Chapter 2 Complex Analysis, Differential Equations and Laplace Transformation

Problem Set 2.1

1. (a) $\dfrac{2+j}{1-2j}=j$, (b) $|j|=1$ and $\dfrac{|2+j|}{|1-2j|}=\dfrac{\sqrt{5}}{\sqrt{5}}=1$

2. (a) $\dfrac{-3-j}{2j}\cdot\dfrac{-j}{-j}=\dfrac{-1+3j}{2}=\dfrac{-1}{2}+\dfrac{3}{2}j$

(b) $\left|\dfrac{-1}{2}+\dfrac{3}{2}j\right|=\sqrt{\left(\dfrac{-1}{2}\right)^2+\left(\dfrac{3}{2}\right)^2}=\sqrt{\dfrac{10}{4}}=\dfrac{\sqrt{10}}{2}$ and $\dfrac{|-3-j|}{|2j|}=\dfrac{\sqrt{10}}{2}$

(c)

```
>> abs(-1/2+(3/2)*j)

ans =
    1.5811

>> abs(-3-j)/abs(2*j)

ans =
    1.5811
```

3. (a) $\dfrac{-3j}{2+3j}=\dfrac{-9}{13}-\dfrac{6}{13}j$, (b) $\left|\dfrac{-9}{13}-\dfrac{6}{13}j\right|=\sqrt{\dfrac{117}{169}}=\dfrac{3\sqrt{13}}{13}$ and $\dfrac{|-3j|}{|2+3j|}=\dfrac{3}{\sqrt{13}}=\dfrac{3\sqrt{13}}{13}$

4. (a) $\dfrac{4}{-4+3j}\cdot\dfrac{-4-3j}{-4-3j}=\dfrac{-16-12j}{25}=\dfrac{-16}{25}-\dfrac{12}{25}j$

(b) $\left|\dfrac{-16}{25}-\dfrac{12}{25}j\right|=\sqrt{\left(\dfrac{-16}{25}\right)^2+\left(\dfrac{-12}{25}\right)^2}=\sqrt{\dfrac{400}{625}}=\dfrac{4}{5}$ and $\dfrac{|4|}{|-4+3j|}=\dfrac{4}{5}$

(c)

```
>> abs(-16/25*j-12/25)

ans =
    0.8000
```

```
>> abs(4)/abs(-4+3*j)

ans =
    0.8000
```

5. $-1+\frac{1}{2}j \overset{\text{2nd quadrant}}{=} \frac{\sqrt{5}}{2}e^{2.6779j}$

6. $1-\frac{3}{2}j \overset{\text{4th quadrant}}{=} \frac{\sqrt{13}}{2}e^{-0.9828j}$

7. $3+j\sqrt{3} \overset{\text{1st quadrant}}{=} 2\sqrt{3}\ e^{j\pi/6}$

8. To calculate phase, we first find $\tan^{-1}\sqrt{3}=\frac{1}{3}\pi$. But $-\sqrt{3}-3j$ is located in the 3rd quadrant. So, the phase is taken as either $\pi+\frac{1}{3}\pi$ in the positive sense (counterclockwise) or $\frac{1}{2}\pi+\frac{1}{6}\pi=\frac{2}{3}\pi$ in the negative (clockwise). In summary, $-\sqrt{3}-3j \overset{\text{3rd quadrant}}{=} 2\sqrt{3}\ e^{-(2\pi/3)j}$.

9. $\frac{1+\frac{2}{3}j}{-\frac{1}{3}+j}=\frac{\frac{\sqrt{13}}{3}e^{0.5880j}}{\frac{\sqrt{10}}{3}e^{1.8925j}}=\frac{\sqrt{13}}{\sqrt{10}}e^{-1.3045j}=0.3-1.1j$

10. $\frac{3-j\sqrt{3}}{\sqrt{3}+3j}=\frac{2\sqrt{3}\ e^{-(\pi/6)j}}{2\sqrt{3}\ e^{(\pi/3)j}}=e^{-(\pi/2)j}=-j$

11. $\frac{3-5j}{2j}=\frac{\sqrt{34}e^{-1.0304j}}{2e^{(\pi/2)j}}=\frac{\sqrt{34}}{2}e^{-2.6012j}=-2.5-1.5j$

12. $\frac{-3j}{-1+j}=\frac{3e^{-(\pi/2)j}}{\sqrt{2}e^{(3\pi/4)j}}=\frac{3\sqrt{2}}{2}e^{-(5\pi/4)j}=-1.5+1.5j$

13. $\left(\frac{1}{3}-j\right)^4=\left(\frac{\sqrt{10}}{3}e^{-1.2490j}\right)^4=\frac{100}{81}e^{-4.9960j}=0.3455+1.1852j$

14. $(-2-j)^5 \overset{\text{3rd quadrant}}{=} \left(\sqrt{5}e^{-2.6779j}\right)^5=25\sqrt{5}e^{-13.3895j}=38-41j$

15. $\dfrac{(1+3j)^3}{(-1+2j)^2}=\dfrac{\left(\sqrt{10}\ e^{1.2490j}\right)^3}{\left(\sqrt{5}\ e^{2.0344j}\right)^2}=\dfrac{10\sqrt{10}\ e^{3.7470j}}{5e^{4.0688j}}=2\sqrt{10}e^{-0.3218j}=6-2j$

16. $\dfrac{-100j}{(1+4j)^3}=\dfrac{100e^{-(\pi/2)j}}{\left(\sqrt{17}e^{1.3258j}\right)^3}=\dfrac{100}{17\sqrt{17}}e^{-5.5482j}=1.0584+0.9567j$

17. $(1+j)^{1/4}=\sqrt{2}^{1/4}\left[\cos\dfrac{\frac{1}{4}\pi+2k\pi}{4}+j\sin\dfrac{\frac{1}{4}\pi+2k\pi}{4}\right],\ k=0,1,2,3$

$1.0696+0.2127j\ ,\ \ -0.2127+1.0696j\ ,\ \ -1.0696-0.2127j\ ,\ \ 0.2127-1.0696j$

18. $(-1+j)^{1/3}=\sqrt{2}^{1/3}\left[\cos\dfrac{\frac{3}{4}\pi+2k\pi}{3}+j\sin\dfrac{\frac{3}{4}\pi+2k\pi}{3}\right],\ k=0,1,2$

$0.7937+0.7937j\ ,\ \ -1.0842+0.2905j\ ,\ \ 0.2905-1.0842j$

19. $(\sqrt{3}-3j)^{1/2}=(2\sqrt{3})^{1/2}\left[\cos\dfrac{-\frac{1}{3}\pi+2k\pi}{2}+j\sin\dfrac{-\frac{1}{3}\pi+2k\pi}{2}\right],\ k=0,1$

$1.6119-0.9306j\ ,\ \ -1.6119+0.9306j$

20. $(2j)^{1/2}=2^{1/2}\left[\cos\dfrac{\frac{1}{2}\pi+2k\pi}{2}+j\sin\dfrac{\frac{1}{2}\pi+2k\pi}{2}\right],\ k=0,1\quad\Rightarrow\quad 1+j\ ,\ \ -1-j$

Problem Set 2.2

1. (*a*) $x(t)=\frac{1}{2}(\sin t-\cos t)$

   (*b*)

```
>> x = dsolve('Dx + x = sin(t)','x(0)= -1/2')

x =

-1/2*cos(t)+1/2*sin(t)
```

2. (*a*) Rewrite the ODE in standard form $\dot{x}+2x=0$ so that $g=2$ and $h=\int 2dt=2t$. Then,

$$x(t)=e^{-2t}\left[\int e^{2t}0\ dt+c\right]=ce^{-2t}$$

By the initial condition $x(0) = c = \frac{1}{2}$. Therefore $x(t) = \frac{1}{2}e^{-2t}$.

(b)

```
>> x = dsolve('(1/2)*Dx + x = 0','x(0) = 1/2')

x =

1/2*exp(-2*t)
```

3. (a) $x(t) = 1 - e^{-t^2/2}$

(b)

```
>> x = dsolve('Dx + t*x = t','x(0)=0')

x =

1-exp(-1/2*t^2)
```

4. (a) $g = 1$ and $h = \int dt = t$. A general solution is then given by

$$y(t) = e^{-t}\left[\int e^t \cos t \; dt + c\right] \overset{\text{Appendix B}}{=} e^{-t}\left[\tfrac{1}{2}e^t(\cos t + \sin t) + c\right] = \tfrac{1}{2}(\cos t + \sin t) + ce^{-t}$$

By the initial condition we have $c = \frac{1}{2}$ and therefore $y(t) = \frac{1}{2}(e^{-t} + \cos t + \sin t)$.

(b)

```
>> y = dsolve('Dy + y = cos(t)','y(0) = 1')

y =

1/2*cos(t)+1/2*sin(t)+1/2*exp(-t)
```

5. (a) Characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ so that $\lambda = -1 \pm j$. Because the ODE is homogeneous, we have

$$x = x_h \overset{\text{Case (3)}}{=} e^{-t}(A\cos t + B\sin t)$$

Applying the initial conditions we find $A = 0$, $B = 1$. Therefore, $x(t) = e^{-t}\sin t$.

(b)

```
>> x = dsolve('D2x + 2*Dx + 2*x = 0','x(0)=0','Dx(0)=1')
```

```
x =
exp(-t)*sin(t)
```

6. (*a*) Characteristic values are $\lambda = -1, -1$ so that $x_h \overset{\text{Case (2)}}{=} (c_1 + c_2 t)e^{-t}$. Based on $f(t) = e^{-2t}$ we pick $x_p = Ke^{-2t}$, no special case, and insert into the ODE to get $K = 1$ hence $x_p = e^{-2t}$. A general solution is $x(t) = (c_1 + c_2 t)e^{-t} + e^{-2t}$. Initial conditions yield $c_1 = 0, c_2 = 1$, and the solution is $x(t) = te^{-t} + e^{-2t}$.

   (*b*)

```
>> x = dsolve('D2x + 2*Dx + x = exp(-2*t)','x(0)=1,Dx(0)=-1')

x =

exp(-t)*t+exp(-2*t)
```

7. (*a*) Characteristic values are $\lambda = 0, -3$ hence $x_h \overset{\text{Case (1)}}{=} c_1 + c_2 e^{-3t}$. Pick $x_p = A\cos t + B\sin t$ and insert into the original ODE to find $(-A + 3B)\cos t + (-B - 3A)\sin t \equiv 10\sin t$. This implies $A = -3$, $B = -1$ so that a general solution is $x = c_1 + c_2 e^{-3t} - 3\cos t - \sin t$. Using the initial conditions, we find $c_1 = 3$, $c_2 = 0$ and thus $x(t) = 3 - \sin t - 3\cos t$.

   (*b*)

```
>> x = dsolve('D2x + 3*Dx = 10*sin(t)','x(0)=0','Dx(0)=-1')

x =

-sin(t)-3*cos(t)+3
```

8. (*a*) Characteristic values are $\lambda = \pm j$ so that $x_h \overset{\text{Case (3)}}{=} c_1 \cos t + c_2 \sin t$. Based on the nature of $f(t)$ we pick $x_p = A\cos 2t + B\sin 2t$, no special case, and insert into the ODE to get $A = 0$, $B = -1$ hence $x_p = -\sin 2t$. A general solution is $x(t) = c_1 \cos t + c_2 \sin t - \sin 2t$. By initial conditions, $c_1 = 1, c_2 = 2$, and the solution is $x(t) = \cos t + 2\sin t - \sin 2t$.

   (*b*)

```
> x = dsolve('D2x + x = 3*sin(2*t)','x(0)=1,Dx(0)=0')

x =

2*sin(t)+cos(t)-sin(2*t)
```

9. (a) $u(t) = (t+1)e^{-t}$

10. (a) Characteristic values are $\lambda = -1, -\frac{1}{2}$ so that $y_h \overset{\text{Case (1)}}{=} c_1 e^{-t/2} + c_2 e^{-t}$. Since the ODE is homogeneous, $y(t) = y_h(t)$. By initial conditions, $c_1 = 1, c_2 = -1$, and the solution is $y(t) = e^{-t/2} - e^{-t}$.

(b)

```
>> y = dsolve('2*D2y + 3*Dy + y = 0','y(0)=0,Dy(0)=1/2')

y =

-exp(-t)+exp(-1/2*t)
```

11. Write $\cos t + 2\sin t = D\sin(t+\phi) = D\sin t\cos\phi + D\cos t\sin\phi$ and compare the two sides to find

$$\begin{matrix} D\sin\phi = 1 \\ D\cos\phi = 2 \end{matrix} \quad \overset{D=\sqrt{5}}{\Rightarrow} \quad \begin{matrix} \sin\phi > 0 \\ \cos\phi > 0 \end{matrix} \quad \Rightarrow \quad \tan\phi = \frac{1}{2} \quad \overset{\text{1st quadrant}}{\Rightarrow} \quad \phi = 0.4636 \text{ rad}$$

Therefore, $\cos t + 2\sin t = \sqrt{5}\sin(t + 0.4636)$.

12. Write $\cos 2t - 3\sin 2t = D\sin(2t+\phi) = D\sin 2t\cos\phi + D\cos 2t\sin\phi$. Comparing the two sides,

$$\begin{matrix} D\sin\phi = 1 \\ D\cos\phi = -3 \end{matrix} \quad \overset{D=\sqrt{10}}{\Rightarrow} \quad \begin{matrix} \sin\phi > 0 \\ \cos\phi < 0 \end{matrix} \quad \Rightarrow \quad \tan\phi = -\frac{1}{3} \quad \overset{\text{2nd quadrant}}{\Rightarrow} \quad \phi = 2.8198 \text{ rad}$$

Therefore $\cos 2t - 3\sin 2t = \sqrt{10}\sin(2t + 2.8198)$.

13. Write $-\cos 2t - \frac{1}{2}\sin 2t = D\sin(2t+\phi) = D\sin 2t\cos\phi + D\cos 2t\sin\phi$. Comparing the two sides,

$$\begin{matrix} D\sin\phi = -1 \\ D\cos\phi = -\frac{1}{2} \end{matrix} \quad \overset{D=\frac{1}{2}\sqrt{5}}{\Rightarrow} \quad \begin{matrix} \sin\phi < 0 \\ \cos\phi < 0 \end{matrix} \quad \Rightarrow \quad \tan\phi = 2 \quad \overset{\text{3rd quadrant}}{\Rightarrow} \quad \phi = -2.0344 \text{ rad}$$

Therefore, $\frac{1}{2}\sqrt{5}\sin(2t - 2.0344)$.

14. Expand $\sin\omega t - \frac{1}{3}\cos\omega t = D\sin(\omega t+\phi) = D\sin\omega t\cos\phi + D\cos\omega t\sin\phi$. Comparison gives

$$\begin{array}{l} D\sin\phi = -\frac{1}{3} \\ D\cos\phi = 1 \end{array} \quad \overset{D=\frac{1}{3}\sqrt{10}}{\Rightarrow} \quad \begin{array}{l} \sin\phi < 0 \\ \cos\phi > 0 \end{array} \quad \Rightarrow \quad \tan\phi = -\frac{1}{3} \quad \overset{\text{4th quadrant}}{\Rightarrow} \quad \phi = -0.3218 \text{ rad}$$

Therefore $\sin\omega t - \frac{1}{3}\cos\omega t = \frac{1}{3}\sqrt{10}\sin(\omega t - 0.3218)$.

Problem Set 2.3

1. (*a*) $\mathcal{L}\left\{e^{-at+b}\right\} = \mathcal{L}\left\{e^{-at}e^{b}\right\} \overset{\text{linearity}}{=} \mathcal{L}\left\{e^{-at}\right\}e^{b} = \dfrac{e^{b}}{s+a}$

(*b*)

```
>> syms a b t
>> laplace(exp(-a*t+b))

ans =

exp(b)/(s+a)
```

2. (*a*) $\mathcal{L}\left\{e^{jat}\right\} = \dfrac{1}{s-ja}$. To convert to rectangular form, multiply and divide by the conjugate of the denominator, that is, $\dfrac{1}{s-ja}\cdot\dfrac{s+ja}{s+ja} = \dfrac{s+ja}{s^2+a^2}$.

(*b*)

```
>> syms a t
>> L = laplace(exp(j*a*t))

L =

1/(s-i*a)
```

3. (*a*)

$$\begin{aligned} \mathcal{L}\left\{\sin(\omega t-\phi)\right\} &= \mathcal{L}\left\{\sin\omega t\cos\phi - \cos\omega t\sin\phi\right\} = \mathcal{L}\left\{\sin\omega t\right\}\cos\phi - \mathcal{L}\left\{\cos\omega t\right\}\sin\phi \\ &= \frac{\omega}{s^2+\omega^2}\cos\phi - \frac{s}{s^2+\omega^2}\sin\phi = \frac{\omega\cos\phi - s\sin\phi}{s^2+\omega^2} \end{aligned}$$

(*b*)

```
>> syms w t p
>> laplace(sin(w*t-p))
```

```
ans =

-(-cos(p)*w+sin(p)*s)/(s^2+w^2)
```

4. (a) We first expand the expression as $\cos(\omega t+\phi)=\cos\omega t\cos\phi-\sin\omega t\sin\phi$. Then, using the linearity of the Laplace transform, we find

$$\mathscr{L}\{\cos(\omega t+\phi)\}=\mathscr{L}\{\cos\omega t\}\cos\phi-\mathscr{L}\{\sin\omega t\}\sin\phi$$
$$=\frac{s}{s^2+\omega^2}\cos\phi-\frac{\omega}{s^2+\omega^2}\sin\phi=\frac{s\cos\phi-\omega\sin\phi}{s^2+\omega^2}$$

(b)

```
>> syms w p t
>> laplace(cos(w*t+p))

ans =

1/(s^2+w^2)*(cos(p)*s-sin(p)*w)

>> pretty(ans)
                              cos(p) s - sin(p) w
                              -------------------
                                     2     2
                                    s   + w
```

5. (a) $g(t)=u(t)-2u(t-1)+u(t-2)$

(b) $G(s)=\dfrac{1-2e^{-s}+e^{-2s}}{s}=\dfrac{(1-e^{-s})^2}{s}$

6. (a) We construct $g(t)$ using the strategy outlined in Figure PS2-3 No6, resulting in

$$g(t)=tu(t)-tu(t-1)$$

(b) We will take the Laplace transform term-by-term. For the 1st term $\mathscr{L}\{tu(t)\}=\dfrac{1}{s^2}$. For the 2nd term, comparison with Eq. (2.18) reveals $f(t-1)=t$, which in turn gives $f(t)=t+1$. Therefore $F(s)=\dfrac{1}{s^2}+\dfrac{1}{s}$ and $\mathscr{L}\{tu(t-1)\}=\left(\dfrac{1}{s^2}+\dfrac{1}{s}\right)e^{-s}$. Combining the two results,

$$G(s)=\frac{1}{s^2}-\left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s}$$

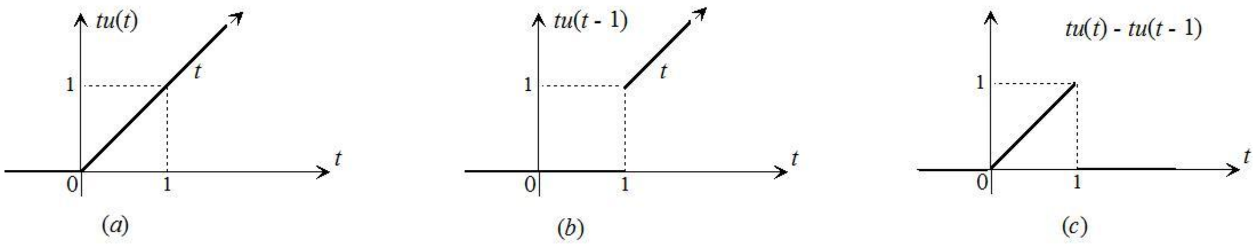

Figure PS2-3 No6

7. (a) $h(t) = tu(t) - tu(t-1) + u(t-1) \overset{\text{simplify}}{=} tu(t) - (t-1)u(t-1)$

(b) $H(s) = \dfrac{1}{s^2} - \dfrac{e^{-s}}{s^2} = \dfrac{1-e^{-s}}{s^2}$

8. (a) Construct $h(t)$ using the strategy shown in Figure PS2-3 No8, leading to

$$h(t) = (1-t)u(t) - (1-t)u(t-1)$$

(b) Rewrite the expression obtained in (a) as $h(t) = u(t) - tu(t) + (t-1)u(t-1)$. Then

$$H(s) = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}$$

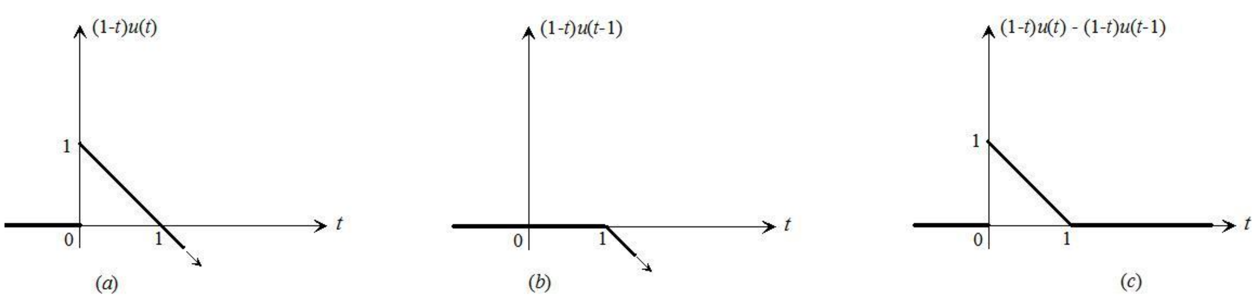

Figure PS2-3 No8

9. $F(s) = \dfrac{(1-e^{-s})^2}{s(1-e^{-2s})} \overset{\text{simplify}}{=} \dfrac{1-e^{-s}}{s(1+e^{-s})}$

10. The period is $P = 1$ so that

$$F(s)=\frac{\int_0^1 e^{-st}f(t)\,dt}{1-e^{-s}} \overset{f(t)=t}{=} \frac{\int_0^1 e^{-st}t\,dt}{1-e^{-s}} \overset{\text{integration by parts}}{=} \frac{\frac{1-e^{-s}-se^{-s}}{s^2}}{1-e^{-s}}=\frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-s})}$$

11. $G(s)=\dfrac{1-e^{-s}-se^{-2s}}{s^2(1-e^{-2s})}$

12. The period is $P=2$ and

$$G(s)=\frac{\int_0^1 -te^{-st}dt+\int_1^2 (2-t)e^{-st}dt}{1-e^{-2s}} \overset{\text{integration by parts}}{=} \frac{e^{-2s}+2se^{-s}-1}{s^2(1-e^{-2s})}$$

13. (*a*) Expand as

$$\frac{3s+2}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}=\frac{(A+B)s+A}{s(s+1)} \quad\Rightarrow\quad \begin{matrix} A+B=3 \\ A=2 \end{matrix} \quad\Rightarrow\quad \begin{matrix} A=2 \\ B=1 \end{matrix}$$

Therefore,

$$\frac{3s+2}{s(s+1)}=\frac{2}{s}+\frac{1}{s+1} \overset{\mathscr{L}^{-1}}{\Rightarrow} 2+e^{-t}$$

(*b*)

```
>> syms s
>> ilaplace((3*s+2)/s/(s+1))

ans =

exp(-t)+2
```

14. (*a*) Partial-fraction expansion leads to

$$\frac{3s^2+s+5}{(s^2+1)(s+2)}=\frac{As+B}{s^2+1}+\frac{C}{s+2}=\frac{(A+C)s^2+Bs+2B+C}{(s^2+1)(s+2)} \quad\Rightarrow\quad \begin{matrix} A+C=3 \\ B=1 \\ 2B+C=5 \end{matrix}$$

Simultaneous solution gives $A=0, B=1, C=3$ so that

$$\frac{3s^2+s+5}{(s^2+1)(s+2)}=\frac{1}{s^2+1}+\frac{3}{s+2}\quad\overset{\mathscr{L}^{-1}}{\Rightarrow}\quad \sin t+3e^{-2t}$$

(*b*)

```
>> syms s
>> ilaplace((3*s^2+s+5)/(s^2+1)/(s+2))

ans =

sin(t)+3*exp(-2*t)
```

15. (*a*) Using partial fractions,

$$\frac{s+5}{s(s^2+2s+5)}=\frac{A}{s}+\frac{Bs+C}{s^2+2s+5}=\frac{(A+B)s^2+(2A+C)s+5A}{s(s^2+2s+5)}$$

Subsequently,

$$\begin{array}{c} A+B=0 \\ 2A+C=1 \\ 5A=5 \end{array}\quad\overset{\text{solve}}{\Rightarrow}\quad\begin{array}{c} A=1 \\ B=-1 \\ C=-1 \end{array}$$

Then

$$\frac{s+5}{s(s^2+2s+5)}=\frac{1}{s}-\frac{s+1}{(s+1)^2+2^2}\quad\overset{\mathscr{L}^{-1}}{\Rightarrow}\quad 1-e^{-t}\cos 2t$$

(*b*)

```
>> syms s
>> ilaplace((s+5)/s/(s^2+2*s+5))

ans =

-exp(-t)*cos(2*t)+1
```

16. (*a*) Forming partial fractions,

$$\frac{4s+5}{s^2(s^2+4s+5)}=\frac{A}{s^2}+\frac{B}{s}+\frac{Cs+D}{s^2+4s+5}=\frac{(B+C)s^3+(A+4B+D)s^2+(4A+5B)s+5A}{s^2(s^2+4s+5)}$$

Then

$$\begin{array}{r} B+C=0 \\ A+4B+D=0 \\ 4A+5B=4 \\ 5A=5 \end{array} \quad \overset{\text{solve}}{\Rightarrow} \quad \begin{array}{r} A=1 \\ B=0 \\ C=0 \\ D=-1 \end{array}$$

Therefore

$$\frac{4s+5}{s^2(s^2+4s+5)} = \frac{1}{s^2} - \frac{1}{(s+2)^2+1^2} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad t - e^{-2t}\sin t$$

(*b*)

```
>> syms s
>> ilaplace((4*s+5)/s^2/(s^2+4*s+5))

ans =

t-exp(-2*t)*sin(t)
```

17. (*a*) $e^{-3t} - te^{-t}$

18. (*a*) Partial-fraction expansion gives

$$\frac{s^2+s-1}{(s+3)(s^2+2s+2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+2} = \frac{(A+B)s^2+(2A+3B+C)s+2A+3C}{(s+3)(s^2+2s+2)}$$

Comparing coefficients of like powers of s results in

$$\begin{array}{r} A+B=1 \\ 2A+3B+C=1 \\ 2A+3C=-1 \end{array} \quad \Rightarrow \quad \begin{array}{r} A=1 \\ B=0 \\ C=-1 \end{array}$$

Finally,

$$\frac{s^2+s-1}{(s+3)(s^2+2s+2)} = \frac{1}{s+3} - \frac{1}{(s+1)^2+1^2} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad e^{-3t} - e^{-t}\sin t$$

(*b*)

```
>> syms s
>> ilaplace((s^2+s-1)/(s+3)/(s^2+2*s+2))

ans =

-exp(-t)*sin(t)+exp(-3*t)
```

19. (a) $e^{-t}(\cos t - 2\sin t)$

20. (a) Taking the Laplace transform of the ODE, taking into account the initial conditions,

$$[s^2 X(s) - s] + 2[sX(s) - 1] + 2X(s) = \frac{1}{s} \quad \overset{\text{Solve for } X(s)}{\Rightarrow} \quad X(s) = \frac{1}{s(s^2+2s+2)} + \frac{s+2}{s^2+2s+2}$$

Term-by-term inverse Laplace transformation (via partial fractions) leads to

$$x(t) = \underbrace{\tfrac{1}{2} - \tfrac{1}{2}e^{-t}(\cos t + \sin t)}_{\text{First term}} + \underbrace{e^{-t}(\cos t + \sin t)}_{\text{Second term}} = \tfrac{1}{2}e^{-t}(\cos t + \sin t) + \tfrac{1}{2}$$

(b)

```
>> x = dsolve('D2x + 2*Dx + 2*x = 1','x(0)=1,Dx(0)=0')

x =

1/2*exp(-t)*sin(t)+1/2*exp(-t)*cos(t)+1/2
```

21. (a) $x(t) = e^{-t} - 3e^{-t/2} + 2$

22. (a) Laplace transform of the ODE and using the initial conditions, we find

$$s^2 X(s) - s + 4X(s) = \frac{1}{s^2+1} \quad \Rightarrow \quad X(s) = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$$

Then $x(t) = \underbrace{-\tfrac{1}{6}\sin 2t + \tfrac{1}{3}\sin t}_{\text{First term}} + \underbrace{\cos 2t}_{\text{2nd term}}$.

(b)

```
>> x = dsolve('D2x + 4*x = sin(t)','x(0)=1,Dx(0)=0')

x =

-1/6*sin(2*t)+1/3*sin(t)+cos(2*t)
```

23. (a) $y(t) = te^{-t} + e^{-2t}$

24. (a) Laplace transformation and solving for $Y(s)$ leads to

$$Y(s) = \frac{9}{s^3(s+3)} + \frac{1}{s(s+3)} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad y(t) = \tfrac{3}{2}t^2 - \tfrac{2}{3}e^{-3t} - t + \tfrac{2}{3}$$

(*b*)

```
>> y = dsolve('D2y+3*Dy=9*t','y(0)=0,Dy(0)=1')

y =

3/2*t^2-2/3*exp(-3*t)-t+2/3
```

25. Poles are at $0, -2$ hence FVT applies. $x_{ss} = \frac{1}{2}$.

26. Poles are at $-3, -2 \pm j$ so that FVT applies. Then

$$x_{ss} = \lim_{s \to 0} \frac{s(2s+3)}{(s+3)(s^2+4s+5)} = 0$$

27. Poles of $X(s)$ are $0, 0, -1, -2$ hence FVT does not apply!

28. Poles of $X(s)$ are at $-1, -1, -2$ so that FVT applies. Then

$$x_{ss} = \lim_{s \to 0} \frac{s(s+3)}{(s+1)^2(s+2)} = 0$$

29. $x(0^+) = \frac{1}{2}$

30. $x(0^+) = \lim_{s \to \infty} \dfrac{s(3s+2)}{(s+1)(s+2)^2} = 0$

Review Problems

1. $\dfrac{33+56j}{169} = 0.1953 + 0.3314j$

2. $\dfrac{4+j}{2j(3-2j)} = \dfrac{4+j}{4+6j}\left(\dfrac{4-6j}{4-6j}\right) = \dfrac{22-20j}{52} = \dfrac{11-10j}{26} = 0.4231 - 0.3846j$

3. $(2-3j)^3 = \left[\sqrt{13}\ e^{-0.9828j}\right]^3 = 13\sqrt{13}\ e^{-2.9484j} = -46 - 9j$

4. $\dfrac{(1+2j)^3}{(-2+j)^4} = \dfrac{\left[\sqrt{5}\ e^{1.1071j}\right]^3}{\left[\sqrt{5}\ e^{2.6779j}\right]^4} = \dfrac{1}{\sqrt{5}} e^{-7.3903j} = 0.2 - 0.4j$

5. Let $z = -1 - \frac{1}{2}j = \sqrt{\frac{5}{4}}\, e^{-2.6779j}$ so that $r = \frac{\sqrt{5}}{2}$ and $\theta = -2.6779$ rad. With $n = 3$, Eq. (2.9) yields

$$\sqrt[3]{z} = \sqrt[3]{r}\left[\cos\frac{\theta + 2k\pi}{3} + j\sin\frac{\theta + 2k\pi}{3}\right] \quad , \quad k = 0,1,2$$

Substituting for r and θ, we find the three roots as

$$0.6511 - 0.8082j \quad , \quad 0.3744 + 0.9680j \quad , \quad -1.0255 - 0.1598j$$

6. Rewrite in standard form $\dot{y} + \frac{2}{3}y = \frac{1}{3}t$ so that by Eq. (2.12),

$$y(t) = e^{-2t/3}\left[\int e^{2t/3}\, \tfrac{1}{3}t\, dt + c\right] = e^{-2t/3}\left[\tfrac{1}{2}te^{2t/3} - \tfrac{3}{4}e^{2t/3} + c\right] = \tfrac{1}{2}t - \tfrac{3}{4} + ce^{-2t/3}$$

Initial condition gives $y(0) = -\frac{3}{4} + c = \frac{1}{4}$ so that $c = 1$. Therefore, $y(t) = \frac{1}{2}t - \frac{3}{4} + e^{-2t/3}$.

7. The characteristic equation $\lambda^2 + 3\lambda = 0$ yields $\lambda = 0, -3$ so that $x_h(t) = A + Be^{-3t}$. Because the function on the right side of the ODE matches a homogeneous solution, pick $x_p = kte^{-3t}$ and insert into the ODE to find $-3K = 10$, hence $K = -\frac{10}{3}$ and $x_p = -\frac{10}{3}te^{-3t}$. Therefore

$$x(t) = A + Be^{-3t} - \tfrac{10}{3}te^{-3t}$$

Using the initial conditions, we find $A = 1$ and $B = 0$, and $x(t) = 1 - \frac{10}{3}te^{-3t}$.

8. (*a*) The characteristic equation $4\lambda^2 + 4\lambda + 5 = (2\lambda + 1)^2 + 2^2 = 0$ yields $\lambda = -\frac{1}{2} \pm j$ so that $x_h(t) = e^{-t/2}(A\cos t + B\sin t)$. The particular solution is $x_p(t) = C + Dt$. Substitute into the original ODE to get $4D + 5C + 5Dt \equiv -1 + 5t$, which implies $C = -1$, $D = 1$, therefore $x_p(t) = t - 1$ and

$$x(t) = e^{-t/2}(A\cos t + B\sin t) + t - 1$$

By the initial conditions, we have $A = 0 = B$, and the solution is $x(t) = t - 1$.

(*b*)

```
>> x=dsolve('4*D2x+4*Dx+5*x=5*t-1','x(0)=-1,Dx(0)=1')

x =

-1+t
```

9. Expand $D\cos(t+\phi) = D\cos t\cos\phi - D\sin t\sin\phi$. Comparing with $\cos t - 2\sin t$, we find

$$\begin{aligned} D\cos\phi &= 1 \\ D\sin\phi &= 2 \end{aligned} \quad \overset{\phi \text{ is in the 1st quadrant}}{\Rightarrow} \quad \tan\phi = 2 \quad \Rightarrow \quad \phi = 1.1071 \text{ rad}$$

The amplitude is $D = \sqrt{1^2+2^2} = \sqrt{5}$. Therefore $\cos t - 2\sin t = \sqrt{5}\cos(t+1.1071)$.

10. Noting $\mathscr{L}\left\{e^{at}\right\} = 1/(s-a)$, we find

$$\mathscr{L}\left\{e^{(2-3j)t}\right\} = \frac{1}{s-(2-3j)} = \frac{1}{s-2+3j}\left(\frac{s-2-3j}{s-2-3j}\right) = \frac{s-2-3j}{(s-2)^2+9} = \frac{s-2}{(s-2)^2+9} - \frac{3j}{(s-2)^2+9}$$

11. (*a*) $g(t) = 2tu(t) - 2tu(t-1) + u(t-1) - u(t-2)$

(*b*) $G(s) = \dfrac{2}{s^2} - \dfrac{2}{s^2}e^{-s} - \dfrac{1}{s}e^{-s} - \dfrac{1}{s}e^{-2s}$. Note that $\mathscr{L}\left\{tu(t-1)\right\} = \left(\dfrac{1}{s^2} + \dfrac{1}{s}\right)e^{-s}$.

12. (*a*) $h(t) = (-t+2)u(t) - (-t+2)u(t-1) + u(t-1) - u(t-2)$

(*b*) The second term is treated as follows. Rewrite

$$(t-2)u(t-1) = (t-1)u(t-1) - u(t-1)$$

so that

$$\mathscr{L}\left\{(t-2)u(t-1)\right\} = \mathscr{L}\left\{(t-1)u(t-1)\right\} - \mathscr{L}\left\{u(t-1)\right\} = \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}$$

Then,

$$H(s) = -\frac{1}{s^2} + \frac{2}{s} + \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s} + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s} \overset{\text{simplify}}{=} -\frac{1}{s^2} + \frac{2}{s} + \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-2s}$$

13. (*a*) Partial-fraction expansion results in

$$\frac{2s^2+1}{s^2(4s^2+1)} \equiv \frac{1}{s^2} - \frac{\frac{1}{2}}{s^2+\frac{1}{4}}$$

Therefore $\mathscr{L}^{-1}\left\{\dfrac{2s^2+1}{s^2(4s^2+1)}\right\} = t - \sin(t/2)$.

(*b*)

```
>> syms s
>> ilaplace((2*s^2+1)/s^2/(4*s^2+1))

ans =

t-sin(1/2*t)
```

14. (*a*) Partial fractions yield

$$\frac{\frac{1}{2}s^2+s-2}{(s+1)(s^2+4)} \equiv \frac{A}{s+1}+\frac{Bs+C}{s^2+4}=\frac{(A+B)s^2+(B+C)s+4A+C}{(s+1)(s^2+4)}$$

Equating the coefficients of like powers of s, we find

$$\begin{cases} A+B=\frac{1}{2} \\ B+C=1 \\ 4A+C=-2 \end{cases} \quad \Rightarrow \quad \begin{matrix} A=-\frac{1}{2} \\ B=1 \\ C=0 \end{matrix}$$

Therefore,

$$\mathscr{L}^{-1}\left\{\frac{\frac{1}{2}s^2+s-2}{(s+1)(s^2+4)}\right\}=\mathscr{L}^{-1}\left\{\frac{-\frac{1}{2}}{s+1}+\frac{s}{s^2+4}\right\}=-\frac{1}{2}e^{-t}+\cos 2t$$

(*b*)

```
>> syms s
>> ilaplace((1/2*s^2+s-2)/(s+1)/(s^2+4))

ans =

-1/2*exp(-t)+cos(2*t)
```

15. (*a*) Take the Laplace transform and solve for $X(s)$ to obtain

$$X(s)=\frac{1}{s(s^2+1)}e^{-s}$$

Let $F(s)=\dfrac{1}{s(s^2+1)}$ so that $X(s)=F(s)e^{-s}$. Also $f(t)=\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\}=1-\cos t$. Using Eq. (2.18) in the opposite direction, we have

$$x(t)=\mathscr{L}^{-1}\left\{F(s)e^{-s}\right\}=f(t-1)u(t-1)=\left[1-\cos(t-1)\right]u(t-1)=\begin{cases} 0 & \text{if} \quad t<1 \\ 1-\cos(t-1) & \text{if} \quad t>1 \end{cases}$$

(*b*)

```
>> dsolve('D2x+x=heaviside(t-1)','x(0)=0,Dx(0)=0')

ans =

-heaviside(-1+t)*(-1+cos(-1+t))
```

16. (*a*) Take the Laplace transform and take into account the initial conditions to find

$$(s^2+3s)X(s)=1 \quad \Rightarrow \quad X(s)=\frac{1}{s(s+3)} \overset{\text{partial fractions}}{\equiv} \frac{A}{s}+\frac{B}{S+3} \quad \Rightarrow \quad \begin{array}{l} A=\frac{1}{3} \\ B=-\frac{1}{3} \end{array}$$

Finally, $x(t)=\frac{1}{3}-\frac{1}{3}e^{-3t}$.

(*b*)

```
>> dsolve('D2x+3*Dx=0','x(0)=0,Dx(0)=1')

ans =

1/3-1/3*exp(-3*t)
```

17. (*a*) Poles are at $0,-1,-1$ so that FVT is applicable:

$$x_{ss}=\lim_{s\to 0}\left\{sX(s)\right\}=\lim_{s\to 0}\frac{1}{(s+1)^2}=1$$

(*b*) Since $x(t)=1-(t+1)e^{-t}$, we find $\lim\limits_{t\to\infty} x(t)=1$, confirming the earlier result.

18. $x(0^+)=\lim\limits_{s\to\infty}\left\{sX(s)\right\}=\lim\limits_{s\to\infty}\frac{s^2}{2(s^2+s+1)}=\frac{1}{2}$