides with the opposite ray BC' of BC. Since (CBD), angles ABC and DBE are vertical angles.



NOTE: The label for the angle measure in Figure P.7 should be $180 - \theta$, not $\theta - 180$.

9. Construct $\angle HEF \cong \angle ABC$ such that $(\overrightarrow{EG}\,\overrightarrow{EH}\,\overrightarrow{EF}) \Rightarrow$ ray EH meets segment DF at G at an interior point of ray EH (crossbar theorem) $\Rightarrow \angle GEF \cong \angle HEF \cong \angle ABC$.

10. Let l be a line passing through A and C. Thus, either (ACB), (ABC), (BAC) or (AB^*C). Since $AB = \alpha$ the second and third cases cannot occur. Also, $B^* = A \Rightarrow (AB^*C)$ cannot hold. Therefore, (ACB).

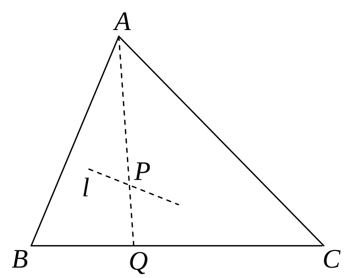
11. Let $\boldsymbol{H}$ be either half-plane determined by line l and containing points A and B; let $C \in l$. If $AB = \alpha$ then A and B are extreme points, and by Corollary **C**, Section **2.6**, (ACB). But this puts A and B on opposite sides of line l, a contradiction.

12. *Definition.* Given $\angle ABC \equiv \angle hk$ (nondegenerate) the *trisectors* of $\angle hk$ are the rays u and v such that $(huvk)$ and $hu = uv = vk$. Proof that the trisectors exist: Let h have coordinate 0 and k positive coordinate $\theta \le 180$. Define u and v those rays concurrent with h and k having coordinates $\frac{1}{3}\theta$ and $\frac{2}{3}\theta$. Then $hu = |0 - \frac{1}{3}\theta| = \frac{1}{3}\theta$, $uv = |\frac{1}{3}\theta - \frac{2}{3}\theta| = \frac{1}{3}\theta$, and $vk = |\theta - \frac{2}{3}\theta| = \frac{1}{3}\theta$. Thus, $hu = uv = vk$, and $hu + uv + vk = \theta = hk \Rightarrow (huvk)$. [$(huv)$ holds since $hu + uv = \frac{2}{3}\theta = |0 - \frac{2}{3}\theta| = hv$, and (hvk) holds in the same manner $\Rightarrow (huvk)$.]

13. Let the rays from B be coordinatized such that $\overrightarrow{BA} \equiv h$ has coordinate 0 and $\overrightarrow{BC} \equiv k$ has coordinate $\theta > 0$; since $\angle BAC$ is acute, $\theta < 90$. Thus $2\theta < 180$ and there exists a unique ray $\overrightarrow{BD} \equiv u$ having coordinate $2\theta \Rightarrow (hku)$ and $hu = 2hk$. That is, $(\overrightarrow{BD}\,\overrightarrow{BC}\,\overrightarrow{BA})$ holds and $m\angle DBA = 2m\angle ABC$.

14. The interior of a triangle is convex by the theorem of Section **2.2**. As stated in the problem, we must show that $X \in \text{Int}\angle A \cap \text{Int}\angle B$ implies that $X \in \text{Int}\angle C$). But if $X \in \text{Int}\angle A \cap \text{Int}\angle B$ then $X \in \text{Int}\angle BAC$ and X lies on the C-side of line AB and on the B-side of line AC; also, $X \in \text{Int}\angle ABC \Rightarrow X$ lies on the A-side of line $BC \Rightarrow X \in \text{Int}\angle ACB$. Hence, $\text{Int}\triangle ABC = \text{Int}\angle A \cap \text{Int}\angle B$.

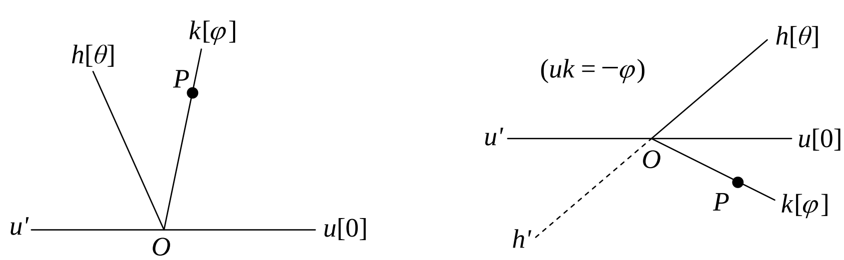
15. Let $P \in \text{Int}\angle BAC$. By definition ray AP meets side BC at an interior point Q (crossbar theorem); if l passes through A, Q, or C we are finished because l would then meet $\triangle ABC$ precisely at the pairs of points (A, Q), (B, R), or (C, S) where R and S are the points $\overrightarrow{BP} \cap \overline{AC}$ or $\overrightarrow{CP} \cap \overline{AB}$. Since $P \in \text{Int}\angle ABC$, $(\overrightarrow{BA}\,\overrightarrow{BP}\,\overrightarrow{BC}) \Rightarrow (APB)$ by Theorem **6**, Section **2.4**. By the Postulate of Pasch, l meets either side AC or side QC (hence BC) $\Rightarrow l$ meets $\triangle ABC$ in at least two points; the Postulate of Pasch then shows that there cannot be more than two points of intersection.



16. **(a)** **Axioms 7−8** allow this

(b) For each ray from O a unique real number (coordinate) in the range $(-180, 180]$ has been assigned to it. Two different rays cannot have the same coordinate by **Axiom 11**, and two different real numbers cannot be assigned to the same ray by the method used.

(c) (See figure.) (The case $\theta > \varphi$ and $|\theta - \varphi| \le 180$.) First, suppose $\varphi > 0$, and let P be an interior point of k. Then P lies on the same side of line $u \cup u'$ as h, since k does, and either $P \in \text{Int}\angle uh$ or $P \in \text{Int}\angle hu' \Rightarrow$ either (hku) or (hku'). If (hku) then $hk = hu - ku = \theta - \varphi$; if (hku') then $hk = hu' - ku' = (180 - hu) - (180 - ku) = ku - hu = \varphi - \theta$. Thus $hk = |\theta - \varphi|$ in either case. Next, suppose $\varphi < 0$. Then h and k lie on opposite sides of line $u \cup u' \Rightarrow$ (as

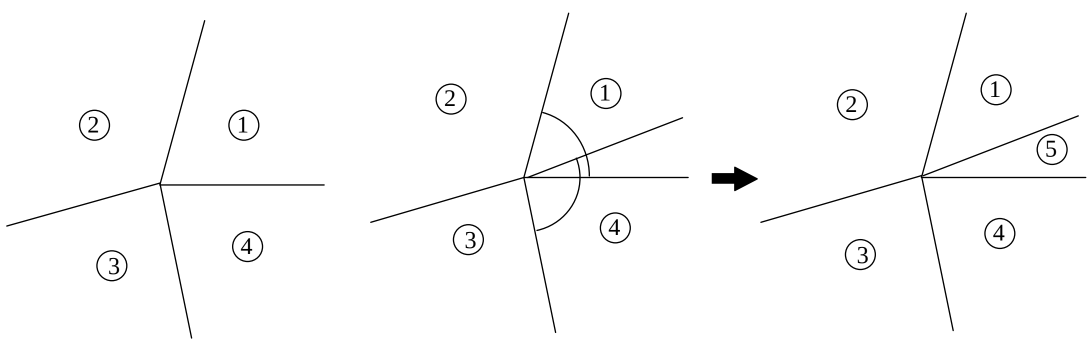


before) either (ukh') or ($h'ku'$); we show that ($h'ku'$) is impossible. If ($h'ku'$) $\Rightarrow$ $(180 - hk) + (180 - ku) = h'u' = hu \Rightarrow 360 - hk = hu + ku = \theta - \varphi = |\theta - \varphi| \le 180 \Rightarrow$ $360 - hk \le 180$, or $hk \ge 180 \Rightarrow k = h'$, a contradiction. Then (ukh') and we have $kh' = uh' - uk \Rightarrow 180 - hk = (180 - \theta) + \varphi \Rightarrow hk = \theta - \varphi = |\theta - \varphi|$. In all cases, $hk = |\theta - \varphi|$.

(d) (The case $\theta > \varphi$ and $|\theta - \varphi| > 180$.) It follows that $|\theta - \varphi| = \theta - \varphi$ and $\varphi < 0$. Once again, k lies on the opposite side of $u \cup u'$ as h and either (ukh') or ($h'ku'$); this time, (ukh') is impossible, since ultimately this implies the contradiction $hk = hu + uk = \theta - \varphi > 180$. Hence ($h'ku'$) holds and $hu = h'u' = h'k + ku' \Rightarrow hu = 180 - hk + 180 - ku \Rightarrow hk = 360 - hu - ku = 360 - \theta + \varphi = 360 - |\theta - \varphi|$.

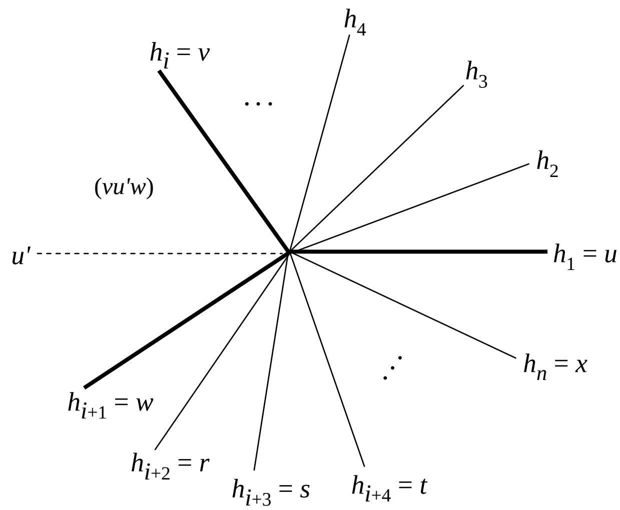
17. By the result of Problem 13, Section **2.4**, either (huk), (hku), (khu), or ($hu'k$). The first three cases are not allowed because the angles $\angle hu$, $\angle hk$, and $\angle uk$ are pairwise adjacent, having disjoint interiors. Hence, ($hu'k$) and $hk = hu' + u'k = (180 - hu) + (180 - ku) \Rightarrow hk + ku + uh = 360$.

18. Since Problem 17 established this for 3 angles, let the number of angles be $n \ge 4$, with consecutive sides $h_1, h_2, h_3, \cdots, h_n$ having origin O, such that h_i is the common side of adjacent angles $\angle h_{i-1}h_i$ and $\angle h_ih_{i+1}$ for each $i = 1, 2, 3, \cdots, n$. [Note that this notation is to be taken cyclically, with $h_0 = h_n$ and $h_{n+1} = h_1$, etc.] Note that the problem implies that we count the *angles* rather than the rays (as in $\angle 1, \angle 2, \angle 3, \cdots, \angle n$), where $\angle 1$ and $\angle 2$, $\angle 2$ and $\angle 3$, etc. are adjacent angles; if two angles overlap, as shown, then one merely changes the notation, as indicated. Thus the extra assumption made under this interpretation, in terms of the previous notation, is that no ray h_j lies in the interior of any angle $\angle h_{i-1}h_i$. This also applies to $h_{n+1} \equiv h_1$ and $\angle h_{i-1}h_i$ for $i \le n$: h_1 cannot lie in the interior of angle $\angle h_{i-1}h_i$. It must



be proven that $m\angle 1 + m\angle 2 + m\angle 3 + \cdots + m\angle n = 360$. Although this result seems trivial enough, proving it logically from the axioms is another matter. The trick is to obtain three rays $u = h_i$, $v = h_j$, and $w = h_k$ such that $i < j < k$ where u, v, and w are the sides of three adjacent angles, then apply the result of Problem 17. This can be done by letting $u = h_1$ and taking $v = h_i$ as the last ray (greatest i) that lies on the h_2-side of the line $h_1 \cup h_1'$, and taking $w = h_{i+1}$. (Not all the rays can lie on one side of l without contradicting the fact that $\angle h_{n-1}h_n$ and $\angle h_nh_1$ are adjacent angles. We also observe that no two consecutive rays can form a straight angle.) Now all the remaining rays lie either between u and v, or between w and u. Moreover, the angle measures are additive since betweenness is determined by index order (this is due to the observation above concerning the subscript rule involving h_j.) For example, if $i > 5$, then $(h_1h_2h_3)$ is true because $(h_1h_3h_2)$ would violate that rule, as would $(h_1h_4h_3)$ $(h_1h_5h_4)$, $\cdots$. Thus, we have the following betweenness relations (see figure below):

$$\begin{aligned}
(h_1h_2h_3) \quad &\Rightarrow \quad h_1h_3 = h_1h_2 + h_2h_3 \\
(h_1h_3h_4) \quad &\Rightarrow \quad h_1h_4 = h_1h_3 + h_3h_4 = h_1h_2 + h_2h_3 + h_4h_5 \\
(h_1h_4h_5) \quad &\Rightarrow \quad h_1h_5 = h_1h_4 + h_4h_5 = h_1h_2 + h_2h_3 + h_3h_4 + h_4h_5
\end{aligned}$$



and so on. The final result is:

$$uv = h_1h_i = \sum_{j=1}^{j=i-1} h_jh_{j+1}$$

Similarly for the angles on the opposite side (using the notation of the figure temporarily)

$$\begin{aligned}
(h_{i+1}h_{i+2}h_{i+3}) \equiv (wrs) \quad &\Rightarrow \quad ws = wr + rs \\
(h_{i+1}h_{i+3}h_{i+4}) \equiv (wst) \quad &\Rightarrow \quad wt = ws + st = wr + rs + st \\
(h_{i+1}h_{i+4}h_n) \equiv (wtx) \quad &\Rightarrow \quad wx = wt + tx = wr + rs + st + tx
\end{aligned}$$

Thus

$$\sum_{j=i+1}^{j=n} h_jh_{j+1} = h_1h_i = wu$$

The final step is to use the result established in Problem 17:

$$\sum_{j=1}^{n} h_jh_{j+1} = \sum_{j=1}^{i-1} h_jh_{j+1} + h_ih_{i+1} + \sum_{j=i+1}^{n} h_jh_{j+1} = uv + vw + wu = 360$$

Section 3.1, pages 99–100

1. $\overline{RS} \cong \overline{ST}$, $\overline{ST} \cong \overline{TR}$, $\overline{RT} \cong \overline{SR}$, $\angle R \cong \angle S$, $\angle S \cong \angle T$, and $\angle T \cong \angle R$; this triangle would be an equilateral, equiangular triangle.