Chapter 2 Permutations and Combinations

2.2.1B A Personal Identification Number (PIN) consists of a sequence of four digits, each drawn from the set $\{0,1,2,3,4,5,6,7,8,9\}$, except that the first digit of a PIN cannot be 0. How many different PINs are there? How many different PINs are there in which no digit is repeated?

Solution: The numbers of the choices for the digits are, respectively, 9, 10, 10, 10, so there are altogether $9\times10\times10\times10=9000$ different PINs. If we are not allowed to repeat digits, the numbers of the choices are 9, 9, 8, 7, respectively. So there are $9\times9\times8\times7=4536$ PINs in which no digit is repeated.

2.2.2B i) How many sequences are there of n digits in which all the digits are different? ii) How many sequences are there of n digits in which no two consecutive digits are the same?

Solution: i) Here we are not excluding 0 for the first digit. As there are only 10 different digits, for $n>10$ there are no sequences of n different digits. For $n\le10$ the numbers of choices for the n successive digits are 10, 9, 8, …., $11-n$, so there are

$10\times9\times\ldots\times11-n=\dfrac{10!}{(10-n)!}$ sequences of n different digits in this case.

ii) In this case there are 10 possibilities for the first digit, and each subsequent digit can be any of the 10 other than the one preceding it, making 9 choices. So there are altogether $10\times9\times9..\times9=10(9^{n-1})$ sequences in which no two consecutive digits are the same.

2.2.3B A password is a sequence of six characters the first three being either an upper- or lowercase letter, the next being a digit, and the final two coming from the set $\{!,£,\$,\%,\hat{},\&,*,(,),\_,+,=,\{,\},[,],@,\#,?\}$ of 19 other symbols occurring on a standard keyboard. How many different passwords are there? How many are there if consecutive characters must be different? How many are there if all the characters must be different?

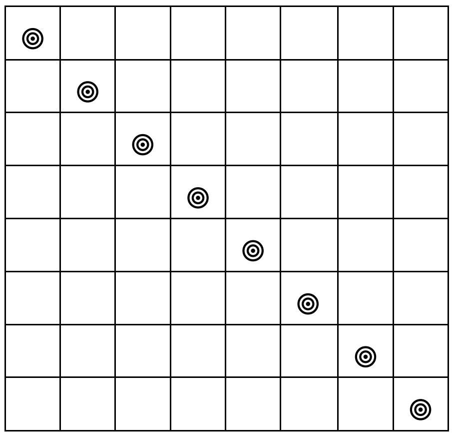
Solution: The first three symbols can each be chosen in 52 ways, the fourth in 10 ways, and the final two can each be chosen in 19 ways. So there are $52\times52\times52\times10\times19\times19=507594880$ different passwords. If consecutive characters must be different, we have $52\times51\times51\times10\times19\times18$ $=462561840$ different passwords. If all the characters must be different, we have

$52\times51\times50\times10\times19\times18$ $=453492000$ different passwords.

2.2.4B In how many different ways may eight red and eight green counters be placed on the squares of an 8×8 chessboard so that there are not two counters on any one square, and there is one red counter and one green counter in each row and column?

Note: This question is rather out of place here, as the simplest method requires ideas derived from later chapters. However, it could be set at this stage as a very challenging problem, as our direct solution shows.

Solution: The eight red counters may be placed on the board so that there is one red counter in each row and column in 8! ways (see Problem 2.3). We now have to calculate, for each of these arrangements, how many ways there are to place eight green counters on the remaining squares with one green counter in each row and column. It should clear that, by permuting the rows, and the columns, we need only consider the case where the red counters are on the main diagonal, as shown in the board, B, below (see Theorem 17.2, if you think this needs proof).



B

The number of ways in which the green counters can be placed on the board with no two in the same row or column is also 8!, but we need to exclude the cases where some of them are lying on top of the red counters. In just one of these cases all 8 green counters are on top of red counters. In no case will there be exactly 7 green counters on top of red counters, since then the 8th green count would have to be on top of the 8th red counter. Now suppose n of the green counters are on top of red counters with $1 \le n \le 6$. There are $C(8, n)$ ways in which the positions of these green counters may be chosen. Then the remaining $8 - n$ green counters can be placed in the exactly the number of ways in which $8 - n$ red counters and $8 - n$ green counters may be placed on an $(8 - n) \times (8 - n)$ board, so that the red counters lie on the diagonal and none of the green counters is on the diagonal.

Thus, if we let a_k be the number of ways of placing *k* red and *k* green counters on a $k \times k$ board, with all the red counters, and none of green counters on the diagonal, and with no two reds in the same row or column, and no two greens in the same row or column, we have that

$$a_8 = 8! - 1 - 0 - C(8,6)a_2 - C(8,5)a_3 - C(8,4)a_4 - C(8,3)a_5 - C(8,2)a_6 \,. \qquad (1)$$

Clearly $a_2 = 1$, as there is just one way to put 2 green counters on a 2×2 board so they are not on the diagonal and they are not in the same row or column, as shown below.

◎	✪
✪	◎

◎ = red counter; ✪ = green counter.

Now by a similar argument to that used to deduce (1), we have that

$a_3 = 3! - 1 - 0 - C(3,1)a_2 = 6 - 1 - 3 = 2$,

$a_4 = 4! - 1 - 0 - C(4,2)a_2 - C(4,1)a_3 = 24 - 1 - 6 \times 1 - 4 \times 2 = 9$,

$a_5 = 5! - 1 - 0 - C(5,3)a_2 - C(5,2)a_3 - C(5,1)a_4 = 120 - 1 - 0 - 10 \times 1 - 10 \times 2 - 5 \times 9 = 44$,

$a_6 = 6! - 1 - 0 - C(6,4)a_2 - C(6,3)a_3 - C(6,2)a_4 - C(6,1)a_5$

$\quad = 720 - 1 - 0 - 15 \times 1 - 20 \times 2 - 15 \times 9 - 6 \times 44 = 265$,

$a_7 = 7! - 1 - 0 - C(7,5)a_2 - C(7,4)a_3 - C(7,3)a_4 - C(7,2)a_5 - C(7,1)a_6$

$\quad = 5040 - 1 - 0 - 21 \times 1 - 35 \times 2 - 35 \times 9 - 21 \times 44 - 7 \times 265 = 1854$, and (at last),

$a_8 = 8! - 1 - 0 - C(8,6)a_2 - C(8,5)a_3 - C(8,4)a_4 - C(8,3)a_5 - C(8,2)a_6 - C(8,1)a_7$

$= 40320 - 1 - 28 \times 1 - 56 \times 2 - 70 \times 9 - 56 \times 44 - 28 \times 265 - 8 \times 1854 = 14833$.

[After reading Chapter 4, it should become evident that this is the same as the number of derangements of 8 objects, that is, $8!\sum_{k=0}^{8} \frac{(-1)^k}{k!} = 14833$.]

This gives the number of ways of placing one green counter in each row and column, for each placing of 8 red counters with one in each row and column. So the total number of ways to place the counters to meet the required conditions is $8! \times 14833 = 598066560$.

2.3.1B A cricket squad consists of six batsmen, eight bowlers, three wicketkeepers and four all-rounders. The selectors wish to pick a team made up of four batsmen, four bowlers, one wicketkeeper and two all-rounders. How many different teams can they pick?

Solution: The number of different teams is

$C(6,4)\times C(8,4)\times C(3,1)\times C(4,2)=15\times 70\times 3\times 6=18900$.

2.3.2B Prove that, for each positive integer *n*, $\sum_{r=0}^{n} C(n,r)^2 = C(2n,n)$.

Solution: Let *X* be a set of $2n$ elements. We partition *X* into disjoint subsets, say *Y*,*Z*, each containing *n* elements. The choice of n elements from *X* corresponds to choosing, for some *r*, with $0\le r\le n$, *r* elements from *Y* and $n-r$ elements from *Z*. Therefore, as

$C(n,n-r)=C(n,r)$, $C(2n,n)=\sum_{r=0}^{n} C(n,r)C(n,n-r)=\sum_{r=0}^{n} C(n,r)^2$.

2.3.3B Let *X* be a finite set. Prove that the number of subsets of *X* that contain an even number of elements is equal to the number of subsets of *X* that contain an odd number of elements. Deduce that for each positive integer *n*, $\sum_{r=0}^{n} (-1)^r C(n,r)=0$.

Solution: Suppose *X* is a set of *n* elements, say $X=\{x_1,x_2,\ldots,x_{n-1,}x_n\}$. We have seen in the solution to Exercise 2.3.2A, that *X* has altogether 2^n subsets. If we wish to select a subset of *X* containing an even number of elements we have a free choice, for $1\le i\le n-1$, whether or not to include x_i. However, when it comes to x_n we no longer have a choice; if we have already selected an even number of elements, we must exclude x_n, and if we have already selected an odd number of elements, x_n must be included. So the number of ways we can choose a subset of *X* containing an even number of elements is $2\times 2\times\ldots\times 2$,with $n-1$ factors, that is, $2^{n-1}=\frac{1}{2}(2^n)$. Thus *X* has 2^{n-1} subsets containing an even number of elements and the same number of subsets containing an odd number of elements.

Therefore $\sum_{\substack{n \text{ even} \\ 0\le n\le r}} C(n,r)=\sum_{\substack{n \text{ odd} \\ 0\le n\le r}} C(n,r)$, and so $\sum_{r=0}^{n} (-1)^r C(n,r)=0$.

2.3.4B Prove that for each positive integer *n,* $\sum_{r=0}^{n} r^2 C(n,r)=n(n+1)2^{n-2}$

Solution: Let *X* be a set of *n* elements. We count the number of ordered pairs (A,B) where *A* is a two-element subset of *X* and *B* is a subset of $X\setminus A$ in two ways. First, for $2\le r\le n$, let Y_r be the set of all such ordered pairs where $\#(A)+\#(B)=r$. We obtain a pair $(A,B) \in Y_r$ by first choosing an r-element subset, say X', of X and then choosing a two element subset A of X', and finally putting $B = X' \setminus A$. X' may be chosen in $C(n,r)$ ways, and then A in $C(r,2) = \frac{1}{2}r(r-1)$ ways. Hence $\#(Y_r) = \frac{1}{2}r(r-1)C(n,r)$. Thus the total number of such ordered pairs (A,B) is $\sum_{r=2}^{n} \frac{1}{2}r(r-1)C(n,r)$, and this is the same as $\sum_{r=0}^{n} \frac{1}{2}r(r-1)C(n,r)$. On the other hand, we can choose such an ordered pair by first choosing a two-element subset, say A, of X, and then any subset B of the $(n-2)$ - element set $X \setminus A$, and this pair of choices may be made in $C(n,2) \times 2^{n-2} = \frac{1}{2}n(n-1)2^{n-2}$ ways. We thus deduce that $\sum_{r=0}^{n} \frac{1}{2}r(r-1)C(n,r) = \frac{1}{2}n(n-1)2^{n-2}$, and hence

$$\sum_{r=0}^{n} r(r-1)C(n,r) = n(n-1)2^{n-2}. \tag{1}$$

By the result of Exercise 2.3.4A ,

$$\sum_{r=0}^{n} rC(n,r) = n2^{n-1} \tag{2}$$

Adding (1) and (2) then gives

$$\sum_{r=0}^{n} r^2 C(n,r) = n(n+1)2^{n-2}.$$

2.4.1B In his autobiography, *What I Remember*, Adolphus Trollope describes the Italian lottery as follows:

> "Ninety numbers, 1-90, are always put into the wheel. Five only of these are drawn out. The player bets that a number named by him shall be one of these (*semplice estratto*); or that it shall be the first drawn (*estratto determinato*); or that two numbers named by him shall be two of the five drawn (*ambo*); or that three so named shall be drawn (*terno*). It will be seen, therefore, that the winner of an *estratto determinato*, ought, if the play were quite even, to receive ninety times his stake. But, in fact, such a player would receive only seventy-five times his stake, the profit of the Government consisting of this pull of

fifteen per ninety against the player. Of course, what he ought to receive in any of the other cases is easily (not by me, but by experts) calculable."

What would be fair odds for the *semplice estratto*, *ambo* and *terno* bets?

Solution: There are $C(90,5)$ different sets of 5 numbers that may be chosen from the numbers 1-90. For $1 \le r \le 3$, the number of these subsets containing *r* specified numbers, is the number of ways of choosing $5-r$ numbers from the remaining $90-r$, that is, $C(90-r,5-r)$. Hence the probability that the 5 drawn numbers include the *r* that you have specified is $\dfrac{C(90-r,5-r)}{C(90,5)}$. We thus obtain the following probabilities:

$$\textit{semplice estratto:}\ \frac{C(89,4)}{C(90,5)}=\frac{1}{18};\quad \textit{ambo:}\ \frac{C(88,3)}{C(90,5)}=\frac{2}{801};\quad \textit{terno:}\ \frac{C(87,2)}{C(90,5)}=\frac{1}{11748},$$

and so fair odds are 17:1, 799:2 and 11747:1, respectively.

2.4.2B A bag contains 2*n* red balls and 2*n* blue balls. What is the probability that if 2*n* balls are drawn at random, the sample will consist of *n* red balls and *n* blue balls?

Solution: The bag contains $4n$ balls altogether. So $2n$ balls can be drawn from the bag in $C(4n,2n)$ ways. There are $C(2n,n)\times C(2n,n)$ ways of choosing *n* red balls and *n* blue balls from this bag. Hence the required probability is

$$\frac{C(2n,n)\times C(2n,n)}{C(4n,2n)}=\frac{((2n)!)^2}{(n!)^4}\Big/\frac{(4n)!}{((2n)!)^2}=\frac{((2n)!)^4}{(n!)^4(4n)!}.$$

[Using Stirling's formula, namely that $n!$ is asymptotic to $\sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$, it can be seen that this probability is asymptotic to $\sqrt{\dfrac{2}{\pi n}}$.]

2.4.3B Suppose there are 2*n* balls in a bag of which *a* are red and *b* are blue, where $a+b=2n$. One ball is removed at random from the bag, and then replaced. Then a second ball is drawn at random from the bag and then replaced. Calculate the probability that either a red ball is drawn twice or a blue ball is drawn twice. Show that this probability is a minimum when $a=b=n$.

Solution: Since the sampling is done with replacement, there are $(2n)^2=4n^2$ ways in which 2 balls can be drawn from the $2n$ balls. Two red balls can be drawn in a^2 ways and two blue balls in b^2 ways. So the probability of getting either two red or two blue balls is $\frac{a^2+b^2}{4n^2}$. Since $b = 2n - a$, we have $\frac{a^2+b^2}{4n^2} = \frac{a^2+(2n-a)^2}{4n^2} = \frac{n^2+(n-a)^2}{2n^2}$.

For a given value of *n* this is a minimum when $n - a = 0$, that is, when $a = n = b$.

2.4.4B Suppose that in a given bridge deal North and South between them have nine spades. What is the probability that the remaining four spades in the other two hands are divided two-two?

Solution: Between them, East and West have 26 cards of which 4 are spades. The 13 cards held by, say, East, can be chosen in $C(26,13)$ ways. There are $C(4,2)\times C(22,11)$ of these hands in which East has 2 of the 4 spades, and 11 of the 22 other cards. So the required probability is $\frac{C(4,2)\times C(22,11)}{C(26,13)} = \frac{6\times 705432}{10400600} = \frac{234}{575}$. This is 0.41 to two decimal places.

2.4.5B Complete the calculation of Exercise 2.4.5A by working out how many poker hands there are that fall into the categories (f) to (i) above. Also work out the probability that a poker hand dealt at random falls into each of these categories.

Solution: (f) *Straight flush.* A straight flush is determined by its suit, which may be chosen in 4 ways, and its lowest card, which may be chosen in 10 ways. So there are $4\times 10 = 40$ of these hands.

(g) *Three of a kind.* The rank of the 3 cards may be chosen in 13 ways, and 3 cards of this rank in $C(4,3) = 4$ ways. The 2 different ranks of the remaining 2 cards may be chosen in $C(12,2) = 66$ ways, and the ranks of these cards in 4×4 ways. So there are $13\times 4\times 66\times 4\times 4 = 54912$ of these hands.

(h) *Two pairs.* The 2 ranks of the two pairs may be chosen in $C(13,2) = 78$. For each rank, we can choose 2 cards of this rank in $C(4,2) = 6$ ways. There are then 44 cards of other ranks from which the 5th card may be chosen. So there are $78\times 6\times 6\times 44$ $= 123552$ of these hands.

(i) *Other hands.* By adding the numbers calculated in the solution to Exercise 2.4.5A, and those given above, we see that there are 1296420 poker hands in categories (a)-(h). There are $C(52,5) = 2598960$ poker hands altogether. So there are $2598960 - 1296420$ $= 1302540$ other hands.

The number of hands of each type, and their probabilities, to 6 decimal places are given the following table.

Kind of hand	*Number*	*Probability*
Straight Flush	40	0.000 015
Four of a Kind	624	0.000 240
Full House	3 744	0.001 441
Flush	5 108	0.001 965
Straight	10 200	0.003 925
Three of a Kind	54 912	0.021 128
Two Pairs	123 552	0.047 539
One Pair	1 098 240	0.422 569
Other Hands	1 302 540	0.501 177
Total	2 598 960	

2.4.6B A coin is biased so that the probability of getting a head is 0.6. If the coin is tossed five times, what is the probability of getting three heads?

Solution: The probability of a given sequence of the outcomes of 5 tosses, 3 of which are heads (such as HTHHT) is $(0.6)^3(0.4)^2 = 0.03456$. The number of such sequences is the number of ways of choosing the positions of the 3 heads, that is, $C(5,3) = 10$. So the probability of getting 3 heads and 2 tails is $10 \times 0.3456 = 0.3456$. [More generally, if a biased coin is tossed *n* times and the probability of getting a head is *p* and the probability of getting a tail is $1 - p$, then, for $0 \le r \le n$, the probability of getting *r* heads is $C(n,r)p^r(1-p)^{n-r}$. This corresponds to what is known as the *binomial distribution* of probabilities.]

2.5.1B If 21 dice are thrown simultaneously, what is the probability that 1 comes up once, 2 comes up twice, 3 comes up three times, 4 comes up four times, 5 comes up five times and 6 comes up six times?

Solution: Since there are 6 outcomes for the result when 1 die is thrown, there are 6^{21} different sequences of outcomes when 21 dice are thrown. By the Multinomial Theorem (Theorem 2.11) the number of such sequences made up of one 1, two 2s, three 3s, four 4s, five 5s and six 6s is $\frac{21!}{1!2!3!4!5!6!}$. Thus the required probability is

$\dfrac{21!}{1!2!3!4!5!6!}\Big/ 6^{21} = \dfrac{2053230379200}{21936950640377856} = \dfrac{396070675}{4231664861184}$, which is 0.0000936 to 3 significant figures.

2.5.2B In how many different ways can one arrange the sequence of letters in the word PROPERISPOMENON ?

Solution: Three of the letters (I,M,S) occur once, three occur twice (E,N,R) and two (P,O) occur three times in this word. So by the Multinomial Theorem (Theorem 2.11) the 15 letters in PROPERISPOMENON can be arranged in order in

$\dfrac{15!}{1!1!1!2!2!2!3!3!} = 4540536000$ ways. [The word *properispomenon* means "a word having a circumflex accent on the penultimate syllable". There are not many examples; one is *fenêtre.*]

2.6.1B Write the following permutation in cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 2 & 8 & 10 & 7 & 6 & 4 & 9 & 3 & 1 \end{pmatrix}.$$

Solution: $(1\ 5\ 7\ 4\ 10)(2)(3\ 8\ 9)(6)$ or just $(1\ 5\ 7\ 4\ 10)(3\ 8\ 9)$.

2.6.2B How many different permutations are there of the numbers $\{1,2,3,4,5,6,7,8,9,10\}$ made up of

i) Four disjoint cycles of lengths 1, 2, 3 and 4?

ii) Four disjoint cycles of which three are of length 2 and one is of length 4?

Solution: Using the same approach as in the solution to Exercise 2.6.2A we obtain the following for the number of permutations of each type.

i) $[C(10,1)\times 0!]\times[C(9,2)\times 1!]\times[C(7,3)\times 2!]\times[C(4,4)\times 3!]$

$$= 10\times 1\times 36\times 1\times 35\times 2\times 1\times 6 = 151200.$$

ii) $\dfrac{1}{3!}(C(10,2)\times 1!\times C(8,2)\times 1!\times C(6,2)\times 1!)\times C(4,4)\times 3! = 45\times 28\times 15 = 18900$.

2.6.3B If a permutation of the set $\{1,2,3,...,n\}$ is chosen at random, what is the probability that it includes exactly one cycle of length 1?

Note: As can be seen, this solution makes use of the idea of a *derangement* from Chapter 4.

Solution: We already know that there are altogether $n!$ permutations of the set $\{1,2,3,...,n\}$ .To obtain a permutation that has just one cycle of length 1 we need first to choose the number which is in the cycle of length 1. This can be done in *n* ways. Then we need to arrange the remaining $n-1$ numbers in a permutation that has no cycles of length 1, that is, so that it is a derangement of the remaining $n-1$ numbers. By Theorem 4.4, there are $(n-1)!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}$ of these derangements. Hence there are $n\times(n-1)!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}=n!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}$ permutations of $\{1,2,3,...,n\}$ containing just one cycle of length 1.

Hence the probability that a permutation of $\{1,2,3,...,n\}$ has just one cycle of length 1 is $\frac{1}{n!}\left(n!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\right)=\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}$. For $n\geq 8$, this is very close to $\frac{1}{e}$, that is, 0.36788 to five decimal places.