Chapter 2

Single Degree-of-Freedom Vibration: Discrete Models

Problems for Section 2.2 – Math Modeling: Deterministic

1. The beam in Figure 2.26 vibrates as a result of loading not shown. State the necessary assumptions to reduce this problem to a one degree-of-freedom oscillator. Then derive the equation of motion.

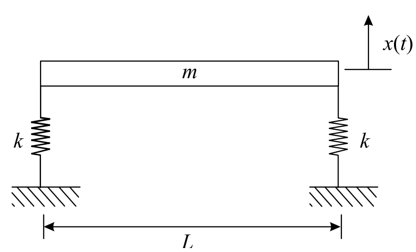


Figure 2.26: Vibrating beam supported by springs.

Solution: In order for the system to be approximated by a one DOF oscillator, one spatial coordinate is required to define its displacement. We need to assume that (i) the beam does not deform but moves as a rigid body, and (ii) the deflection at each end of the beam is the same, implying a symmetric loading (or only a moment is applied and the rotation at one end is equal and opposite to that at the other end).
Under these limitations, a SDOF system can be used to model the behavior of this beam. Let $F(t)$ be the external force acting upward on the beam. Then, using Newton's second law,

$$+\uparrow \sum F_{vertical} = F(t) - 2kx = m\frac{d^2x}{dt^2}$$

giving the equation of motion

$$m\frac{d^2x}{dt^2} + 2kx = F(t).$$

The system has the natural frequency $\omega_n = \sqrt{2k/m}$.

2. If a beam is supported continuously on a foundation, as shown in Figure 2.27, damping must be added to an idealized model to represent the viscous effects of the mat foundation. How would you idealize this system as a one degree-of-freedom oscillator? Derive the equation of motion.

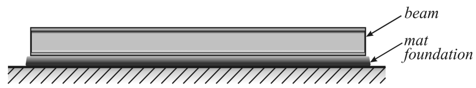


Figure 2.27: Vibrating beam on mat foundation.

Solution: To be modeled as a one DOF system, it must be possible to describe the motion using one coordinate. We assume the beam is rigid and does not deform. We also assume that the mat behaves uniformly and has the same physical characteristics along the whole length of the beam. Finally, the loading must be symmetrical so that the beam will translate only (although a pure rotation instead is also a possible behavior).
Given these assumptions, the equivalent system is the single mass oscillating in a vertical direction, where k is the equivalent total stiffness of the mat, and c is the total damping of the mat. Let $x(t)$ be the displacement from the static equilibrium, and $F(t)$ be the external force, both defined positive upward. The force must be distributed uniformly to avoid rotation. Then,

$$+\uparrow \sum F_{vertical} = m\frac{d^2x}{dt^2}$$
$$F(t) - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t).$$

The system has a natural frequency $\omega_n = \sqrt{k/m}$.

3. An idealized one degree-of-freedom model is tested many times in order to estimate its natural frequency. It is relatively straightforward to measure its mass m, but stiffness k can only be measured approximately. How might one use the natural frequency data to estimate k?

Solution: If we have test data on the natural frequency ω_n, and we know m, we can then use this data to estimate the value of k by using the average frequency $\overline{\omega}_n$ in the equation $k = \overline{\omega}_n^2 m$. How do we get the frequency data? By weighing the object, we can obtain the mass $m = W/g$, where W is the weight. Then we hang the weight on the spring, displace it and measure the period. This is actually the damped period. However, if damping is small, the undamped period, T, will be approximately equal to the damped period. Then $\omega_n = 2\pi/T$. From this we have k again.

4. The cantilever beam in Figure 2.28 undergoes harmonic oscillation, being driven by a force of amplitude A with frequency nominally equal to ω. An examination of a long time-history of the response shows slight fluctuations about an exact harmonic response. If we divide this long time-history into segments of one period $(2\pi/\omega)$ and superpose these, we obtain the set of curves shown in Figure 2.29.

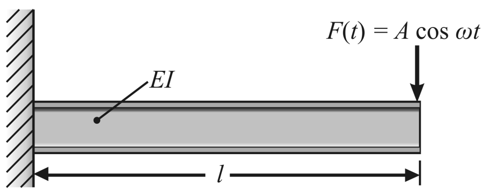


Figure 2.28: Cantilever beam.

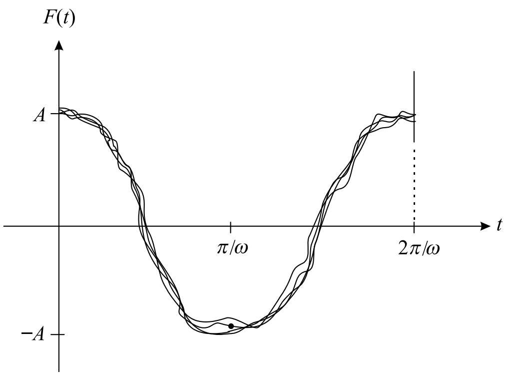


Figure 2.29: Overlapping time-histories.

How serious are such fluctuations in the response of the beam? Is there a way to relate the magnitude of the fluctuation to the maximum response? What would be a reasonable way to specify the value of ω?

Solution: These fluctuations do not seem significant based on a qualitative assessment of deviations. Although there may be applications where the variations in loading might be important, for most applications the beam motion is not heavily dependent on such minor deviations.

We may quantify the magnitude of the fluctuations by measuring the amplitudes at one instant of time. For example, at one instant of time, the largest and smallest amplitudes might be a_l and a_s. The fluctuation at this instant is defined as $\Delta = (a_l - a_s)$. We can obtain a measure of the importance of the fluctuation by relating it to the amplitude of the response A, a simple way being Δ/A. Then, $\Delta/A \times 100$ provides a percent fluctuation. A small percent signifies minor fluctuations. The frequency ω can be estimated from the curves by averaging the periods and then using the average period $\overline{T}$ to calculate the frequency $\omega = 2\pi/\overline{T}$.

5. For each nonlinear equation of motion, linearize the equation and discuss the range of validity of the linearized equation. For example: the linearized equation of motion has an $x\%$ error in the nonlinear term $\cos\theta$ for $\theta > \theta_0$. Analytically solve the linearized equations of motion and numerically solve the fully nonlinear equation using a program such as MATLAB. Plot the linear and nonlinear time-histories on the same graph and discuss the comparisons.
(a) $\ddot{\theta} + 3\cos\theta = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$
(b) $\ddot{\theta} + 3\sin\theta = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$
(c) $\ddot{\theta} + 3\cos^2\theta = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$
(d) $\ddot{\theta} + 3\sin^2\theta = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$.

Solution:
(a) $\ddot{\theta} + 3\cos\theta = 0, \quad \theta(0) = 0.5, \; \dot{\theta}(0) = 0$

To linearize this equation implies that θ is small and that the following approximation can be made: $\cos\theta \simeq 1$. The equation of motion becomes $\ddot{\theta}+3 = 0$ with initial conditions, $\theta(0) = 0.5$ and $\dot{\theta}(0) = 0$. The solution is given by $\theta(t) = 0.5 - 1.5t^2$.
To solve the original nonlinear equation requires a numerical integration. Most math programs have built-in differential equation solvers. In MAPLE, one would write,

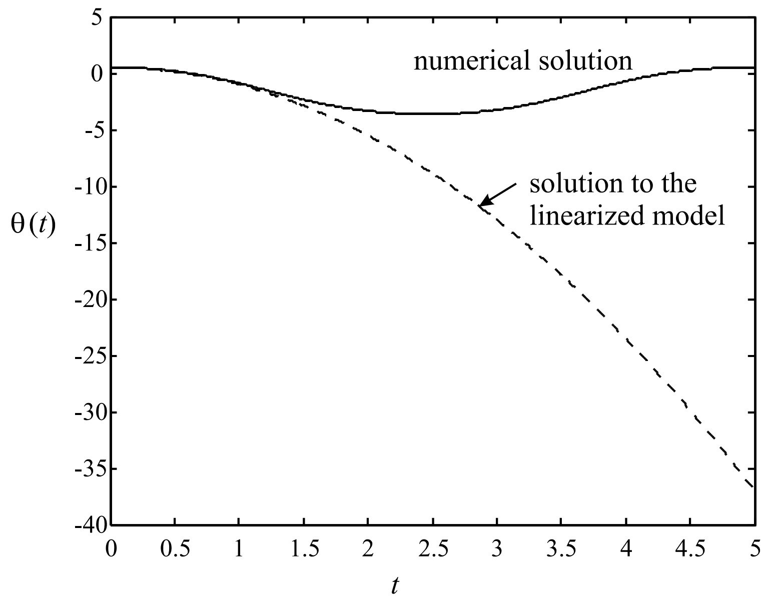
```
dsolve({diff(theta(t),t$2)+3*cos(theta(t)),theta(0)=0.5,D(theta)(0)=0});
```

Below is a sample MATLAB program that solves and plots the results. Note that MATLAB requires the equations of motion to be in a state-space form.

```
function problem2_5
clear
[t,Y]=ode45(@odefun, [0:0.01:5], [0.5 0]);
plot(t, Y(:,1), t, 0.5-1.5*t.^2,'–')


function dot=odefun(t,Y)
dot=[Y(2);
        -3*cos(Y(1))];
end
end
```

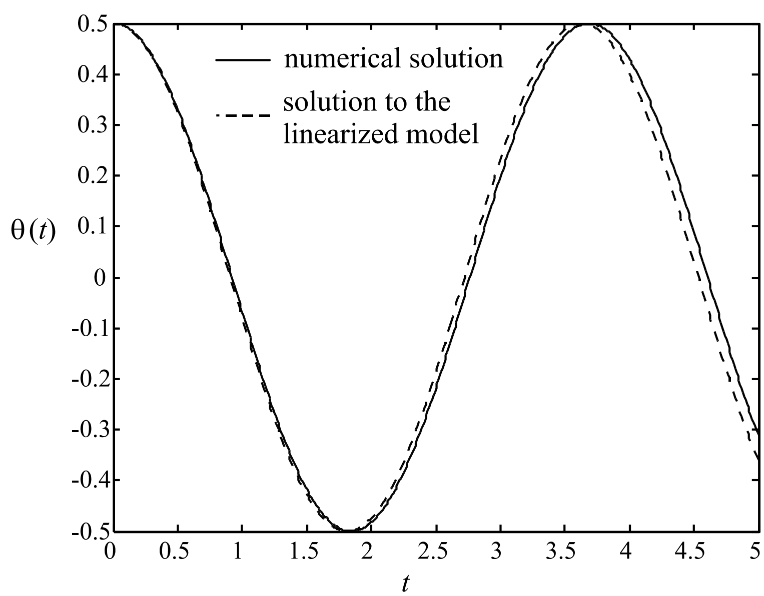
The figures below show the results of the numerical simulation as compared to the linearized solution.



The error is within 10% for $t < 1$. Afterwards, the linearized solution develops greater errors.

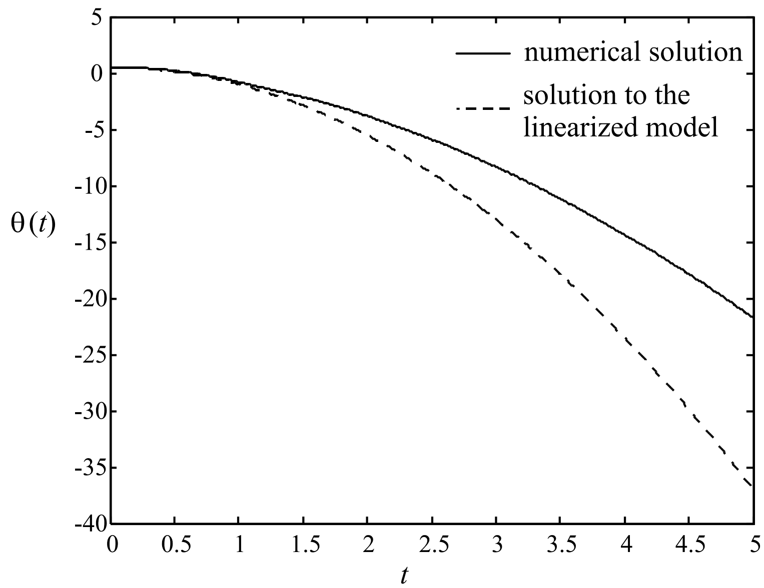
(b) $\ddot{\theta} + 3\sin\theta = 0, \quad \theta(0) = 0.5, \ \dot{\theta}(0) = 0$

The linearized equation, with $\sin\theta \simeq \theta$, becomes $y'' + 3y = 0$ with $y(0) = 0.5$ and $y'(0) = 0$ with solution $y\,(t) = 0.\,5\cos 1.\,7321t$. Let us plot the numerical solution to the nonlinear equation and the solution to the linearized equation. Note that the solution to the linearized equation works very well.



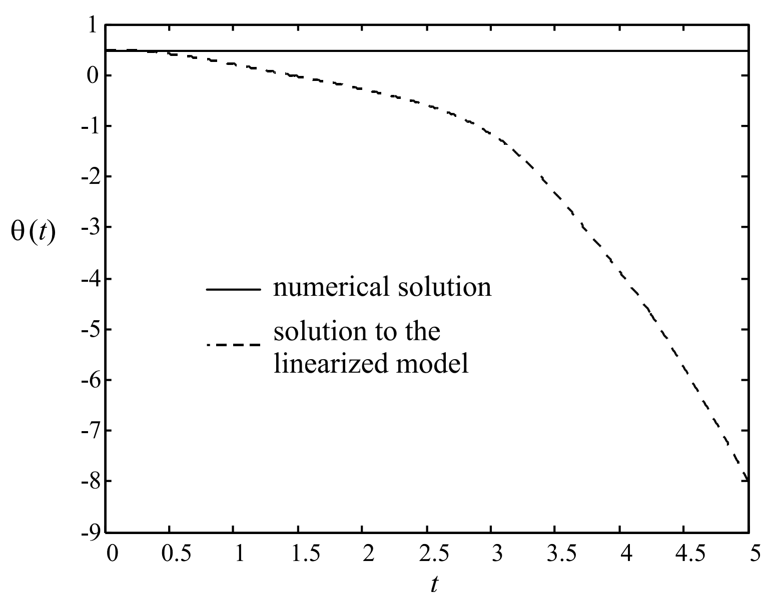
(c) $\ddot{\theta} + 3\cos^2\theta = 0, \quad \theta(0) = 0.5, \ \dot{\theta}(0) = 0$

Recall the trigonometric identity $\cos^2\theta = (1 + \cos 2\theta)\,/2$. Applying this identity to the above differential equation and substituting the small angle approximation $\cos 2\theta \simeq 1$ leaves us with the expression $\ddot{\theta} + 3 = 0$, which is the same equation as in part (a). So, we arrive at the approximate solution $\theta(t) = 0.5 - 1.5t^2$. The following figure plots the linear and numerical solutions. Note that the linearized solution is a good approximation for only the first two or three seconds of the motion.



(d) $\underline{\ddot{\theta} + 3\sin^2\theta = 0, \quad \theta(0) = 0.5, \ \dot{\theta}(0) = 0}$

Recall the trigonometric identity $\sin^2\theta = (1 - \cos 2\theta)/2$. Applying this identity to the above differential equation and substituting the small angle approximation as above leaves us with the expression $\ddot{\theta} = 0$, implying a constant solution since the slope of the solution at time $t = 0$ is zero. Two integrations satisfying the initial conditions leads to the result $\theta(t) = 0.5$. Plotting the approximate and the numerical solutions against each other gives the following figure.



The linear approximation valid here for the first 2.5 seconds of the motion, and then large errors develop.

6. For each idealized model in Figures 2.30 to 2.33, draw a free-body diagram and derive the equation of motion using (a) Newton's second law of motion and (b) the energy method. The block in Figure 2.31 slides on a frictionless surface. State whether the oscillations are linear or nonlinear. Determine the natural frequency of each model.

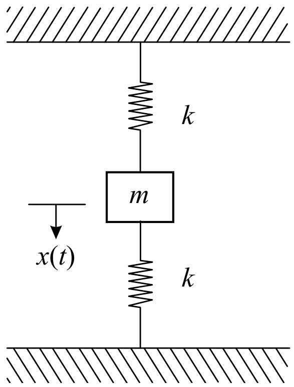


Figure 2.30: Oscillating mass.

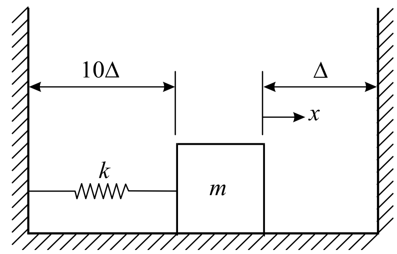


Figure 2.31: Sliding oscillation of mass.

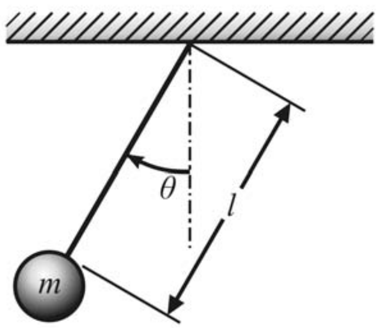


Figure 2.32: A simple pendulum.

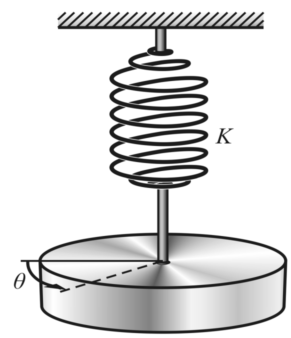


Figure 2.33: Torsional vibration.

Solution: Fig.2.30 For an oscillator vibrating in the gravitational direction x, about static equilibrium, we have

$$
\begin{aligned}
+ \rightarrow \sum F_x &= m\ddot{x} \\
-kx - kx &= m\ddot{x}.
\end{aligned}
$$

Therefore, the equation of motion is $m\ddot{x} + (k+k)x = 0$ with natural frequency $\omega_n = \sqrt{2k/m}$. Using an energy approach to the same problem, we first find the kinetic and potential energies, and then apply the principle of energy conservation $T + V =$ constant,

$$\begin{aligned} T &= \frac{1}{2} m \dot{x}^2 \\ V &= \frac{1}{2}(k+k)x^2 \\ \frac{1}{2} m \dot{x}^2 + \frac{1}{2}(2k)x^2 &= const. \end{aligned}$$

Differentiating the last expression with respect to time, we have

$$m\dot{x}\ddot{x} + 2kx\dot{x} = 0.$$

Since $\dot{x} \neq 0$, we can divide it out of the expression, leaving the equation of motion $m\ddot{x} + 2kx = 0$.

Fig.2.31 Sum the forces in the x direction to obtain

$$\begin{aligned} + \rightarrow \sum F_x &= m\ddot{x} \\ -kx &= m\ddot{x}. \end{aligned}$$

The equation of motion is $m\ddot{x} + kx = 0$. When the mass reaches the wall, it bounces back with a new velocity. The new velocity depends on the coefficient of restitution, e. The velocity after the bounce is $v_f = v_i e$ in the opposite direction. We re-solve the same equation of motion with this new initial velocity in the opposite direction.
Using an energy approach to the same problem, we first find the kinetic and potential energies, and then apply the principle of energy conservation $T + V =$ constant,

$$\begin{aligned} T &= \frac{1}{2} m \dot{x}^2, \ V = \frac{1}{2} kx^2 \\ E &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} kx^2 = const. \end{aligned}$$

Differentiating the total energy, E, and dividing the expression by $\dot{x}$, we obtain the same equation of motion $m\ddot{x} + kx = 0$.

Fig.2.32 For the oscillating pendulum, from the free-body diagram, we take the sum of the moments and set these equal to the mass moment of inertia about the base multiplied by the angular acceleration,

$$\begin{aligned} + \circlearrowright \sum M_o &= I\ddot{\theta} \\ -mgl \sin\theta &= I\ddot{\theta}. \end{aligned}$$

The negative sign is there because the moment is in the opposite sense of the positive direction given to θ. I is the mass moment of inertia about the point of contact of the string. For the point mass shown, $I = ml^2$. Therefore, the equation of motion is $I\ddot{\theta} + mgl\sin\theta = 0$. The natural frequency is given by $\omega_n = \sqrt{mgl/I}$.

By the energy method, we substitute the kinetic and potential energies into the principle of the conservation of energy.

$$\begin{aligned} T &= \frac{1}{2} I \dot{\theta}^2, \ V = -mgl\cos\theta \\ T + V &= \frac{1}{2} I \dot{\theta}^2 - mgl\cos\theta = \text{const.} \end{aligned}$$

Differentiating the total energy with respect to time,

$$\begin{aligned} I\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta} &= 0 \\ \dot{\theta}(I\ddot{\theta} + mgL\sin\theta) &= 0. \end{aligned}$$

Since $\dot{\theta} \neq 0$, we can divide it out of the expression, leaving us with the equation of motion $I\ddot{\theta} + mgL\sin\theta = 0$.

<u>Fig.2.33</u> Take the moment about the vertical axis through the center of the disk to obtain

$$\begin{aligned} + \circlearrowleft \sum M_G = J\ddot{\theta} \\ -K\theta &= J\ddot{\theta}, \end{aligned}$$

where J is the mass polar moment of inertia. For a solid disk $J = mR^2/2$. The equation of motion is $J\ddot{\theta} + K\theta = 0$, and the natural frequency is then $\omega_n = \sqrt{K/J}$.
Using the energy approach, we first find the kinetic and potential energies, and then apply the principle of energy conservation $T + V =$ constant.

$$\begin{aligned} T &= \frac{1}{2} J\dot{\theta}^2, \ V = \frac{1}{2} K\theta^2 \\ T + V &= \frac{1}{2} J\dot{\theta}^2 + \frac{1}{2} K\theta^2 = const. \end{aligned}$$

Differentiating the total energy and dividing the expression by $\dot{\theta}$, we obtain the same equation of motion $J\ddot{\theta} + K\theta = 0$.

7. A disk of mass m is mounted between two shafts with different properties, as shown in Figure 3.34. (a) What is the natural frequency of the system? (b) If the disk is rotated θ, where $\theta \ll 1$ rad, and then released, what will its angular position be at an arbitrary time t?

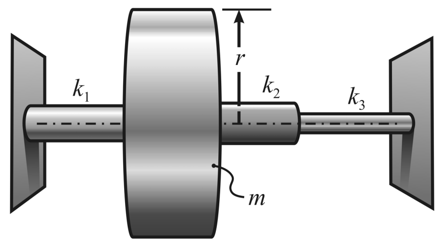


Figure 2.34: A disk mounted between two shafts.

Solution: Note that k_2 and k_3 are in series. The equivalent stiffness is

$$k'_{eq} = \left(\frac{1}{k_2} + \frac{1}{k_3}\right)^{-1} = \frac{k_2 k_3}{k_2 + k_3}.$$

Let us assume a rotation of θ, then the springs exert moments in the opposite direction of the assumed motion. Summing the moment about the rotational axis, we have

$$\begin{aligned} \sum M &= I_{disk}\ddot{\theta} \\ -k_1\theta - k'_{eq}\theta &= I_{disk}\ddot{\theta}. \end{aligned}$$

The equation of motion is then

$$I_{disk}\ddot{\theta} + \underbrace{\left(k_1 + \frac{k_2 k_3}{k_2 + k_3}\right)}_{k_{eq}}\theta = 0.$$

The natural frequency is

$$\omega_n = \sqrt{\frac{1}{I_{disk}}\left(k_1 + \frac{k_2 k_3}{k_2 + k_3}\right)}.$$

Note that the polar mass moment of inertia of a solid disk is $I_{disk} = \frac{1}{2}mr^2$, where r is the radius of the disk. The response is purely sinusoidal, and

$$\theta\,(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t.$$

For a disk released from an initial position θ_0 with zero velocity, the response at an arbitrary time is given by

$$\theta\,(t) = \theta_0 \cos \omega_n t.$$

8. Calculate the equivalent torsional spring constant for the shaft on the disk shown in Figure 2.35.

Solution: The stiffness of a torsional bar is GJ/l, where J is the area polar moment of inertia given by

$$J = \frac{1}{2}\pi r^4 = \frac{1}{32}\pi d^4.$$

In this case, two torsional bars are in series so that the equivalent stiffness is

$$\begin{aligned} k_{eq} &= \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1} = \frac{k_1 k_2}{k_1 + k_2} \\ &= \frac{\frac{G_1 J_1}{l_1}\frac{G_2 J_2}{l_2}}{\frac{G_1 J_1}{l_1} + \frac{G_2 J_2}{l_2}} \\ &= \frac{G_1 G_2 J_1 J_2}{G_1 J_1 l_2 + G_2 J_2 l_1}. \end{aligned}$$

Substituting for J_1 and J_2, we obtain the equation for the equivalent stiffness in terms of the diameters,

$$k_{eq} = \frac{\pi}{32}\frac{G_1 G_2 d_1^4 d_2^4}{G_1 d_1^4 l_2 + G_2 d_2^4 l_1}.$$

9. A valve mechanism is drawn schematically in Figure 2.36. The mechanism is in equilibrium when the rocker arm is horizontal. The system is assumed to be frictionless. Use an energy method to determine the natural angular frequency for small vibration about the equilibrium.

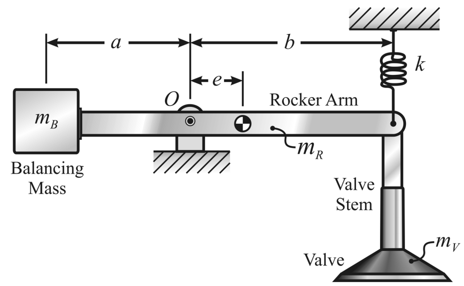


Figure 2.36: A rocker-arm valve system.

Solution: Let θ be the angle of rotation of the rocker arm in the counterclockwise direction measured from the static equilibrium. The kinetic energy of the system is given by

$$T = \underbrace{\frac{1}{2}m_B\left(a\dot{\theta}\right)^2}_{T_{mass}} + \underbrace{\frac{1}{2}I_O\dot{\theta}^2}_{T_{rod}} + \underbrace{\frac{1}{2}m_V\left(b\dot{\theta}\cos\theta\right)^2}_{T_{valve}}$$

where $I_O = I_G + m_B e^2 = \frac{1}{12}m_B(a+b)^2 + m_B e^2$ using the parallel axis theorem. The potential energy of the system is

$$V = -\underbrace{m_B g a\sin(\theta+\theta_s)}_{V_{mass}} + \underbrace{m_B g e\sin(\theta+\theta_s)}_{V_{rod}} + \underbrace{m_V g b\sin(\theta+\theta_s)}_{V_{valve}} + \underbrace{\frac{1}{2}k\left(b\sin(\theta+\theta_s)\right)^2}_{V_{spring}},$$

where θ_s is the angle of rotation under static equilibrium. The total energy, E, is the sum of the kinetic and potential energies. The system is conservative, and the total energy is constant. Differentiate the total energy and we obtain

$$\begin{aligned}0 &= \frac{dE}{dt} = m_B a^2\dot{\theta}\ddot{\theta} + I_O\dot{\theta}\ddot{\theta} + m_V b^2\dot{\theta}\cos\theta\left(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta\right)\\ &\quad -m_B g a\cos(\theta+\theta_s)\dot{\theta} + m_B g e\cos(\theta+\theta_s)\dot{\theta} + m_V g b\cos(\theta+\theta_s)\dot{\theta} + kb^2\sin(\theta+\theta_s)\dot{\theta}.\end{aligned}$$

Divide by $\dot{\theta}$ to obtain

$$\begin{aligned}0 &= m_B a^2\ddot{\theta} + I_O\ddot{\theta} + m_V b^2\cos\theta\left(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta\right)\\ &\quad -m_B g a\cos(\theta+\theta_s) + m_B g e\cos(\theta+\theta_s) + m_V g b\cos(\theta+\theta_s) + kb^2\sin(\theta+\theta_s).\end{aligned}$$

Assuming small rotations $\left(\theta^2 \ll 1,\ \text{or}\ |\theta| \ll 1\right)$ and we obtain

$$0 = \left(m_B a^2 + I_O + m_V b^2\right)\ddot{\theta} + kb^2(\theta+\theta_s) - m_B g a + m_B g e + m_V g b.$$

The equilibrium position θ_s satisfies $kb^2\theta_s - m_B g a + m_B g e + m_V g b = 0$. Then, the equation of motion is simplified to

$$0 = \left(m_B a^2 + I_O + m_V b^2\right) \ddot{\theta} + kb^2\theta.$$

The natural frequency of this system is

$$\omega_n = \sqrt{\frac{kb^2}{m_B a^2 + I_O + m_V b^2}}.$$

10. A rod is supported on two rotating grooved rollers, as depicted in Figure 2.39. The tubes rotate in opposite directions and the coefficient of friction between the tubes and the rod is μ_k. Find the natural frequency of the system and describe the behavior of the rod if it is disturbed in the horizontal direction.

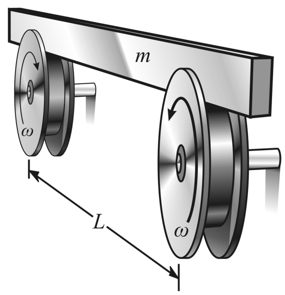
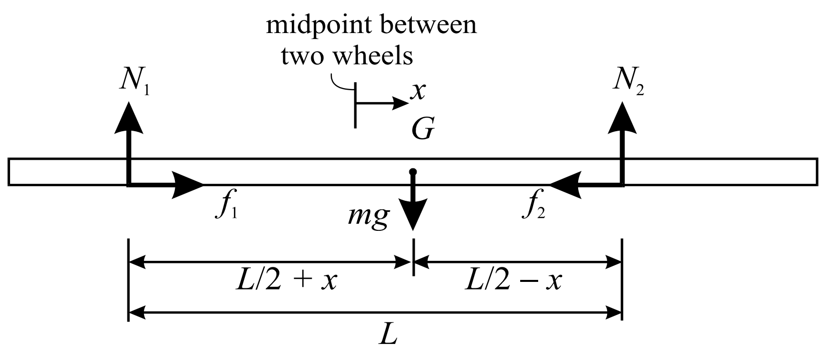


Figure 2.39: Rod supported by two rotating grooved rollers.

Solution: The rod oscillates because of the variation in the frictional force by each wheel. Let us consider a free-body diagram shown below.



The rod does not move up and down nor rotate. Therefore, we have

$$+\uparrow \sum F_y = N_1 + N_2 - mg = 0$$

$$+\circlearrowleft \sum M_G = N_1\left(\frac{L}{2}+x\right) - N_2\left(\frac{L}{2}-x\right) = 0.$$

Solving, we can find the normal forces as functions of x:

$$N_1 = mg\left(\frac{1}{2}-\frac{x}{L}\right) \text{ and } N_2 = mg\left(\frac{1}{2}+\frac{x}{L}\right).$$

The force balance equation in the x direction is given by

$$\begin{aligned} +\rightarrow \sum F_x = f_1 - f_2 = m\ddot{x} \\ \mu_k N_1 - \mu_k N_2 &= m\ddot{x} \\ \mu_k mg\left(\frac{1}{2}-\frac{x}{L}-\frac{1}{2}-\frac{x}{L}\right) &= m\ddot{x}. \end{aligned}$$

The equation of motion is then

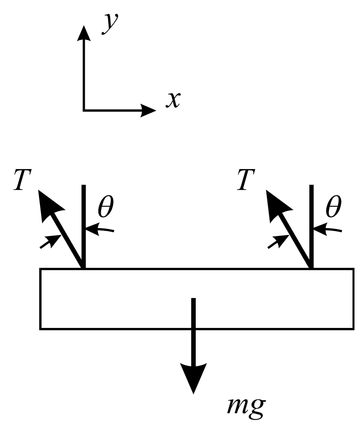
$$m\ddot{x} + 2\frac{\mu_k mg}{L}x = 0.$$

The rod oscillates about the midpoint at the natural frequency of

$$\omega_n = \sqrt{2\frac{\mu_k mg}{L}}.$$

11. A 25 kg block is suspended by two cables, as depicted in Figure 2.38. Assume small displacements. (a) What is the frequency of oscillation in Hz of the block in the x direction if it is slightly displaced in this direction? (b) What is the period of oscillation in the z direction if the block is slightly displaced in this direction?

Solution: (a) Assuming the block oscillates on the xy plane, the corresponding free-body diagram is shown below.



The block does not rotate, but translates. Summing the forces in the x and y direction, we obtain

$$+ \rightarrow \sum F_x = -2T\sin\theta = m\ddot{x} \tag{1}$$

$$+ \uparrow \sum F_y = 2T\cos\theta - mg = m\ddot{y}. \tag{2}$$

The accelerations in the x and y directions are written in terms of θ and derivatives of θ. Note that the accelerations at the center of gravity are the same as those anywhere else on the block. We write the position of one of the points where the string is attached. Taking a second derivative with respect to time gives the accelerations,

$$x = l\sin\theta \Longrightarrow \ddot{x} = l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta \tag{3}$$

$$y = l - l\cos\theta \Longrightarrow \ddot{y} = l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta. \tag{4}$$

Substituting Equation (3) into (1) and (4) into (2), the force balance equations are now

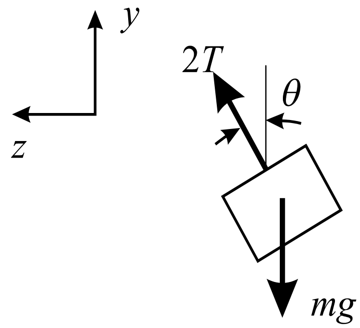
$$\begin{aligned} -2T\sin\theta &= m\left(l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) \\ 2T\cos\theta - mg &= m\left(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta\right). \end{aligned}$$

There are two unknowns, θ and T, both functions of time. We eliminate the tension to obtain a differential equation in terms of θ. Multiplying the first by $\cos\theta$ and the second equation by $\sin\theta$ and adding them, we obtain

$$ml\ddot{\theta} + mg\sin\theta = 0,$$

which is identical to a pendulum equation. The reason why the equation of motion is that of a pendulum is because the block does not rotate. The natural frequency for small oscillations in units of Hz is $f = \omega_n/2\pi = \sqrt{g/L}/2\pi$.

(b) If the block oscillates in the yz plane, the free-body diagram is shown below.



Taking a moment about the fixed point at the ceiling, we have

$$\begin{aligned} +\circlearrowleft \sum M &= \left(I_G + ml^2\right)\ddot{\theta} \\ -mgl\sin\theta &= \left(I_G + ml^2\right)\ddot{\theta}, \end{aligned}$$

where $I_G = \frac{1}{12}m\left(d^2 + h^2\right)$. The equation of motion is given by

$$\left(I_G + ml^2\right)\ddot{\theta} + mgl\sin\theta = 0.$$

For small oscillations, the natural frequency is

$$\omega_n = \sqrt{\frac{\left(\frac{d^2+h^2}{12} + l^2\right)}{gl}} = \frac{1}{2}\sqrt{\frac{(d^2 + h^2 + 12l^2)}{3gl}}.$$

In Hz,

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{4\pi}\sqrt{\frac{(d^2 + h^2 + 12l^2)}{3gl}}.$$

12. Derive the equation of motion and natural frequency for a mass m on the string that is under constant tension T as shown in Figure 2.39. Assume small displacements and m is much greater than the mass of the string.

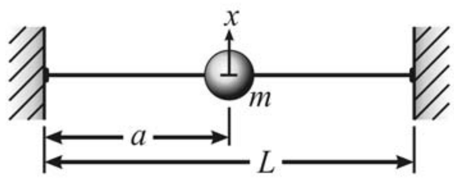
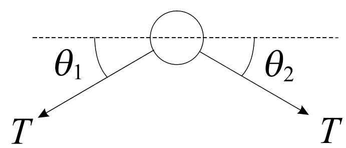


Figure 2.39: Vibration of a mass on a string under tension.

Solution: When the string with the mass is displaced from equilibrium, it is the vertical components of the string tension forces that restore the mass to its original position. The free-body diagram is shown below.



Summing the forces in the vertical direction we obtain

$$+\uparrow \sum F = m\ddot{x}$$
$$-T\sin\theta_1 - T\sin\theta_2 = m\ddot{x}.$$

For small displacements, we can make the approximation that $\sin\theta \simeq \tan\theta$. For the triangle with base a, $\tan\theta = x/a$, and for the triangle with base $L-a$, $\tan\theta \simeq x/(L-a)$. Therefore, the equation of motion becomes

$$\begin{aligned} -T\frac{x}{a} - T\frac{x}{l-a} &= m\ddot{x} \\ m\ddot{x} + T\left(\frac{L}{a\,(L-a)}\right)x &= 0. \end{aligned}$$

The natural frequency is therefore

$$\omega_n = \sqrt{\frac{T}{m}\frac{L}{a\,(L-a)}}.$$

13. Continuing Problem 2.12, the string is stretched to the position shown in Figure 2.40. Calculate the natural frequency of the system using the following parameter values: $W = 2$ lb, $T = 50$ lb, and $l = 4$ ft, where T is the tension in the string for the configuration shown.

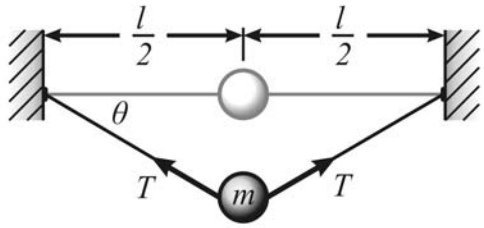


Figure 2.40: Mass on a stretched string.

Solution: From the previous problem, we found that the natural frequency is

$$\omega_n = \sqrt{\frac{T}{m}\frac{l}{a\,(l-a)}}.$$

Substitute the following values:

$$\begin{aligned} T &= 50 \text{ lb} \\ a &= 2 \text{ ft} \\ l &= 4 \text{ ft} \\ m &= \frac{W}{g} = \frac{2}{32.2}\ \frac{\text{lb}}{\text{ft/s}^2}. \end{aligned}$$

The natural frequency is

$$\begin{aligned} \omega_n &= \sqrt{\frac{T}{m}\frac{l}{a\,(l-a)}} \\ &= \sqrt{\frac{50}{(2/32.2)}\frac{4}{2\,(4-2)}} \\ &= 28.4 \text{ rad/s.} \end{aligned}$$

14. Derive the equation of motion for a uniform stiff rod restrained from vertical motion by a torsional spring of stiffness K as shown in Figure 2.41. The torsional spring constant is determined by the application of a moment M and the measurement of the angular displacement θ, that is, $M = K\theta$. Calculate the natural frequency of oscillation. Let J define the moment of inertia of the rod about the point of oscillation. State any assumptions.

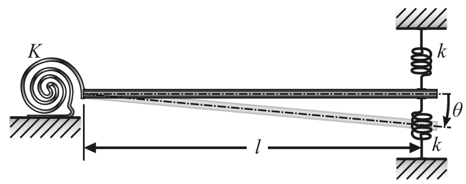
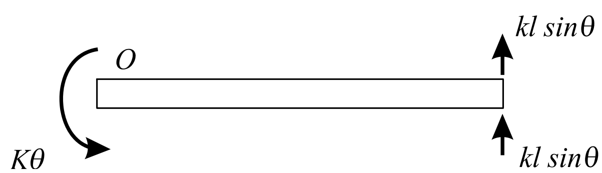


Figure 2.41: Restrained rigid rod.

Solution: Let J be the mass moment of inertia of the rod about its base, O. The free-body diagram is shown below.



We assume small displacements at the end. In this way we can assume that essentially the springs remain vertical. Newton's second law of motion, in moment form, can be applied for the sum of the external moments about the left end of the rod. It should be noted that the moment is taken positive in the positive θ direction.

$$+\curvearrowright \sum M_o = I_o\ddot{\theta}$$
$$-K\theta - (kl\sin\theta)(l) - (kl\sin\theta)(l) = I_o\ddot{\theta},$$

where $I_o = ml^2/3$, $l\sin\theta$ is the deflection of the right end and l is the moment arm of the spring force. Then,

$$\frac{ml^2}{3}\ddot{\theta} + K\theta + 2kl^2\sin\theta = 0$$

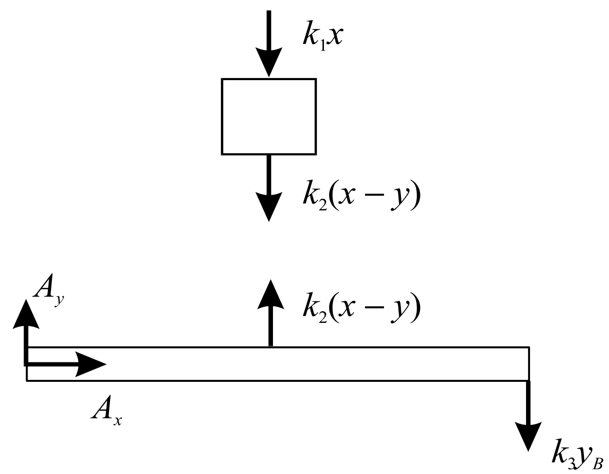
or

$$\frac{ml^2}{3}\ddot{\theta} + \left(2kl^2 + K\right)\theta = 0,$$

where for small angles $\sin\theta \simeq \theta$ gives the equation of linearized motion. The natural frequency for small motions is $\omega_n = \sqrt{3\left(2kl^2 + K\right)/ml^2}$.

15. A uniform rigid and massless rod is pinned at one end and connected to ground via a spring at the other end. At midpoint on the rod, a spring is connected to a mass which is then connected to a fixed point via another spring, as shown in Figure 2.42. (a) Derive the equation of free vibration for the system, and (b) find the natural frequency. (c) Calculate the natural frequency given the following parameters: $k_1 = 20$ lb/in, $k_2 = 30$ lb/in, $k_3 = 40$ lb/in, $W = 50$ lb, and $l = 8$ ft.

Solution: (a) Equation of motion
The free-body diagrams corresponding to positive displacements are shown below.



For the mass, the force balance equation in the vertical direction is

$$+\uparrow \sum F = m\ddot{x}$$
$$-k_1x - k_2(x-y) = m\ddot{x}. \tag{1}$$

Note that the deflections are measured from static equilibrium so that gravity can be omitted from the problem. (We can always do this if the only thing that the gravity does is to cause a static deflection.) The moment balance equation for the rod is

$$+\circlearrowleft \sum M_A = I_A\ddot{\theta}$$
$$k_2(x-y)\frac{l}{2} - k_3y_Bl = 0,$$

where $I_A = 0$ because, from the problem statement, the rod is massless, and y_B is twice y (using similar triangles). Then, the moment balance equation for the rod is reduced to

$$k_2(x-y) - 4k_3y = 0.$$

Solving for y, we find $y = (k_2/(k_2+4k_3))\,x$. Substituting this back into Equation (1), the equation of motion is given by

$$\begin{aligned} -k_1x - k_2\left(x - \frac{k_2}{k_2+4k_3}x\right) &= m\ddot{x} \\ m\ddot{x} + \left(k_1 + k_2 + \frac{k_2^2}{k_2+4k_3}\right)x &= 0. \end{aligned}$$

(b) The general expression for the natural frequency is

$$\omega_n = \sqrt{\left(k_1 + k_2 + \frac{k_2^2}{k_2 + 4k_3}\right)\frac{1}{m}}.$$

(c) The numerical value for the natural frequency given $k_1 = 20$ lb/in, $k_2 = 30$ lb/in, $k_3 = 40$ lb/in, $W = 50$ lb, and $l = 8$ ft is

$$\begin{aligned}\omega_n &= \sqrt{\left(20 + 30 + \frac{30^2}{30 + 4\,(40)}\right)\frac{32.2 \times 12}{50}} \\ &= 20.6 \text{ rad/s.}\end{aligned}$$

16. (a) A rod of nonuniform cross-section but uniform material is pinned at one end and supported by two springs as shown in Figure 2.43. The rod is displaced slightly from equilibrium and released and observed to oscillate with a period of $T = 1.5$ s. Calculate the moment of inertia of the rod with respect to the hinge axis of rotation if $k = 500$ N/m and $l = 1$ m. (b) Calculate the period of oscillation if both springs were on the same side of the rod and all parameter values are the same as in the previous part.

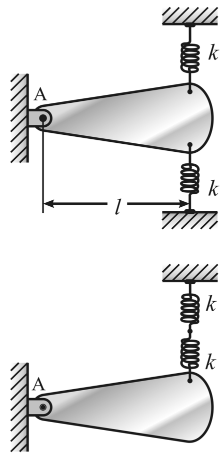
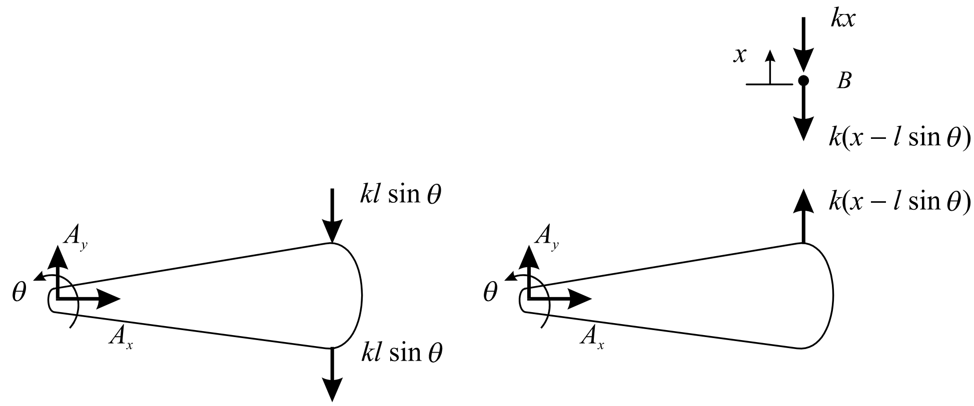


Figure 2.43: A rigid rod supported by two springs.

Solution: Assuming a positive rotation in the counterclockwise direction and positive displacement upward, we draw free-body diagrams for both cases below.



(a) First Case - We sum the moment about A, finding

$$+\circlearrowleft \sum M_A = I_A\ddot{\theta}$$
$$-2kl^2 \sin\theta = I_A\ddot{\theta}.$$

The linearized equation of motion is

$$I_A\ddot{\theta} + 2kl^2\theta = 0.$$

The natural frequency is $\omega_n = \sqrt{2kl^2/I_A}$, and the natural period is $T = 2\pi/\omega_n = 2\pi\sqrt{I_A/2k^2l}$. Substituting $k = 500$ N/m, $l = 1$ m, and $T = 1.5$ s, we find

$$I_A = \left(\frac{T}{2\pi}\right)^2 2kl^2 = 57.0 \text{ kg-m}^2.$$

(b) Second Case - If the springs are attached in series, the equivalent stiffness becomes $k_{eq} = (1/k + 1/k)^{-1} = k/2$. This can be seen from the free-body diagram. We sum the moments about A to obtain

$$+\circlearrowleft \sum M_A = I_A\ddot{\theta}$$
$$kl\,(x - l\sin\theta) = I_A\ddot{\theta}. \tag{1}$$

We sum the forces about B to obtain

$$+\uparrow \sum F = 0 \text{ (massless point B)}$$
$$kx + kx - kl\sin\theta = 0.$$

Solving for x, we obtain $x = l\sin\theta/2$. Substituting this into Equation (1), we obtain

$$kl\left(\frac{l\sin\theta}{2} - l\sin\theta\right) = I_A\ddot{\theta}$$
$$I_A\ddot{\theta} + \frac{k}{2}l^2\sin\theta = 0.$$

The linearized equation of motion is

$$I_A\ddot{\theta} + \frac{k}{2}l^2\theta = 0.$$

The stiffness of the second system is reduced by a factor of four. Then, the natural frequency is reduced by a factor of $\sqrt{4}$ or 2. This will increase the period by a factor of two. The period of the second system is 3 s.

17. A body of mass m is suspended by a spring of constant k and attached to an elastic beam of length l, as shown in Figure 2.44. When the mass is attached to the spring measurements are taken of the spring extension δ_s and the end deflection of the beam x_b. Neglect the masses of the beam and the spring. (a) Estimate the natural frequency of the system. (b) Calculate the natural frequency if the mass is attached to the beam directly without a spring. (c) Calculate the natural frequency if the beam is assumed to be rigid. (d) Calculate the respective numerical values given $\delta_s = 12$ mm, $x_b = 2$ mm, and $l = 0.5$ m.

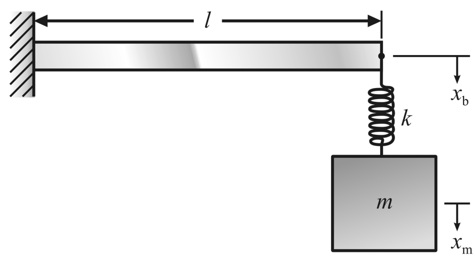
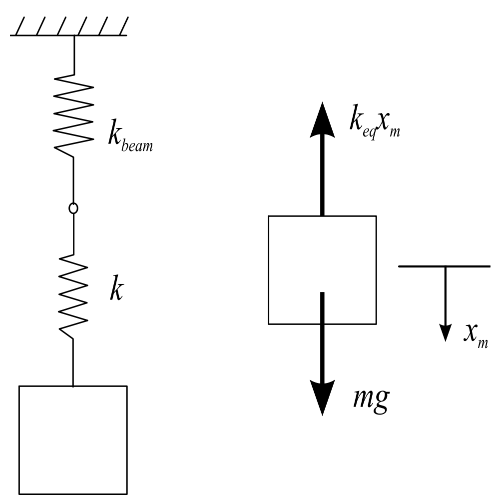


Figure 2.44: Mass suspended by a spring from a beam.

Solution: (a) In this system, a leaf spring is connected to a coil spring in series. The system in Figure 2.44 can be simplified to



The beam can be modeled as a spring with stiffness $k_{beam.}$ This leaf spring has a stiffness of $k_{beam} = 3EI/L^3$. The combined stiffness is

$$\begin{aligned} k_{eq} &= \left(\frac{1}{k} + \frac{1}{k_{beam}}\right)^{-1} \\ &= \frac{k_{beam}k}{k + k_{beam}}. \end{aligned}$$

The equation of motion is

$$m\ddot{x}_m + k_{eq}x_m = 0,$$

where x_m is measured from the equilibrium position. The natural frequency is

$$\omega_n = \sqrt{\frac{k_{beam}k}{(k + k_{beam})\,m}}.$$

Note that we could have omitted mg from the free-body diagram if we measure the deflection from static equilibrium.
(b) If the mass is attached to the beam directly without a spring, the natural frequency is obtained by letting $k_{eq} = k_{beam}$ or let $k \to \infty$ in the natural frequency expression obtained in part (a):

$$\omega_n = \sqrt{\frac{k_{beam}}{m}}.$$

(c) If the beam is assumed rigid, we let $k_{beam} \to \infty$ in the natural frequency expression obtained in part (a):

$$\omega_n = \sqrt{\frac{k}{m}}$$

(d) If a mass is placed, we find $\delta_s = 12$ mm and $x_b = 2$ mm. For springs in series, the forces in the springs are equal. That is,

$$k_{beam}x_b = mg \text{ and } k\delta_s = mg.$$

Then,

$$k_{beam} = \frac{mg}{0.002} \text{ and } k = \frac{mg}{0.012}.$$

The numerical answer to parts (a), (b) and (c) are

$$\text{(a)} \qquad \omega_n = \sqrt{\frac{k_{beam}k}{(k + k_{beam})\,m}} = \sqrt{\frac{\frac{1}{x_b}\frac{1}{\delta_s}}{\left(\frac{1}{x_b} + \frac{1}{\delta_s}\right)}g} = 26.5 \text{ rad/s}$$

$$\text{(b)} \qquad \omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{g}{x_b}} = 70.0 \text{ rad/s}$$

$$\text{(b)} \qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_s}} = 28.6 \text{ rad/s}.$$

Note that l is needed to calculate k_{beam}.

18. A solid cylinder floating in equilibrium in a liquid of specific gravity γ is depressed slightly and released into motion. A schematic is shown in Figure 2.45. Find the equilibrium position and solve for the natural frequency of oscillation assuming the cylinder remains upright at all times. Suppose the assumption that the cylinder remains upright is not reasonable. Then what difficulties do you foresee in the calculations, and how might they be resolved?

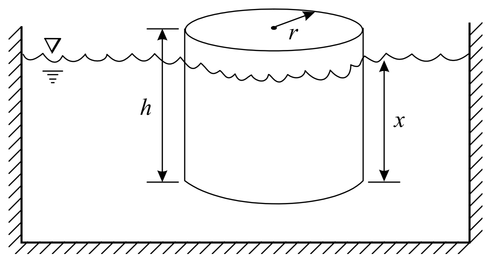


Figure 2.45: Solid body oscillating in a liquid.

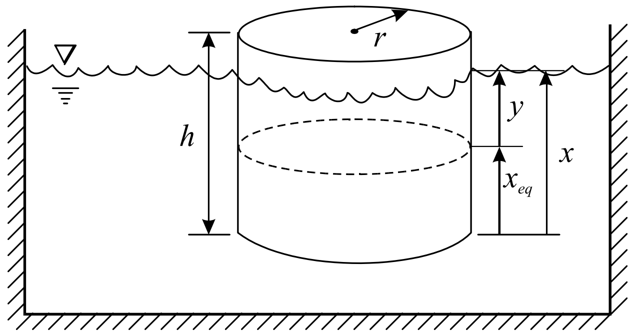
Solution: We know from Archimedes' principle that for a body immersed in a fluid, a buoyancy force acts in the opposite direction of gravity with a magnitude equal to the product of displaced volume and mass density of the fluid. For water this mass density equals $\rho_{water} = 64$ lb/ft^3. The displaced volume equals $\pi r^2 x$, where x is the length of the cylinder that is immersed and r is the radius of the cylinder. We assume the cylinder remains vertical as it oscillates. The mass of the cylinder is $\pi r^2 h\rho$, where h is the length of the cylinder and ρ is the mass density of the cylinder. The equilibrium position x_{eq} is given by equating the buoyancy force to the weight of the cylinder:

$$\begin{aligned}
F_{buoyancy} &= W_{cylinder} \\
\pi r^2 x_{eq}\rho_{water} g &= (\pi r^2 h\rho)g \\
x_{eq} &= \frac{h\rho}{\rho_{water}} = \gamma h.
\end{aligned}$$

The forces in the vertical direction are those due to gravity: the weight of the cylinder and the buoyancy force. If the oscillation is taken about the equilibrium position, weight can be ignored. The buoyancy force acts like a spring force. Newton's second law of motion in the vertical direction is given by

$$\begin{aligned}
\sum F_{vertical} &= m_{cylinder} a \\
-\pi r^2 y\rho_{water} g &= \pi r^2 h\rho\ddot{x},
\end{aligned}$$

where y is the displacement measured from the static (equilibrium) position so that $x = y + x_{eq}$. The figure below is referred to for the definitions of x, x_{eq}, and y.



In standard form this equation of motion becomes

$$h\rho\ddot{y} + \rho_{water} g y = 0,$$

or

$$\ddot{y} + \frac{\rho_{water} g}{h\rho} y = 0,$$
$$\ddot{y} + \frac{g}{\gamma\rho} y = 0,$$

where the natural frequency is given by

$$\omega_n = \sqrt{\frac{g}{\gamma\rho}}.$$

The specific gravity γ is the ratio of the density of a fluid to the density of water. If the cylinder does not remain vertical during oscillation, then in addition to vertical oscillations there will be rotational oscillations and the restoring force will be more complicated to evaluate.

19. The uniform rod is restrained by four translational springs and a torsional spring as shown in Figure 2.46. Determine the natural frequency of the system using the energy approach.

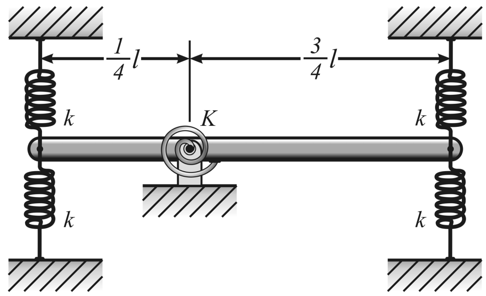


Figure 2.46: A rigid rod supported by four translational springs and a torsional spring.

Solution: The kinetic energy is given by

$$T = \frac{1}{2} I_O \dot{\theta}^2,$$

where O is where the rod is attached to the hinge, and $I_O = I_G + md^2$, where in this case, $I_G = ml^2/12$ and $d = l/4$. Then,

$$I_O = \frac{7}{48} ml^2.$$

The potential energies are stored in the four translational springs and the torsional spring. The total potential energy is given by

$$\begin{aligned} V &= k\left(\frac{l}{4}\theta\right)^2 + k\left(\frac{3l}{4}\theta\right)^2 + \frac{1}{2}K\theta^2 \\ &= \frac{5}{8}kl^2\theta^2 + \frac{1}{2}K\theta^2. \end{aligned}$$

Note that gravitational potential energy can be omitted if θ is measured from the static equilibrium. The total energy is $E = T + V$, and it is conserved. Taking a derivative with respect to time, we have

$$\frac{dE}{dt} = 0 = \frac{7}{48} ml^2 \dot{\theta}\ddot{\theta} + \frac{10}{8} kl^2 \theta\dot{\theta} + K\theta\dot{\theta}.$$

Dividing by $\dot{\theta}$, we have the equation of motion,

$$\frac{7}{48} ml^2 \ddot{\theta} + \left(\frac{5}{4}kl^2 + K\right)\theta = 0.$$

The natural frequency is

$$\omega_n = \sqrt{\frac{\left(\frac{5}{4}kl^2 + K\right)}{\frac{7}{48}ml^2}}.$$

20. Derive Equation 2.23,

$$x(t) = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t.$$

beginning with the equation of motion.

Solution: Begin with Equation 2.20 with $F(t)$ set to zero:

$$\ddot{x} + \omega_n^2 x = 0.$$

The solution is given in the form,

$$x(t) = C_1 \sin\omega_n t + C_2 \cos\omega_n t.$$

We can evaluate the arbitrary constants by satisfying initial conditions: $x(0)$ and $\dot{x}(0)$ to find

$$\begin{aligned} x(0) &= C_2, \\ \dot{x}(0) &= C_1\omega_n. \end{aligned}$$

Therefore, $C_2 = x(0)$ and $C_1 = \dot{x}(0)/\omega_n$. Equation 2.23 is found:

$$x(t) = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t.$$

Problems for Section 2.3 – Undamped Free Vibration

21. Derive Equation 2.26,

$$x(t) = B_1 \cos \omega_n t + B_2 \sin \omega_n t,$$

beginning with the equation of motion.

Solution: The equation of motion is $\ddot{x} + \omega_n^2 x = F(t)$. Assume the general solution

$$x(t) = A \exp(rt).$$

Differentiate this function twice and substitute into the ordinary differential equation to find the characteristic equation

$$r^2 + \omega_n^2 = 0,$$

where the common factor $A \exp(rt)$ has been divided out of the expression. This provides us with two roots:

$$r_{1,2} = \pm i\omega_n.$$

Each of these is one solution to the governing ordinary differential equation. Therefore,

$$x(t) = A_1 \exp(r_1 t) + A_2 \exp(r_2 t).$$

We substitute the two roots in this expression, and proceed as in the text, where the rest of the derivation is provided before our goal: Equation 2.26.

22. Show that the period of free vibration of a load weighing W suspended from two parallel springs, as shown in Figure 2.47, is given by

$$T = 2\pi\sqrt{W/g(k_1 + k_2)},$$

and show that the equivalent stiffness is $k = k_1 + k_2$. Discuss the need to hang the weight asymmetrically, that is $a_1 \neq a_2$ if $k_1 \neq k_2$, so that the extension of the springs is identical and that the ratio $a_1/a_2 = k_2/k_1$.

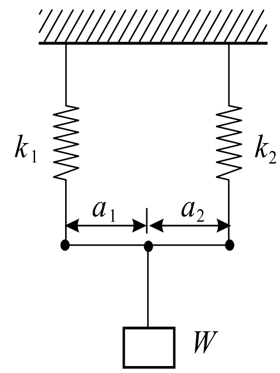
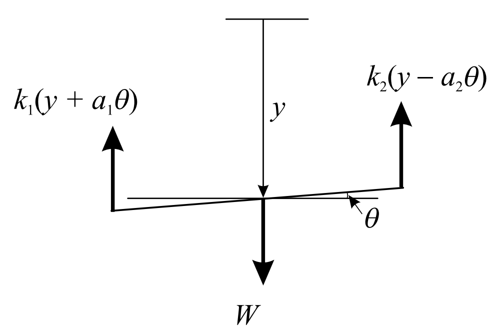


Figure 2.47: Weight hanging from two parallel springs.

Solution: Let y be the deflection of the mass defined positive downward, and let θ be the rotation of the rigid bar defined as positive in the counter clockwise direction. For positive y and θ, we draw the free-body diagram shown below.



Free-body diagram.

The deflection of the left spring is approximately $y + a_1\theta$ and of the right spring $y - a_2\theta$. We have arbitrarily assumed that rotation of the bar is counter clockwise resulting that the beam deflects more at the left side than the right side. If for particular parameter values this is not true, then the signs will change in the resulting expressions to reflect this. By Newton's second law of motion,

$$+\downarrow \sum F_y = \frac{W}{g}\ddot{y}$$
$$-k_1\left(y + a_1\theta\right) - k_2\left(y - a_2\theta\right) = \frac{W}{g}\ddot{y}.$$

If the weight is hung asymmetrically such that $a_1/a_2 = k_2/k_1$, then the equation of motion becomes

$$\begin{aligned} \frac{W}{g}\ddot{y} + (k_1 + k_2)y &= 0 \\ \ddot{y} + \frac{(k_1 + k_2)g}{W}y &= 0, \end{aligned}$$

where $\omega_n^2 = (k_1 + k_2)g/W$ and the period is then $T = 2\pi/\omega_n = 2\pi\sqrt{W/g(k_1 + k_2)}$. If the above assumptions were not made, then we would need a second equation of motion that governs θ.

23. For the body suspended between two springs as in Figure 2.48, show that the period of oscillation is $T = 2\pi\sqrt{W/g(k_1 + k_2)}$.

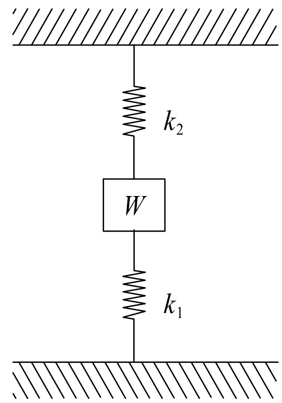
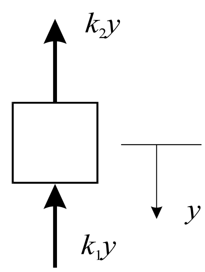


Figure 2.48: Body suspended between two springs.

Solution: Let y be the downward displacement of the body measured from the static position. The corresponding free-body diagram is shown below.



Applying Newton's second law of motion in the vertical y direction results in

$$-(k_1 + k_2)y = \frac{W}{g}\ddot{y}.$$

Note that gravity is omitted because y is measured from static equilibrium. Therefore, the equation of motion is

$$\ddot{y} + \frac{g(k_1 + k_2)}{W}y = 0.$$

The natural frequency ω_n and the period T are, respectively,

$$\omega_n = \sqrt{\frac{g(k_1 + k_2)}{W}}, \qquad T = \frac{2\pi}{\omega_n}.$$

24. A compound pendulum in the shape of a rectangle is supported at point O and allowed to oscillate. The dimensions of the rectangle are given in Figure 2.49. Calculate the natural frequency for small oscillations.

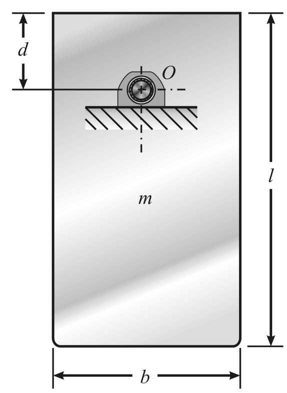


Figure 2.49: A compound pendulum

Solution: Use a free-body diagram based on a counter clockwise positive θ rotation. Taking a moment about O, we have

$$+\circlearrowleft \sum M_O = I_O\ddot{\theta}$$
$$-mg\left(\frac{l}{2}-d\right)\sin\theta = I_O\ddot{\theta},$$

where I_O is the mass moment of inertia of the plate about O given by

$$\begin{aligned} I_O &= I_G + m\overline{OG}^2 \\ &= \frac{1}{12}m\left(l^2+b^2\right) + m\left(\frac{l}{2}-d\right)^2, \end{aligned}$$

where G is the center of mass. The equation of motion is given by

$$I_O\ddot{\theta} + mg\left(\frac{l}{2}-d\right)\sin\theta = 0,$$

which can be linearized for small rotations by replacing $\sin\theta$ by θ.
The natural frequency is then

$$\omega_n = \sqrt{\frac{mg\left(\frac{l}{2}-d\right)}{I_O}} = \sqrt{\frac{mg\left(\frac{l}{2}-d\right)}{\frac{1}{12}m\left(l^2+b^2\right)+m\left(\frac{l}{2}-d\right)^2}}.$$

25. Two springs are joined in series as shown in Figure 2.50. If these are to be replaced by an equivalent spring, find the equivalent stiffness as well as the period of oscillation. The solution is

$$k = \frac{k_1 k_2}{k_1 + k_2}, \quad T = 2\pi\sqrt{\frac{W}{g}\frac{(k_1 + k_2)}{k_1 k_2}}.$$

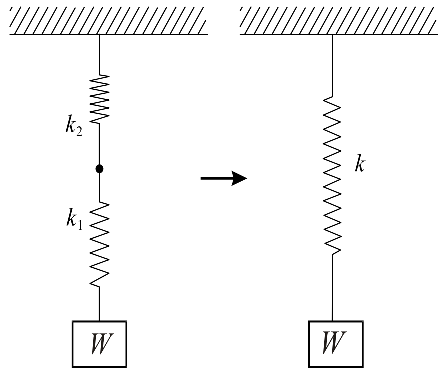
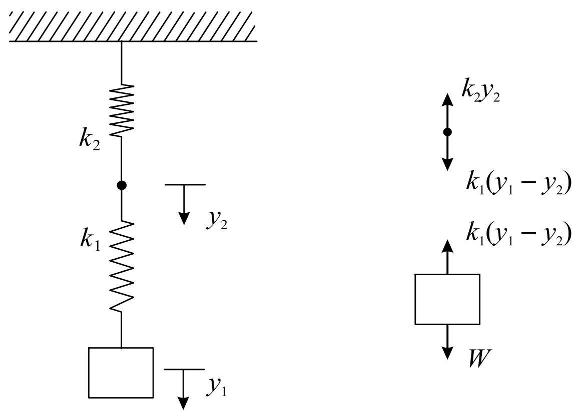


Figure 2.50: Two springs in series at left.

Solution: We need to find the equivalent stiffness of two springs in series. To do this attach a second coordinate to the point where the two springs are attached to each other. Call the deflection of this location, y_2, and call the displacement of the weight y_1 as shown below.



To find the equivalent stiffness we consider the static equilibrium problem. From a free-body of W and the lower spring, $W = k_1(y_1 - y_2)$. From a free-body of the upper spring, we have $W = k_2(y_2)$. Since y_2 is an intermediate displacement, we can eliminate y_2 using the second equation. Solving for y_2 and substituting it into the first equation we find

$$\begin{aligned} W &= k_1(y_1 - y_2) \\ &= k_1(y_1 - \frac{W}{k_2}); \end{aligned}$$

or

$$\begin{aligned} W(1 + \frac{k_1}{k_2}) &= k_1 y_1 \\ W(\frac{k_1 + k_2}{k_2}) &= k_1 y_1 \\ W &= \frac{k_1 k_2}{k_1 + k_2} y_1. \end{aligned}$$

The equivalent stiffness is then $k_1 k_2/(k_1 + k_2)$. The equation of motion is

$$\frac{W}{g}\ddot{y}_1 + \frac{k_1 k_2}{k_1 + k_2} y_1 = 0,$$

The natural frequency and period are

$$\omega_n = \sqrt{\frac{k_1 k_2 g}{(k_1 + k_2)W}}, \quad T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{(k_1 + k_2)W}{k_1 k_2 g}}.$$

26. A pendulum of mass m and mass moment of inertia I_O is suspended from a hinge, as shown in Figure 2.51. The center of gravity O is located a distance h from the hinge. For small oscillations, the period equals T. Suppose the pendulum swings with an amplitude of angle α from the vertical, find the force exerted on the hinge when the pendulum is (a) at its maximum rotation and (b) at its vertical position.

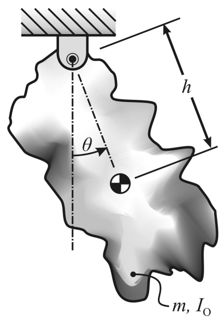


Figure 2.51: Compound pendulum rotating about pinned connection.

Solution: *** In the following G is used for the center of gravity and O for the point of contact.***

(a) Taking the sum of the moments in a free-body diagram about the point of contact, we find the equation of motion given by

$$I_O\ddot{\theta} + mgh\sin\theta = 0.$$

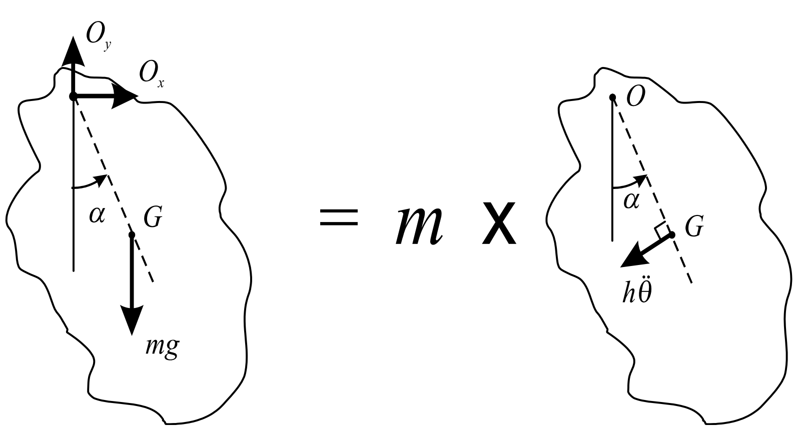
The linearized equation is

$$I_O\ddot{\theta} + mgh\theta = 0$$

The problem states that for small oscillations the period is T. That is,

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_O}{mgh}},$$

which can be used to calculate I_O of the compound pendulum if m and h are also known.
At $\theta = \alpha$, the acceleration of the pendulum is $\dot{v}$ or $h\ddot{\theta}$ in the tangential direction toward the equilibrium position.



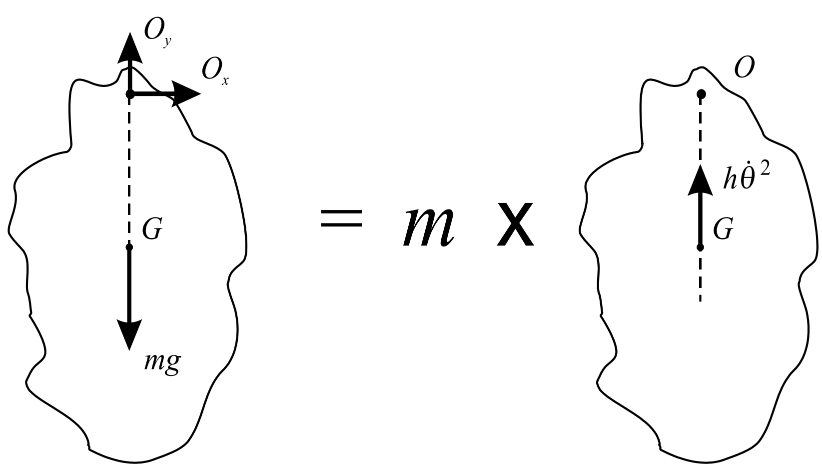
$$\sum \vec{F} = m\vec{a} \text{ when } \theta = \alpha.$$

From the equation of motion, we know $\ddot{\theta}\,(\theta = \alpha) = -mgh\alpha/I_O$. We write

$$\begin{aligned}
\text{At } \theta = \pm\alpha \qquad \sum \vec{F} &= m\vec{a}\,(\theta = \alpha) = mh\ddot{\theta}\,(-\cos\alpha\hat{\imath} - \sin\alpha\hat{\jmath}) \\
\vec{F}_O - mg\hat{\jmath} &= mh\left(\frac{mgh\alpha}{I_O}\right)(\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}) \\
\vec{F}_O\,(\theta = \alpha) &= \frac{m^2h^2g\alpha}{I_O}\cos\alpha\hat{\imath} + \left(\frac{m^2h^2g\alpha}{I_O}\sin\alpha + mg\right)\hat{\jmath}.
\end{aligned}$$

(Answer to part (a))

(b) When the pendulum passes through the vertical position, the acceleration is $h\dot{\theta}^2$ in the normal direction only.



$$\sum \vec{F} = m\vec{a} \text{ when } \theta = 0.$$

Summing the forces at these locations, we have

$$\text{At } \theta = 0 \qquad \sum \vec{F} = m\frac{v^2}{h}\hat{\jmath}.$$

The normal velocity at $\theta = 0$ is obtained from the conservation of total energy. At $\theta = 0$, the total energy at $\theta = \alpha$ is converted to the kinetic energy. That is

$$mg\,(h - h\cos\alpha) = \frac{1}{2}mv^2.$$

The velocity squared is then

$$mv^2\,(\theta = 0) = 2mgh\,(1 - \cos\alpha)\,.$$

The force balance equation is then

$$\begin{aligned}\sum \vec{F} &= m\frac{v^2}{\rho}\hat{\jmath}\\ \vec{F}_O - mg\hat{\jmath} &= \frac{1}{h}\left(2mgh\left(1-\cos\alpha\right)\right)\hat{\jmath}.\end{aligned}$$

The reaction force when $\theta = 0$ is

$$\vec{F}_O\left(\theta = 0\right) = \left(3mg - 2mg\cos\alpha\right)\hat{\jmath}. \qquad \text{(Answer to part (b))}$$

27. A *bifilar* pendulum of length $2a$ is suspended with two vertical strings, each of length l, as shown in Figure 2.52. (Bifilar means fitted with or involving the use of two threads or wires.) Assuming small rotations of the strings, such that the bar is essentially horizontal with half its weight supported by each string, show that the period is given by $T = \frac{2\pi a}{b}\sqrt{l/3g}$.

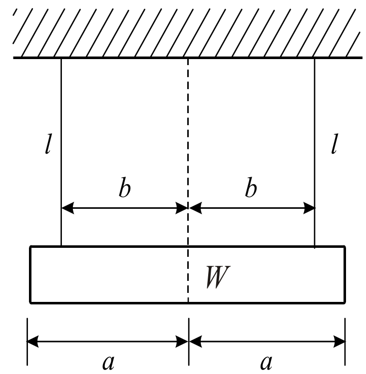
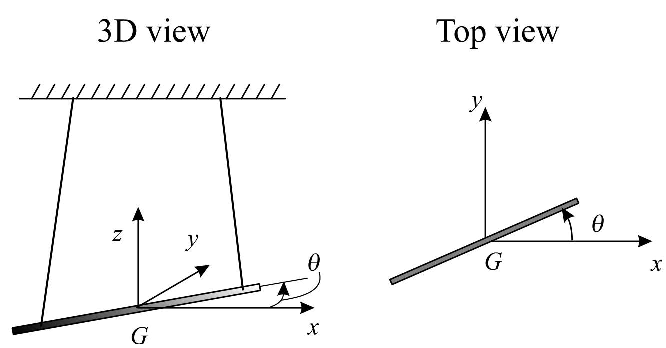


Figure 2.52: A *bifilar* compound pendulum.

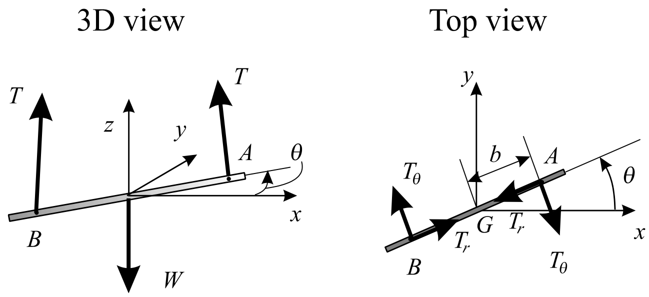
Solution: Note that if we assume that the bar moves as a pendulum in-plane, then the equation of motion will be the same as that for a simple pendulum. Here, we take the motion to be out of plane as shown below.



Out-of-plane oscillation of a bifilar pendulum.

Let us visualize the motion: the bar begins to rotate about its centerline, and as it rotates its ends also raise. However, we assume that this raising can be neglected if the angle of rotation out of the plane of the paper θ is small. By neglecting the motion in the z direction, we can also approximate the tension in the each string as $mg/2$. For a small angle of rotation, the tension in each string develops a restoring moment (like a spring).

From the free-body diagram below



the restoring moment about the z axis due to the tension in the each string is $T_\theta b$. The component of the tension perpendicular to the rod in the x-y plane is approximately

$$T_\theta = Tb\theta/l = mgb\theta/2l$$

and the restoring moment due to the two strings is $mgb^2\theta/l$. The restoring moment is set equal to the moment of inertia of beam about its center of mass (and rotation) I_G times the angular acceleration $\ddot{\theta}$:

$$\begin{aligned} I_G\ddot{\theta} &= \frac{-mgb^2}{l}\theta \\ \frac{ma^2}{3}\ddot{\theta} + \frac{mgb^2}{l}\theta &= 0, \end{aligned}$$

where we have used $I_G = \frac{1}{12}m(2a)^2$. Simplifying the equation of motion,

$$\begin{aligned} \ddot{\theta} + \frac{b^2}{a^2}\frac{3g}{l}\theta &= 0 \\ \ddot{\theta} + \omega_n^2\,\theta &= 0, \end{aligned}$$

where $\omega_n^2 = 3b^2g/a^2l$. The period is given by $T = 2\pi/\omega_n$, which equals the given expression.
Note that using vector notation may be more convenient. Taking the moment about the z axis through the center of mass, we have, (sum of the moments equals the time rate of change of the angular momentum)

$$\sum \vec{M}_G = \frac{d}{dt}\vec{H}_G,$$

where the sum of the moments is given by

$$\sum \vec{M}_G = \vec{r}_{B/G} \times \vec{T} + \vec{r}_{A/G} \times \vec{T},$$

where

$$\begin{aligned} \vec{r}_{A/G} &= b\cos\theta\vec{i} + b\sin\theta\vec{j} \\ \vec{r}_{A/G} &= -b\cos\theta\vec{i} - b\sin\theta\vec{j} \\ \vec{T} &= T((b - b\cos\theta)\,\vec{i} - b\sin\theta\vec{j} + l\vec{k})/l. \end{aligned}$$

Considering the z component only, we obtain the same equation of motion,

$$-2T\frac{b^2}{l}\sin\theta = I_{CM}\ddot{\theta}.$$

28. An inverted hinged pendulum with a mass m at the top is suspended between two springs with constants k, as shown in Figure 2.53. The rod can be assumed rigid and massless, and in the vertical position the springs are not stretched. For small motion, the springs can be assumed to remain horizontal. Show that the period of oscillation is $T = 2\pi/\sqrt{2k/m - g/l}$.

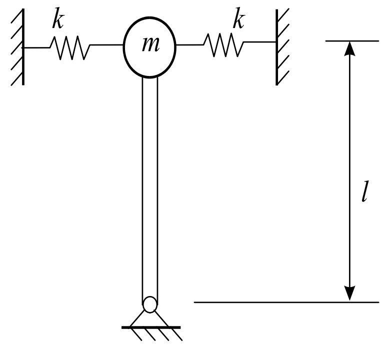
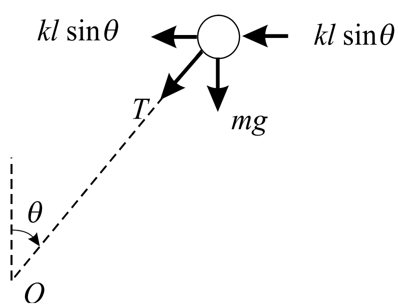


Figure 2.53: Inverted pendulum.

Solution: From the free-body diagram of the mass m, we see that a displacement to the right in the x direction results in forces in the opposite sense due to each spring of $-kl\sin\theta$ each.



Even though we assume the springs to be horizontal, the motion of the rigid beam with the mass at the end is a rotational motion about the hinge. Therefore, use Euler's equation,

$$+\circlearrowright \sum M_O = I_O\ddot{\theta},$$

where θ is the angle of the beam from the vertical and I_O equals the moment of inertia about the hinge: $I_O = I_G + ml^2 \simeq ml^2$ since I_G is very small when compared to ml^2. If you are wondering why the rod only transmits tension but not shear, it is because the mass is assumed to have no inertia about its own center of gravity. If the pendulum has mass moment of inertia, shear must be included.

Taking the moment about the hinge in the clockwise direction, we have:

$$\begin{aligned} -(2kl\sin\theta)l + (mg)l\sin\theta &= ml^2\ddot{\theta} \\ \ddot{\theta} + \frac{2kl^2}{ml^2}\sin\theta - \frac{g}{l}\sin\theta &= 0. \end{aligned}$$

Even though we have not explicitly assumed small angles so far, we have assumed small angles in reality when we assumed that the spring forces act horizontally. Therefore, the above equations are not yet valid until we make the substitution: $\sin\theta \simeq \theta$:

$$\begin{aligned}\ddot{\theta} + \frac{2kl^2}{ml^2}\theta - \frac{g}{l}\theta &= 0 \\ \ddot{\theta} + \left[\frac{2k}{m} - \frac{g}{l}\right]\theta &= 0,\end{aligned}$$

where

$$\omega_n^2 = \frac{2k}{m} - \frac{g}{l}.$$

The term in the brackets is the natural frequency squared for small oscillations ω_n^2, and the period is given by $T = 2\pi/\omega_n$.

29. Consider the inverted simple pendulum shown in Figure 2.54. Initially the pendulum is in an exact vertical position. If it is very slightly displaced from the vertical, what are the stability characteristics of the system? What kind of motion is expected and what is the equation of motion? Discuss the long-term behavior.

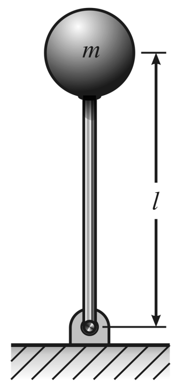


Figure 2.54: Inverted pendulum.

Solution: We can start with the known pendulum equation given by

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0,$$

which is based on Figure 2.32. For an inverted pendulum, we expand $\sin\theta$ about $\theta = \pi$ rad. The Taylor series of $\sin\theta$ about $\theta = \pi$ rad is $-\sin\theta$ (retaining only the first term). Then, the governing equation becomes

$$\ddot{\theta} - \frac{g}{l}\theta = 0.$$

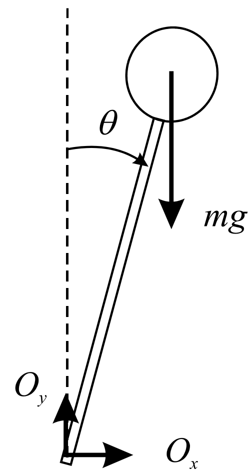
The solution is given by

$$\theta(t) = C_1 \exp\left(\sqrt{\frac{g}{l}}t\right) + C_1 \exp\left(-\sqrt{\frac{g}{l}}t\right).$$

The response is unstable about the equilibrium when it is in the upright position. We cannot discuss a long term behavior based on the linearized model because the model is valid only at the upright position.

However, from energy conservation – there is no damping in the system, we know that the pendulum will not be able to reach a height higher than initial height. If the pendulum is released from 5° from the vertical with zero initial velocity, it will oscillate between 5° and 355° as measured from vertical.

We can also derive the equation of motion based on an inverted pendulum. Let θ be measured from vertical this time as shown below.



Taking moments about O in the positive clockwise direction, we obtain

$$
\begin{aligned}
+\circlearrowright \sum M_O &= mgl\sin\theta \\
I_O\ddot{\theta} &= mgl\sin\theta \\
\ddot{\theta} - \frac{g}{l}\sin\theta &= 0.
\end{aligned}
$$

In order to investigate the behavior near $\theta = 0$, we expand $\sin\theta$ near $\theta = 0$. The one-term Taylor series of $\sin\theta$ near $\theta = 0$ is θ. Then, the equation of motion becomes

$$
\ddot{\theta} - \frac{g}{l}\theta = 0,
$$

which is identical to the linearized equation of motion obtained based on Figure 2.32.

30. For the system of Figure 2.55, what should be the value of k_3 if $k_1 = 2k_2 = 3k_3$, for a period of free vibration of 400 ms for $m = 2.5$ kg?

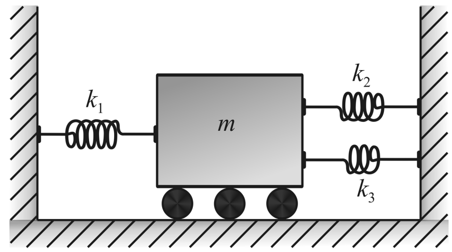


Figure 2.55: Mass constrained by several springs.

Solution: Springs k_1, k_2 and k_3 are in parallel. This becomes more obvious when we consider what happens when the mass moves a distance x. The corresponding spring force is $k_1x + k_2x + k_3x$ in the direction opposite to the assumed displacement.
The equivalent stiffness is

$$\begin{aligned} k_{eq} &= k_1 + k_2 + k_3 \\ &= 3k_3 + \frac{3}{2}k_3 + k_3 \\ &= 5.5k_3. \end{aligned}$$

The natural period is

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k_{eq}}}.$$

For $T = 0.4$ s and $m = 2.5$ kg, $k_{eq} = 616.23$ N/m. Then, $k_3 = k_{eq}/5.5 = 112.04$ N/m.

31. A slender bar of mass m and length l is supported at its base by a torsional spring of stiffness K, as per Figure 2.56. The bar rests in the vertical position when in equilibrium with the spring not stretched. Show that the following are true: The differential equation of the rotation θ from equilibrium is given by

$$\frac{ml^2}{3}\ddot{\theta} + K\theta - \frac{mgl}{2}\sin\theta = 0.$$

For small vibration, that is, $\theta \ll 1$, show that the natural frequency is given by

$$\omega_n = \frac{(k - mgl/2)}{ml^2/3}.$$

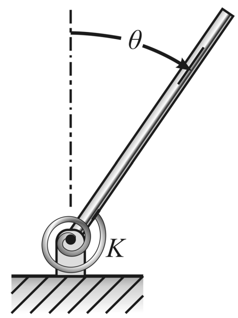
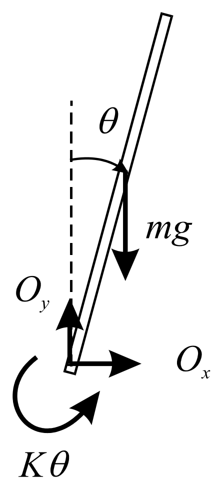


Figure 2.56: A slender bar of mass is supported at its base by a torsional spring.

Solution: Draw the free-body diagram first.



Take moments about point O to obtain

$$+\circlearrowright \sum M_O = I_O\ddot{\theta}$$
$$mg\frac{l}{2}\sin\theta - K\theta = \frac{ml^3}{3}\ddot{\theta}.$$

We assume small rotations ($\theta^2 \ll 1$) so that $\sin\theta$ can be approximated as θ. The linearized equation of motion is

$$\frac{ml^3}{3}\ddot{\theta} + \left(K - mg\frac{l}{2}\right)\theta = 0.$$

The natural frequency is

$$\omega_n = \sqrt{\frac{(K - mgl/2)}{ml^3/3}}.$$

Note that K must be greater than $mgl/2$ for the system to be stable, that is, so that the spring is sufficiently stiff to balance out the torque applied by the mass of the rod.

32. A bar supported by a hinge at its base is held in place by a spring connected to a collar, as per Figure 2.57. The spring is not stretched when the bar is vertical. As the bar is displaced from equilibrium by an angle θ, the collar slides on the frictionless bar so that the spring remains horizontal. Derive the governing equation of motion using an energy method, and then reduce the model for small amplitude motion. Identify the frequency of oscillation.

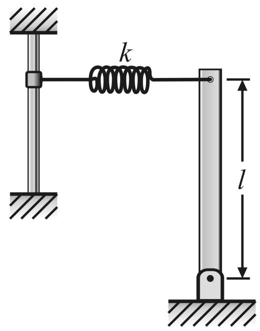


Figure 2.57: An inverted bar is held in place by a spring connected to a collar.

Solution: The kinetic energy is given by

$$T = \frac{1}{2} I_O \dot{\theta}^2,$$

where $I_O = ml^2/3$ where O is the hinge where the bar is supported. The potential energy is given by

$$V = mg\frac{l}{2}\cos\theta + \frac{1}{2}k\left(l\sin\theta\right)^2.$$

The total energy is constant and is given by

$$E = T + V = \frac{1}{2} I_O \dot{\theta}^2 + mg\frac{l}{2}\cos\theta + \frac{1}{2}k\left(l\sin\theta\right)^2.$$

Take the derivative with respect to time (not forgetting the chain rule) to obtain

$$0 = I_O\ddot{\theta}\theta - mg\frac{l}{2}\sin\theta\dot{\theta} + k\left(l\sin\theta\right)l\cos\theta\dot{\theta}.$$

Divide by $\dot{\theta} \neq 0$ to obtain the equation of motion,

$$I_O\ddot{\theta} - mg\frac{l}{2}\sin\theta + kl^2\sin\theta\cos\theta = 0.$$

We assume small angles ($\theta^2 \ll 1$) so that $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. The linearized equation of motion is then

$$I_O\ddot{\theta} + \left(kl^2 - mg\frac{l}{2}\right)\theta = 0.$$

If $k > mg/2l$, the natural frequency is

$$\omega_n = \sqrt{\frac{kl - mg/2}{ml/3}}.$$

33. Solve the previous problem for the initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = \dot{\theta}_0$ for two cases of the stiffness:

$$k > \frac{mg}{2l} \quad \text{and} \quad k < \frac{mg}{2l}.$$

What is the solution for $k = mg/2l$?

Solution: If $k > mg/2l$, the equivalent stiffness of the linearized system is positive, and $\theta\,(t)$ will oscillate about the static equilibrium position (vertical position). For the given initial conditions, the response is

$$\theta\,(t) = \frac{\dot{\theta}_0}{\omega_n} \sin \omega_n t,$$

where

$$\omega_n = \sqrt{\frac{kl - mg/2}{ml/3}}.$$

If $k < mg/2l$, the equivalent stiffness is negative, and the response will grow with time. The solution near $\theta = 0$ is given by

$$\theta\,(t) = C_1 \exp\left(\sqrt{\frac{mg/2 - kl}{ml/3}}t\right) + C_2 \exp\left(-\sqrt{\frac{mg/2 - kl}{ml/3}}t\right).$$

Upon substituting the initial conditions, we obtain

$$C_1 = \frac{\dot{\theta}_0}{2}\sqrt{\frac{ml/3}{mg/2 - kl}} \text{ and } C_2 = -\frac{\dot{\theta}_0}{2}\sqrt{\frac{ml/3}{mg/2 - kl}}.$$

The response is therefore

$$\theta\,(t) = \frac{\dot{\theta}_0}{2}\sqrt{\frac{ml/3}{mg/2 - kl}} \exp\left(\sqrt{\frac{mg/2 - kl}{ml/3}}t\right) - \frac{\dot{\theta}_0}{2}\sqrt{\frac{ml/3}{mg/2 - kl}} \exp\left(-\sqrt{\frac{mg/2 - kl}{ml/3}}t\right).$$

We must keep in mind that this solution is good for small θ only ($\theta \ll 1$).

34. A simple pendulum of initial length l_0 and initial angle θ_0 is released from rest. If the length is a function of time according to $l = l_0 + \varepsilon t$, find the position (l, θ) of the pendulum at any time assuming small oscillations. (Hint: This problem requires the use of Bessel functions.) The governing equation of motion will turn out to be

$$(l_0 + \varepsilon)\ddot{\theta} + 2\varepsilon\dot{\theta} + g\theta = 0,$$

and in transformed Bessel form,

$$x^2\theta'' + 2x\theta' + \frac{xg}{\varepsilon^2}\theta = 0,$$

where $x = l_0 + \varepsilon t$.

Solution: Let us first derive the equation of motion shown. We use Newton's second law. The energy method using the total energy will not give the right answer because this is a *non-natural* system (a system with prescribed motion).
Summing the moments about the support point O,

$$\frac{d}{dt}(H_O) + mg(l_0 + \varepsilon t)\sin\theta = 0,$$

where $H_O = I_O\dot{\theta}$ and $I_O = m(l_0 + \varepsilon t)^2$. The moment balance equation becomes

$$m(l_0 + \varepsilon t)^2\ddot{\theta} + 2m\varepsilon(l_0 + \varepsilon t)\dot{\theta} + mg(l_0 + \varepsilon t)\sin\theta = 0$$

Divide this equation by $m(l_0 + \varepsilon t)$ and assume that a small angle of rotation will result in the equation of motion in the problem statement: $(l_0 + \varepsilon t)\ddot{\theta} + 2\varepsilon\dot{\theta} + g\theta = 0$. This equation can be transformed into a Bessel equation by change of variables.
Let us start with

$$(l_0 + \varepsilon t)^2\ddot{\theta} + 2\varepsilon(l_0 + \varepsilon t)\dot{\theta} + g(l_0 + \varepsilon t)\theta = 0.$$

Let $x = l_0 + \varepsilon t$, and the derivatives are given by

$$\frac{d\theta}{dx} = \frac{d\theta}{dx}\frac{dx}{dt} = \varepsilon\frac{d\theta}{dx} \text{ and } \frac{d^2\theta}{dt^2} = \frac{d}{dx}\left(\frac{d\theta}{dx}\frac{dx}{dt}\right)\frac{dx}{dt} = \varepsilon^2\frac{d^2\theta}{dx^2}.$$

Then,

$$\begin{aligned} x^2\varepsilon^2\theta'' + 2\varepsilon^2 x\theta' + gx\theta &= 0 \\ x^2\theta'' + 2x\theta' + \frac{g}{\varepsilon^2}x\theta &= 0, \end{aligned}$$

which is the second equation in the problem statement.
From the theory of Bessel functions, the solution of differential equation

$$x^2y'' + (2k+1)xy' + (\alpha^2x^{2r} + \beta^2)y = 0,$$

where k, α, r, β are constants, is the general equation

$$y = x^{-k}\left[c_1 J_{\kappa/r}(\alpha x^r/r) + c_2 Y_{\kappa/r}(\alpha x^r/r)\right],$$

where $\kappa = \sqrt{k^2 - \beta^2}$. In the above problem, we have $2k + 1 = 2$, $\beta = 0, r = 1/2, \alpha^2 = g/\varepsilon^2, \kappa = k = 1/2$. Making these substitutions, our solution for θ becomes

$$\begin{aligned}
\theta &= x^{-1/2}\left[AJ_1(2\frac{\sqrt{g}}{\varepsilon}x^{1/2}) + BY_1(2\frac{\sqrt{g}}{\varepsilon}x^{1/2})\right] \\
\theta &= (l_0 + \varepsilon t)^{-1/2}\left[AJ_1(2\frac{\sqrt{g}}{\varepsilon}(l_0 + \varepsilon t)^{1/2}) + BY_1(2\frac{\sqrt{g}}{\varepsilon}(l_0 + \varepsilon t)^{1/2})\right].
\end{aligned}$$

The first initial condition is that $\theta(t = 0) = \theta_0$. The second initial condition is that $\dot{\theta}(t = 0) = 0$. Therefore,

$$\theta_0 = (l_0)^{-1/2}\left[AJ_1(2\frac{\sqrt{gl_0}}{\varepsilon}) + BY_1(2\frac{\sqrt{gl_0}}{\varepsilon})\right]$$

and

$$\begin{aligned}
\dot{\theta}(t) &= -\frac{\varepsilon}{2(l_0 + \varepsilon t)^{3/2}}\left[AJ_1(2\frac{\sqrt{g}}{\varepsilon}(l_0 + \varepsilon t)^{1/2}) + BY_1(2\frac{\sqrt{g}}{\varepsilon}(l_0 + \varepsilon t)^{1/2})\right] \\
&\quad + \frac{\sqrt{g}}{l_0 + \varepsilon t}\left[AJ_1'(2\frac{\sqrt{g}}{\varepsilon}(l_0 + \varepsilon t)^{1/2}) + BY_1'(2\frac{\sqrt{g}}{\varepsilon}(l_0 + \varepsilon t)^{1/2})\right] \\
\dot{\theta}(0) &= 0 = -\frac{\varepsilon}{2(l_0)^{3/2}}\left[AJ_1(2\frac{\sqrt{gl_0}}{\varepsilon}) + BY_1(2\frac{\sqrt{gl_0}}{\varepsilon})\right] \\
&\quad + \frac{\sqrt{g}}{l_0}\left[AJ_1'(2\frac{\sqrt{gl_0}}{\varepsilon}) + BY_1'(2\frac{\sqrt{gl_0}}{\varepsilon})\right] \\
&= -\frac{\varepsilon}{2l_0}\theta_0 + \frac{\sqrt{g}}{l_0}\left[AJ_1'(2\frac{\sqrt{gl_0}}{\varepsilon}) + BY_1'(2\frac{\sqrt{gl_0}}{\varepsilon})\right]
\end{aligned}$$

We can use this last equation to find

$$\frac{\varepsilon\theta_0}{2\sqrt{g}} = AJ_1'(2\frac{\sqrt{gl_0}}{\varepsilon}) + BY_1'(2\frac{\sqrt{gl_0}}{\varepsilon}).$$

Solve this equation simultaneously with the equation for θ_0 above for A and B, (omit the argument $2\sqrt{gl_0}/\varepsilon$) :

$$\begin{aligned}
A &= \frac{\sqrt{l_0}Y_1' - (\varepsilon/(2\sqrt{g}))Y_1}{J_1Y_1' - Y_1J_1'}\theta_0 \\
B &= \frac{(\varepsilon/(2\sqrt{g}))J_1 - \sqrt{l_0}J_1'}{J_1Y_1' - Y_1J_1'}\theta_0.
\end{aligned}$$

Use the identity $J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = 2/(\pi x)$, for $n = 1$,

$$J_1Y_1' - J_1'Y1 = \frac{\varepsilon}{\pi\sqrt{gl_0}},$$

and then

$$\begin{aligned}
A &= \frac{\pi\sqrt{g}l_0\theta_0}{\varepsilon}Y_1' - \frac{\pi\sqrt{l_0}\theta_0}{2}Y_1 \\
B &= \frac{\pi\sqrt{l_0}\theta_0}{2}J_1 - \frac{\pi\sqrt{g}l_0\theta_0}{\varepsilon}J_1'.
\end{aligned}$$

From the identity $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$, and a similar identity for $Y_{n,}'$ for $n = 1$, we have

$$\begin{aligned}
A &= -\frac{\pi\sqrt{l_0}\theta_0}{2}Y_2(\frac{2\sqrt{gl_0}}{\varepsilon}) \\
B &= \frac{\pi\sqrt{l_0}\theta_0}{2}J_2(\frac{2\sqrt{gl_0}}{\varepsilon}).
\end{aligned}$$

The equation for θ becomes

$$\theta = \frac{\pi\sqrt{l_0}\theta_0}{2\sqrt{l_0+\varepsilon t}}\left[J_2(\frac{2\sqrt{gl_0}}{\varepsilon})Y_1(\frac{2\sqrt{gl}}{\varepsilon}\sqrt{l_0+\varepsilon t}) - Y_2(\frac{2\sqrt{gl_0}}{\varepsilon})J_1(\frac{2\sqrt{gl}}{\varepsilon}\sqrt{l_0+\varepsilon t})\right].$$

35. A homogeneous disk of weight W and radius r is supported by two identical cylindrical steel shafts of length l, as shown in Figure 2.58. From solid mechanics, the relation between the moment on the disk M and the angle of rotation of this disk θ is

$$M = \frac{GJ}{l}\theta,$$

where G is the shear modulus of the material, J is the polar moment of inertia of the cross-section of each shaft, and GJ is known as the torsional rigidity. Suppose this system is being designed for a particular application that requires the frequency of vibration to be f. Find the value of r in terms of the parameters to satisfy this requirement.

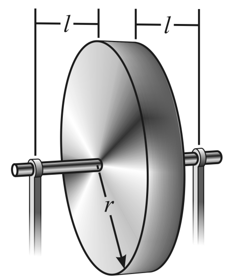


Figure 2.58: A homogeneous disk of weight is supported by two identical cylindrical steel shafts.

Solution: The mass moment of inertia of the disk about its center of gravity is $I_G = mr^2/2$ and the stiffness of the problem is $2GJ/l$. The equation of motion is given by

$$I_G\ddot{\theta} + 2\frac{GJ}{l}\theta = 0.$$

The natural frequency is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2GJ}{lI_G}}.$$

Writing I_G in terms of r, we have

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2GJ}{mr^2l/2}} = \frac{1}{\pi}\sqrt{\frac{GJ}{ml}}\frac{1}{r}.$$

The radius of the disk in terms of all the other parameters is then

$$r = \frac{1}{f_n}\frac{1}{\pi}\sqrt{\frac{GJ}{ml}}.$$

36. When a craft is sent into space it is necessary to know the mass properties of the system, including the moment of inertia of astronauts on the flight, in order to properly calculate its trajectory and fuel expenditures.
The device in Figure 2.59 is one way to measure an astronaut's moment of inertia. The horizontal platform is pinned at O and supported on the other end by a linear spring with constant k. When the astronaut is not present, the frequency of small vibration of the platform about O is measured to be f. When the astronaut is lying on the platform, the frequency of small vibration is measured to be f_a. (a) Find the astronaut's moment of inertia about the z axis assuming he or she is l m tall. (b) Assume reasonable parameter values for the system, with the astronaut of height 2.0 m, and calculate the moment of inertia.

Figure 2.59: System to measure astronaut's moment of inertia.

Solution: (a) In absence of the astronaut, the platform oscillates at frequency f. The equation of motion is given by

$$+\circlearrowleft \sum M_O = -M_p - kl^2\theta = I_{O,platform}\ddot{\theta},$$
$$I_{O,platform}\ddot{\theta} + kl^2\theta = -M_p$$

where M_p is the moment that the platform exerts due to its weight. This can be omitted if θ is measured from static equilibrium. The natural frequency is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{kl^2}{I_{O,platform}}}. \tag{1}$$

If the astronaut lies on the platform, the moment equation becomes

$$+\circlearrowleft \sum M_O = -M_p - M_a - kl^2\theta = \left(I_{O,platform} + I_{O,astronaut}\right)\ddot{\theta},$$

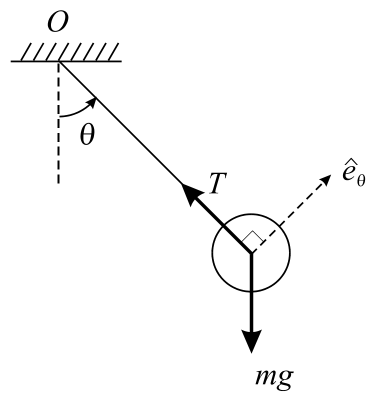
where M_a is the moment that the astronaut exerts due to his weight. This can again be omitted if θ is measured from static equilibrium. The natural frequency of the combined system is

$$f_a = \frac{1}{2\pi}\sqrt{\frac{kl^2}{I_{O,platform} + I_{O,astronaut}}}. \tag{2}$$

Assume that we know the spring constant, k, the length, l, and both frequencies, f_n and f_a, but we do not know the moments of inertia. Then, we first find the moment of inertia of the platform, $I_{O,platform}$, using Equation (1) and find $I_{O,astronaut}$ using Equation (2).
(b) Let us assume that $k = 10,000$ N/m, $f = 7.00$ Hz, and $f_a = 3.00$ Hz. Then, $I_{O,platform} = 20.7$ kg·m^2 and $I_{O,astronaut} = 91.9$ kg-m^2.

37. A small pendulum is mounted in a rocket that is accelerating at a rate of $4g$, as sketched in Figure 2.60. The pendulum is composed of a massless rod of length $l = 1$ m that supports a block of mass $m = 0.5$ kg. Assuming small oscillations, what is the rotational natural frequency of the pendulum?

Solution: Draw the free-body diagram.



Summing the forces in the transverse direction ($\hat{e}_\theta$), we have

$$\sum F_\theta = ma_\theta,$$

where the left-hand side of the equation is given by $\sum F_\theta = -mg\sin\theta$ and the right-hand side is given by $a_\theta = l\ddot{\theta} + 3g\sin\theta$. Then, the equation of motion becomes

$$\begin{aligned} m\left(l\ddot{\theta} + 3g\sin\theta\right) + mg\sin\theta &= 0 \\ \ddot{\theta} + 4\frac{g}{l}\sin\theta &= 0. \end{aligned}$$

For small rotation ($\theta^2 \ll 1$, $\sin\theta \simeq \theta$), we have

$$\ddot{\theta} + \frac{4g}{l}\theta = 0,$$

where the natural frequency is given by $\omega_n = 2\sqrt{g/l}$.

38. For the undamped rocker arm sketched in Figure 2.61, determine the natural frequency of the system undergoing small amplitude oscillation using Newton's second law of motion. Assume that the mass of the T-bar is negligible compared to the mass m of the block.

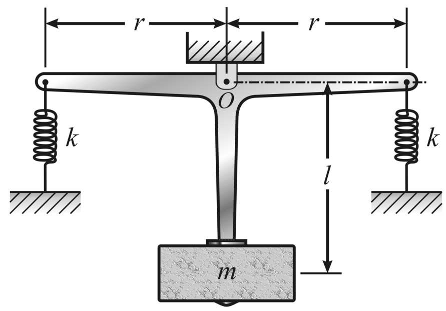
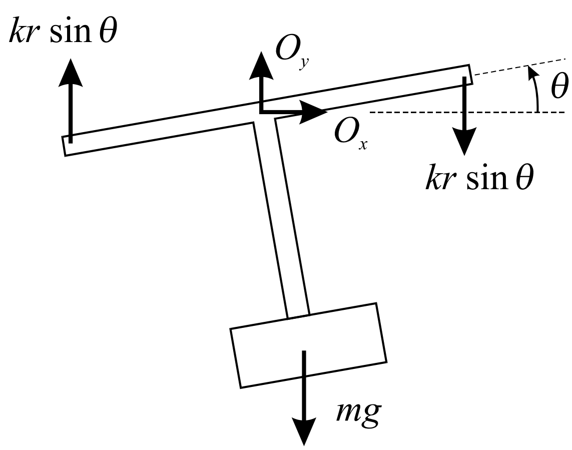


Figure 2.61: Vibrating rocker arm.

Solution: Draw a free-body diagram based on a rotation defined as positive in the counterclockwise direction.



The moment about O is given by

$$+\circlearrowleft \sum M_O = I_O \ddot{\theta},$$

where $I_O = ml^2$ and

$$+\circlearrowleft \sum M_O = -2kr \sin\theta \,(r\cos\theta) - mgl\sin\theta.$$

The equation of motion is then

$$ml^2\ddot{\theta} + 2kr^2 \sin\theta\cos\theta + mgl\sin\theta = 0.$$

Assuming small rotation $\left(\theta^2 << 1,\ \sin\theta \simeq \theta,\ \cos\theta \simeq 1\right)$, we obtain the linearized equation of motion:

$$ml^2\ddot{\theta} + \left(2kr^2 + mgl\right)\theta = 0.$$

The natural frequency is then $\omega_n = \sqrt{\left(2kr^2 + mgl\right)/ml^2}$.

39. For the bar system of Figure 2.62, derive the equation of motion for small oscillations about the horizontal equilibrium position assuming the support pin is frictionless. Based on the equation of motion, determine the effective mass and effective stiffness constant of the system.

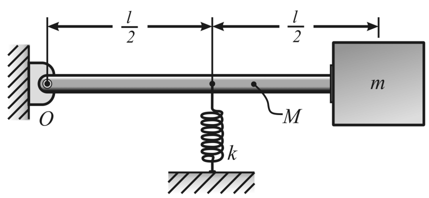
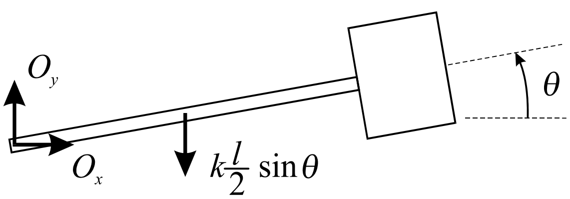


Figure 2.62: Pinned bar-spring system.

Solution: It is assume that the equilibrium position is when the bar is horizontal. We measure rotation from this static equilibrium and the gravity can be omitted from the problem. Draw a free-body diagram.



Summing the moments about O, we have

$$+\circlearrowleft \sum M_O = I_O\ddot{\theta},$$

where $I_O = \frac{1}{3}Ml^2 + ml^2$ and therefore

$$+\circlearrowleft \sum M_O = -k\left(\frac{l}{2}\right)^2 \sin\theta\cos\theta.$$

Assuming small rotations, the linearized equation of motion is given by

$$\left(\frac{1}{3}Ml^2 + ml^2\right)\ddot{\theta} + k\left(\frac{l}{2}\right)^2 \theta = 0.$$

Dividing the above equation by l and rewriting the equation of motion in terms of the vertical displacement of the block, $y = l\theta$, we obtain

$$\left(\frac{1}{3}M + m\right)\ddot{y} + \frac{k}{4}y = 0.$$

The equivalent stiffness is $k/4$ and the equivalent mass is $m + M/3$.

40. We consider a generalization of Example 2.9. A mass moves to the left with speed v on a platform, disconnected from two springs, as shown in Figure 2.63. Assuming that there is no friction, find the period of oscillation of the mass, given the data: $k_1 = 36$ N/cm, $k_2 = 18$ N/cm, $m = 25$ kg, $b = 30$ cm, $d = 100$ cm, and $v = 6$ m/s.

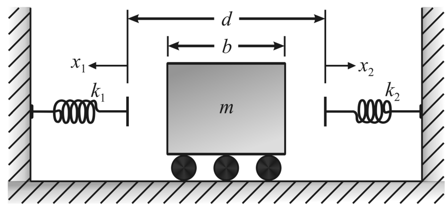


Figure 2.63: Mass disconnected from springs oscillating.

Solution: The mass moves to the left with a constant velocity of v. Let $t = 0$ be defined as when the mass comes into contact with k_1. The equation of motion is given by

$$m\ddot{x}_1 + k_1 x_1 = 0 \text{ with } x_1(0) = 0 \text{ and } \dot{x}_1(0) = v,$$

where x_1 is the position of the left face of the block defined positive to the left.
The solution to this equation is valid until $x_1(t)$ becomes zero again as the block leaves the spring on the left. The block is in contact with the spring on the left for half the normal period or $\pi/\sqrt{k_1/m}$. The block travels the distance of $(d - b)$ at velocity v to the right until it comes in contact with the spring on the right. The time of travel is $(d - b)/v$. The block is in contact with the second spring for half the normal period or $\pi/\sqrt{k_2/m}$.
After it loses contact on the right, it travels the distance $(d - b)$ at a constant velocity v to reach where the block reaches $x_1 = 0$. The period is therefore

$$\begin{aligned} T &= \frac{\pi}{\sqrt{k_1/m}} + \frac{d-b}{v} + \frac{\pi}{\sqrt{k_2/m}} + \frac{d-b}{v} \\ &= \frac{\pi}{\sqrt{36/25}} + \frac{0.7}{6} + \frac{\pi}{\sqrt{18/25}} + \frac{0.7}{6} \\ &= 6.55 \text{ s.} \end{aligned}$$

Note that we can write a single equation of motion in terms of x, defined from the middle:

$$m\ddot{x} + k(x)\,x = 0,$$

where

$$k(x) = \begin{cases} 0 \text{ for } & -\dfrac{d-b}{2} \le x \le \dfrac{d-b}{2} \\ k & \text{elsewhere.} \end{cases}$$

41. A spring-mass system is suspended from the ceiling. The governing equation of motion is

$$m\ddot{x} + kx = 0,$$

where x is measured from the static equilibrium position. Assume the initial conditions are $x(0) = 1$ in and $\dot{x}(0) = 1$ in/s. Solve for the response $x(t)$ for two cases: (a) the spring is assumed massless, and (b) the inertia effects of the spring are included.
Plot both solutions and comment on the importance of including the spring inertia. Consider three sets of parameter values: (a) $W/g = 1$ lb, $k = 1$ lb/in, (b) $W/g = 1$ slug, $k = 10$ lb/in, (c) $W/g = 1$ slug, $k = 0.1$ lb/in. What conclusions can be drawn?

Solution: Massless Spring:
The equation of motion is given by

$$\ddot{x} + \frac{k}{m}x = 0,$$

and its solution is given by

$$x(t) = B_1 \cos\omega_n t + B_2 \sin\omega_n t,$$

where $\omega_n = \sqrt{k/m}$. We use the initial conditions $x(0) = 1$ in, $\dot{x}(0) = 1$ in/s to find the constants: $B_1 = 1$, $B_2 = 1/\omega$. Thus, the solution to our differential equation is

$$x(t) = \cos\omega_n t + \frac{1}{\omega}\sin\omega_n t,$$

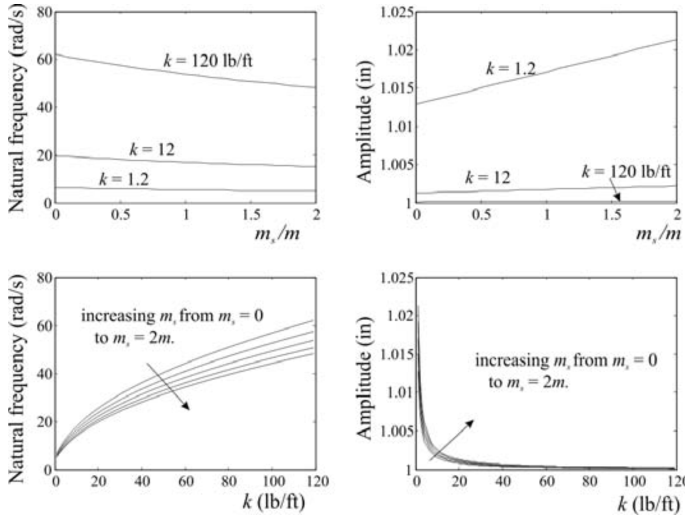
with ω_n defined as above. In the amplitude-phase form, we can write

$$x(t) = \sqrt{1 + \left(\frac{1}{\omega_n}\right)^2}\cos\left(\omega_n t - \phi\right).$$

Massive Spring:
The analysis including the mass of the spring is identical to that for the massless spring, except we replace m by $m_{eq} = m + \frac{m_s}{3}$, where m_s is the spring mass. The natural frequency is then $\omega_{eq} = \sqrt{k/m_{eq}}$, and the solution is given by $x(t) = \cos\omega_{eq}t + \frac{1}{\omega_{eq}}\sin\omega_{eq}t$. In amplitude-phase form, we have

$$x(t) = \sqrt{1 + \left(\frac{1}{\omega_{eq}}\right)^2}\cos\left(\omega_{eq}t - \phi_{eq}\right).$$

Note that the units of the mass and stiffness must be converted to slug and lb/ft to obtain the natural frequency in terms of rad/s. (The $\mathrm{lb_m}$ and slug are related by 1 slug = 32.2 $\mathrm{lb_m}$.) The figures below show the variation in the natural frequencies and the response amplitudes as functions of the spring mass and stiffness for $m = 1$ $\mathrm{lb_m}$.



The natural frequency and amplitude as functions of mass (top) and stiffness (bottom). The mass is kept at 1/32.2 slug.

42. Two systems with discontinuities are shown in Figure 2.64. Each system oscillates but at some distance $\pm x_c$, the mass comes into contact with springs. For each system, derive the equation of motion for a complete cycle.

Solution: (a) The equation of motion is given by

$$m\ddot{x} + k(x)\,x = 0,$$

where

$$k(x) = \begin{cases} 0 & \text{for } -x_c \leq x \leq x_c \\ k & \text{elsewhere.} \end{cases}$$

(b) The equation of motion for this system is

$$m\ddot{x} + k(x)\,x = 0,$$

where

$$k(x) = \begin{cases} 2k & \text{for } -x_c \leq x \leq x_c \\ 3k & \text{elsewhere.} \end{cases}$$

The easiest way to solve for the response and the period is to solve them in different segments.

Problems for Section 2.4 – Harmonic Forcing with No Damping

43. Derive Equation 2.33,

$$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t + \frac{x_{st}}{1-(\omega/\omega_n)^2}[\cos\omega t - \cos\omega_n t].$$

Solution: We begin with our known relationships, namely:

$$\begin{aligned} \ddot{x} + \omega_n^2 x &= F(t) \\ F(t) &= \frac{A}{m}\cos\omega t. \end{aligned}$$

The solution to this equation consists of two parts,

$$x(t) = x_h(t) + x_p(t),$$

the homogeneous and particular solutions, and they are given by

$$\begin{aligned} x_h(t) &= C_1\cos\omega_n t + C_2\sin\omega_n t \\ x_p(t) &= B_1\cos\omega t. \end{aligned}$$

The constant coefficients of integration, C_1 and C_2, are determined when the initial conditions are applied to the total solution. The coefficient B_1 in the particular solution must be such that the assumed solution, $x_p(t)$, satisfies the differential equation. Upon substituting the assumed solution into the differential equation we obtain

$$B_1\left(\omega_n^2 - \omega^2\right)\cos\omega t = \frac{A}{m}\cos\omega t.$$

Using the fact that $\omega_n = \sqrt{k/m}$, we find that

$$B_1 = \frac{A/k}{1-(\omega/\omega_n)^2}.$$

Note that $x_{st} = A/k$, the static response of the spring. Finally, using the complete solution for the response $x(t)$, we can satisfy the initial conditions, to find

$$\begin{aligned} C_1 &= x_0 - \frac{x_{st}}{1-(\omega/\omega_n)^2} \\ C_2 &= \frac{\dot{x}_0}{\omega_n}. \end{aligned}$$

Substituting C_1 and C_2 into our expression for $x(t)$ gives the solution,

$$x(t) = \left(x_0 - \frac{x_{st}}{1-(\omega/\omega_n)^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{x_{st}}{1-(\omega/\omega_n)^2}\cos\omega t,$$

that we are looking for: Equation 2.33.

44. Solve the governing equation of motion,

$$\ddot{x} + x = \frac{1}{m} F(t),$$

for four cases: (a) $F(t) = \cos 0.5t$, (b) $F(t) = \cos 0.99t$, (c) $F(t) = \cos t$, and (d) $F(t) = \cos 2t$. Draw comparisons between the results.

Solution: Each of these loading cases has a homogeneous response and a particular response. The homogeneous solution is given by $x_h(t) = C_1 \cos t + C_2 \sin t$. Note that the natural frequency equals 1 rad/s. We will only solve for the particular solution below.
(a,b,d) Assume the particular solution

$$x_p(t) = A \cos \omega_f t + B \sin \omega_f t,$$

where $F(t) = \cos \omega_f t$. Differentiating this assumed solution twice and substituting it into the governing differential equation of motion we find

$$\cos \omega_f t \left(-\omega_f^2 A + A\right) + \sin \omega_f t \left(-\omega_f^2 B + B\right) = \frac{1}{m} \cos \omega_f t.$$

This identity is actually two equations:

$$\begin{aligned} \left(1 - \omega_f^2\right) A &= 1 \\ \left(1 - \omega_f^2\right) B &= 0. \end{aligned}$$

Since $\omega_f \neq 0$, we find $B = 0$, and $A = 1/\left(1 - \omega_f^2\right)$. The particular response is then

$$x_p(t) = \frac{1}{\left(1 - \omega_f^2\right)} \cos \omega_f t.$$

The total solution is given by

$$x\left(t\right) = C_1 \cos t + C_2 \sin t + \frac{1}{m\left(1 - \omega_f^2\right)} \cos \omega_f t.$$

(c) In this problem, the loading is at the same frequency as the natural frequency of the system. Since the system is undamped, we know to expect oscillation with unbounded growth. Assume the particular solution

$$x_p(t) = At \cos t + Bt \sin t.$$

Differentiating this assumed solution twice and substituting it into the governing differential equation of motion we find

$$\cos t\,(2B) + \sin t\,(-2A) = \frac{1}{m} \cos t.$$

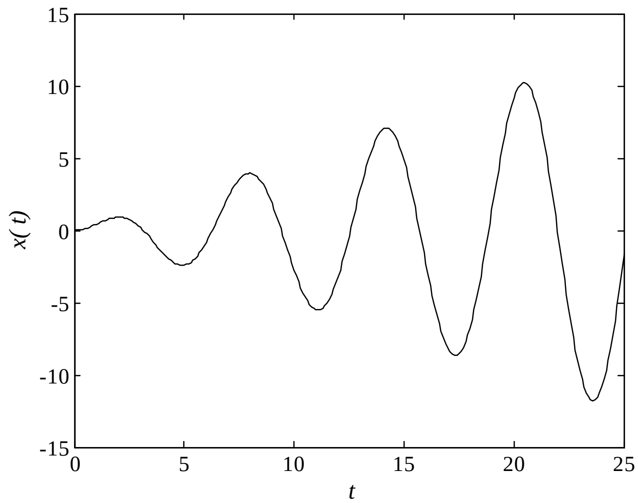
This identity is actually two equations: $2B = 1/m$ and $-2A = 0$, resulting in $A = 0$ and $B = 1/2m$. The particular solution is then

$$x_p(t) = (1/2m)\, t \sin t$$

and the total solution is

$$x(t) = C_1 \cos t + C_2 \sin t + \frac{1}{2m} t \sin t.$$

The following plot shows the unstable part of the response for $m = 1$.



Resonant response, $x(t) = 0.5t \sin t$.

It should be noted that, in many of the vibration texts, the transient and homogeneous solution are used synonymously, and the steady-state and the particular solution are used synonymously. This is true for a damped system because the homogeneous solution decays with time.
In this case, the homogeneous solution is the transient solution. However, in an undamped system, the homogeneous solution persists so that it is no longer the transient response (there is no transient response in an undamped system). This is not a gross error because there is no physical system that is truly undamped. Therefore, from an engineering perspective, the transient and homogeneous solution are the same, and likewise for the steady-state and the particular solutions.

45. For the oscillator that is beating according to the governing equation

$$\ddot{x} + 16x = \frac{A}{m}\sin(4-\varepsilon)t,$$

where ε is small but not equal to zero, what do you expect to happen as $\varepsilon \to 0$? Relate the rate of growth of the response amplitude to the value of ε or to the value of $4-\varepsilon$.

Solution: We know from our studies of resonance that a structure loaded harmonically by a force that oscillates at the natural frequency of the structure will oscillate with ever-increasing amplitudes. Therefore, even before analyzing the above governing differential equation, we know that for $\varepsilon = 0$, we have a resonance condition. Even for small ε, we will have large amplitude motion.
We can solve the differential equation for the forced response to find:

$$\begin{aligned} x(t) &= \frac{A}{m}\frac{1}{[4-\varepsilon]^2 - 16}\sin(4-\varepsilon)t \\ &+ C_1\cos 4t + C_2 \sin 4t. \end{aligned}$$

Applying the initial conditions, we find $C_1 = x_0$ and

$$C_2 = \frac{1}{4}\left(v_o - \frac{A}{m}\frac{4-\varepsilon}{[4-\varepsilon]^2-16}\right).$$

The total solution is given by

$$\begin{aligned} x(t) &= \underbrace{\frac{A}{m}\frac{1}{[4-\varepsilon]^2-16}\left(\sin(4-\varepsilon)t - \frac{4-\varepsilon}{4}\sin 4t\right)}_{\text{forced solution}} \\ &+ \underbrace{x_0\cos 4t + \frac{1}{4}v_o\sin 4t}_{\text{free solution}}. \end{aligned}$$

The free solution, the part of the solution due to the initial conditions, does not grow with time. In this current form we can see the term $-16 + [4-\varepsilon]^2$ in the denominator of the forced response goes to zero as $\varepsilon \to 0$, resulting in a response of unbounded amplitude given enough time. However, it does not show how the solution behaves in finite time.
In order to investigate the behavior of the solution in finite time, consider the term $\sin(4-\varepsilon)t$ in the forced solution. This term is expanded as $\cos\varepsilon t \sin 4t - \sin\varepsilon t\cos 4t$. The terms

$$\frac{\cos\varepsilon t}{[4-\varepsilon]^2-16}\sin 4t \qquad \text{and} \qquad \frac{\sin\varepsilon t}{[4-\varepsilon]^2-16}\cos 4t$$

describe sinusoidal functions that oscillate at 4 rad/s and with magnitudes that vary slowly with frequency ε rad/s. The magnitude of the envelop function is $1/([4-\varepsilon]^2-16)$, which approaches infinity as $\varepsilon \to 0$. Therefore, resonance can be thought of beating with an infinite beat period and an envelope that grows to infinity in time.

In order to investigate how fast the response grows as $\varepsilon \to 0$, we do the following. Using the method of partial fraction expansion, we write

$$\frac{1}{[4-\varepsilon]^2 - 16} = \frac{1}{8}\left(\frac{1}{\varepsilon - 8} - \frac{1}{\varepsilon}\right).$$

The term, $\sin(4-\varepsilon)t$, is expanded as $\cos \varepsilon t \sin 4t - \sin \varepsilon t \cos 4t$. Then the forced response becomes

$$\begin{aligned} x_{force}(t) &= \frac{A}{m}\frac{1}{8}\left(\frac{1}{\varepsilon-8} - \frac{1}{\varepsilon}\right)\left(\cos\varepsilon t \sin 4t - \sin\varepsilon t\cos 4t - \frac{4-\varepsilon}{4}\sin 4t\right) \\ &= \frac{A}{m}\frac{1}{8}\left(\frac{1}{\varepsilon-8} - \frac{1}{\varepsilon}\right)\left[\left(\cos\varepsilon t - \frac{4-\varepsilon}{4}\right)\sin 4t - \sin\varepsilon t \cos 4t\right] \\ &= \frac{A}{m}\frac{1}{8}\left(\frac{\cos\varepsilon t - \frac{4-\varepsilon}{4}}{\varepsilon - 8}\sin 4t - \frac{\sin\varepsilon t}{\varepsilon-8}\cos 4t - \frac{\cos\varepsilon t - \frac{4-\varepsilon}{4}}{\varepsilon}\sin 4t + \frac{\sin\varepsilon t}{\varepsilon}\cos 4t\right). \end{aligned}$$

As $\varepsilon \to 0$, the forced solution becomes

$$\lim_{\varepsilon\to 0} x_{force}(t) = \frac{A}{m}\frac{1}{8}\left(0 - 0 - \frac{1}{4}\sin 4t + t\cos 4t\right),$$

where the L'Hopital's rule was used to evaluate the last two terms,

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{\cos\varepsilon t - \frac{4-\varepsilon}{4}}{\varepsilon} &= \lim_{\varepsilon\to 0}\frac{-t\sin\varepsilon t + \frac{1}{4}}{1} = \frac{1}{4} \\ \lim_{\varepsilon\to 0}\frac{\sin\varepsilon t}{\varepsilon} &= \lim_{\varepsilon\to 0}\frac{t\cos\varepsilon t}{1} = t. \end{aligned}$$

Finally, the total solution is given by

$$x(t) = \left(\frac{1}{4}v_o - \frac{1}{32}\frac{A}{m}\right)\sin 4t + x_0\cos 4t + \frac{A}{m}\frac{1}{8}t\cos 4t.$$

As $\varepsilon \to 0$, the solution grows linearly with time. This is a long way to find the resonant solution. The fast way was to set $\varepsilon = 0$ from the beginning in the equation of motion.

46. The block given in Figure 2.65 is acted on by the force

$$F(t) = 100 + 25 \sin 75t \text{ N}$$

and, after the transients have died out, it oscillates with an amplitude of 0.6 mm about a position 55 mm to the left of the static equilibrium position corresponding to the condition when no force is present. What is the mass of the block?

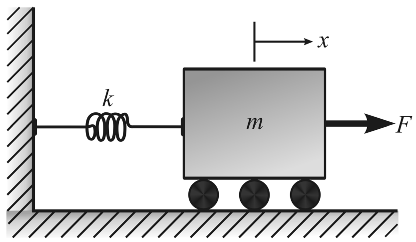


Figure 2.65: Block subjected to external force.

Solution: Let us assume that there is very small damping. It is small enough that the damping can be ignored when the oscillation amplitude is calculated, but the transient response can still die out in time.
The new static equilibrium position is (static force)$/k$ = 100 N$/k$ = 0.055 m. Then, k = 100/0.055 = 1818.2 N/m. The amplitude of the steady-state solution is given by

$$|X| = \frac{F}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}.$$

Assuming that the damping is very small the amplitude is approximately given by

$$|X| = \frac{F}{k} \left| \frac{1}{1-r^2} \right|.$$

From the problem, the amplitude of the dynamic portion of the force is 25 N and the response amplitude is 0.0006 m. We just found that $k = 1818$ N/m. Then, we find $r = 4.891$. The natural frequency is then,

$$\omega_n = \frac{\omega_f}{r} = \frac{75}{4.891} = 15.34 \text{ rad/s}.$$

The mass is

$$m = \frac{k}{\omega_n^2} = \frac{1818}{(15.34)^2} = 7.732 \text{ kg}.$$

47. For Example 2.11, what range of frequencies of the motion $y(t)$ must be excluded to keep the maximum force at C less than 10 N?

Solution: The equation of motion is

$$m\ddot{x} + kx = \frac{1}{2}kA\sin\omega t.$$

The force at C is $mg + k\,(x - y/2)$ downward.
Assuming that very small damping is included in the model, the steady-state response is

$$x_{ss}\,(t) = \frac{1}{2}kA\frac{1}{k - m\omega^2}\sin\omega t.$$

The force at C is

$$F_C = mg + k\left(\frac{A}{2}\frac{1}{1 - \omega^2/\omega_n^2} - \frac{A}{2}\right)\sin\omega t,$$

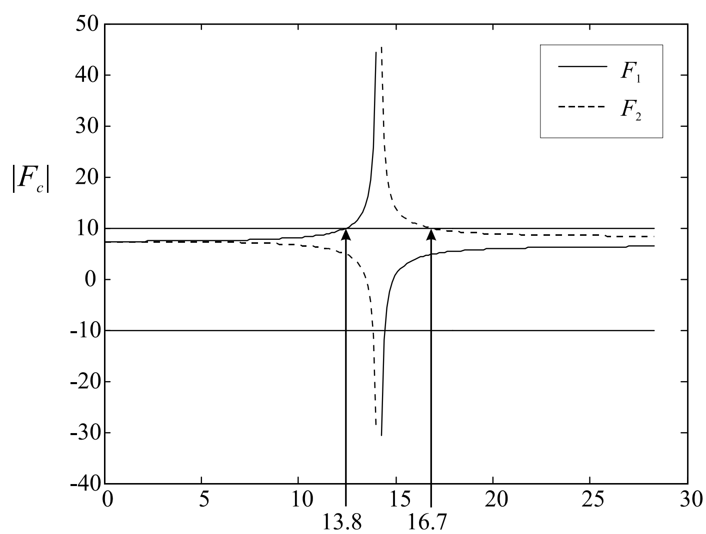
where F_C oscillates between

$$F_1 = mg + k\left(\frac{A}{2}\frac{1}{1 - \omega^2/\omega_n^2} - \frac{A}{2}\right)$$

and

$$F_2 = mg - k\left(\frac{A}{2}\frac{1}{1 - \omega^2/\omega_n^2} - \frac{A}{2}\right).$$

We consider two cases. If $\omega > \omega_n$, $|F_1|$ is larger than $|F_2|$. Then, the frequency must be such that $|F_1| < 10$ N. This is when the motion of the rod and the mass are in phase. The frequency must be such that $\omega < 13.8$ rad/s to have the force on C be less than 10 N.
If $\omega < \omega_n$, $|F_2|$ is larger than $|F_1|$. Then, the frequency must be such that $F_2 < 10$ N. This is when the motion of the rod and the mass are 180° out of phase. In this case, the frequency that satisfies this is $\omega > 16.7$ rad/s. To avoid forces larger that 10 N, we must avoid the frequency range $13.8 < \omega < 16.7$ rad/s.
It is much easier if we plot F_1 and F_2 as shown below.



48. A gantry for a crane is shown schematically in Figure 2.66. In this figure, the gantry is represented by the wheeled vehicle and the crane by the simple pendulum of length l with mass m at its end. The gantry is oscillating with the displacement $x(t) = A \sin \omega t$. If A is very small, what should ω be so that the crane has an amplitude of motion equal to $2A$? Assume $l = 1$ m.

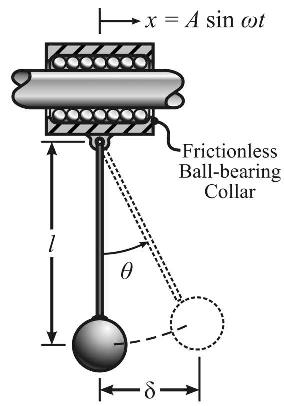
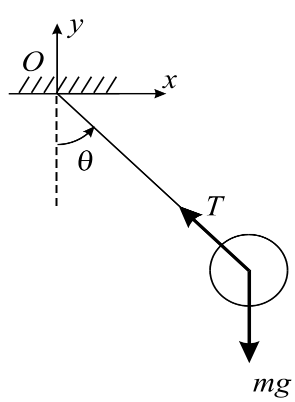


Figure 2.66: A model of a gantry for a crane.

Solution: The amplitude of motion depends on the initial conditions, especially if the system is undamped. Let us assume that the system has very small damping so that the transient effects die out. Then, we want to find the frequency of the base motion so that the steady-state response has an amplitude of $2A$ in the horizontal direction.

Draw a free-body diagram and sum the forces in the horizontal and vertical directions.



From Newton's second law

$$+ \rightarrow \sum F_x = ma_x \Longrightarrow -T \sin \theta = m \frac{d^2}{dt} \left(l \sin \theta + A \sin \omega t \right)$$

$$+ \uparrow \sum F_y = ma_y \Longrightarrow T \cos \theta - mg = m \frac{d^2}{dt} \left(-l \cos \theta \right).$$

Multiplying the first by $\cos\theta$ and second by $\sin\theta$, and adding the two equations, we have

$$m\frac{d^2}{dt}\left(l\sin\theta + A\sin\omega t\right)\cos\theta + m\frac{d^2}{dt}\left(-l\cos\theta\right)\sin\theta + mg\sin\theta = 0.$$
$$ml\ddot{\theta} + mg\sin\theta = mA\omega^2\sin\omega t\cos\theta.$$

Assuming small rotations, the equation of motion becomes

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{A\omega^2}{l}\sin\omega t,$$

with a steady-state solution

$$\theta_{ss}\left(t\right) = \frac{A\omega^2}{l}\frac{1}{\frac{g}{l}-\omega^2}\sin\omega t.$$

The horizontal motion is $x_{total} = l\sin\theta + A\sin\omega t \approx l\theta + A\sin\omega t$,

$$x_{total}\left(t\right) = \left(\frac{A\omega^2}{\dfrac{g}{l}-\omega^2} + A\right)\sin\omega t.$$

For the state-state amplitude of the bob to be $2A$,

$$\frac{A\omega^2}{\dfrac{g}{l}-\omega^2} + A = 2A,$$

the base must have a frequency of $\omega = 2.21$ rad/s (for $l = 1$ m).

49. Solve the equation of motion and discuss the results physically,

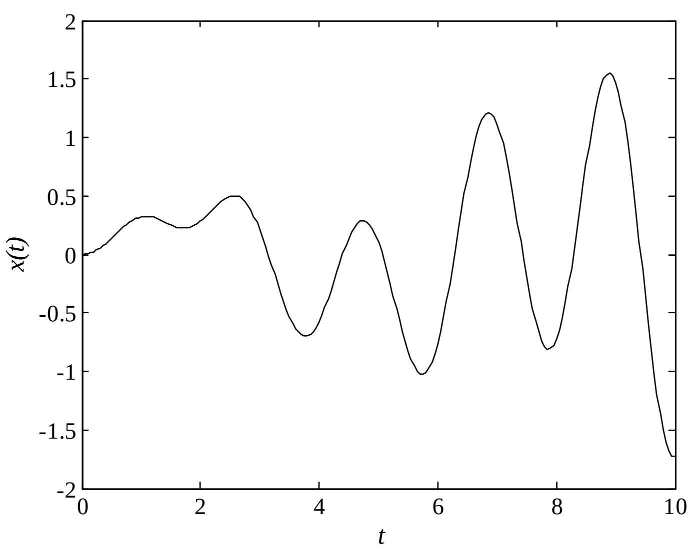
$$\ddot{x} + 9x = 3\sin t + \cos 3t$$
$$x(0) = 0, \quad \dot{x}(0) = 0.$$

Is it possible to predict some specific response behavior without solving the governing equation?

Solution: From the equation of motion we know that the system has a natural frequency of 3 rad/s, which coincides with one of the forcing frequencies. Therefore, the response will consist of a harmonic response at the natural frequency of 3 rad/s, another harmonic response at the forcing frequency of 1 rad/s, and a harmonic response that grows linearly with time at 3 rad/s. The exact solution is

$$x_1(t) = -\frac{1}{8}\sin 3t + \frac{3}{8}\sin t + \frac{1}{6}t\sin 3t,$$

and is shown next.

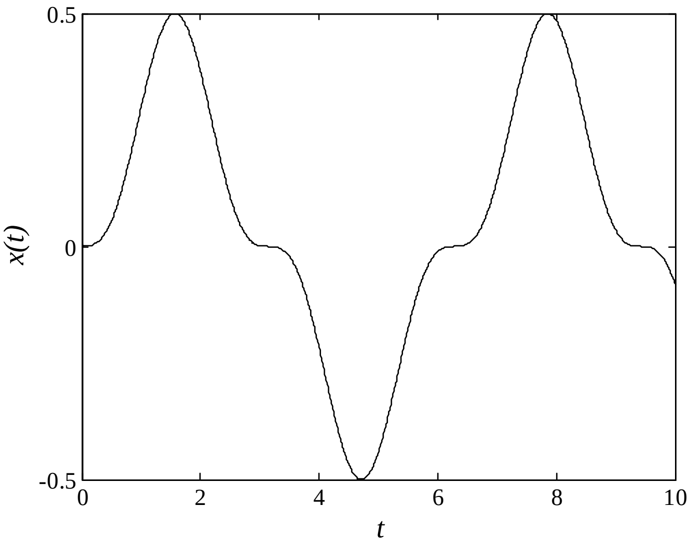


$x_1(t) = -\frac{1}{8}\sin 3t + \frac{3}{8}\sin t + \frac{1}{6}t\sin 3t$

Let us also solve for the response due to each term on the right-hand side separately.

$$x'' + 9x = \sin t$$
$$x(0) = 0 \qquad \text{Exact solution is : } x(t) = -\tfrac{1}{8}\sin 3t + \tfrac{3}{8}\sin t$$
$$x'(0) = 0$$

The harmonic loading is at a frequency of 1 rad/s, while the natural frequency is 3 rad/s. Therefore, we expect a periodic response as shown next.



$-\frac{1}{8}\sin 3t + \frac{3}{8}\sin t$

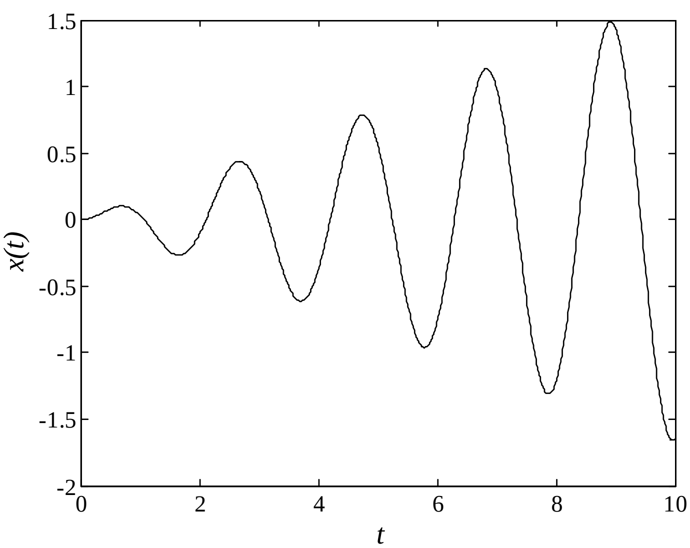
The second loading case is given by:

$$\boxed{\begin{array}{c} x'' + 9x = \cos 3t \\ x(0) = 0 \\ x'(0) = 0. \end{array}}$$

Here the loading is at the natural frequency, and thus, we expect unbounded growth. The exact solution is obtained by assuming a solution of the form

$$x(t) = At\cos 3t + Bt\sin 3t.$$

Differentiating and substituting into the differential equation of motion we find $x(t) = \frac{1}{6}t\sin 3t$.



$\frac{1}{6}t\sin 3t$

(Note that MAPLE evaluates the solution to be $\frac{1}{9}\cos^3 t - \frac{1}{12}\cos t + \frac{2}{3}t\sin t\cos^2 t - \frac{1}{6}t\sin t - \frac{1}{36}\cos 3t$. Believe it or not, this expression and one we found are identical. Plot the difference and you will obtain zero. This is just a word of warning on being too dependent on canned programs. We depend on them, but it is always a good idea to study the results and try to verify the results using back-of-the-envelope computations.)

50. Solve for the response of the governing equation of motion,

$$\ddot{x} + \omega_n^2 x = \frac{A}{m} \sin \omega_n t,$$

for arbitrary initial conditions.

Solution: The free vibration problem has been previously solved, resulting in the general solution

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t,$$

where ω_n is the natural frequency. We will only solve for the particular solution next.
In this problem, the loading is at the same frequency as the natural frequency of the system. Since the system is undamped, we know to expect oscillation with unbounded growth. Assume a particular solution of the form

$$x_p(t) = Ct \cos \omega_n t + Bt \sin \omega_n t.$$

Differentiate this assumed solution twice and substitute it into the governing differential equation of motion to find:

$$\begin{aligned} & \frac{d^2}{dt^2}[Ct \cos \omega_n t + Bt \sin \omega_n t] \\ = \; & \left[-2C (\sin \omega_n t)\, \omega_n - Ct (\cos \omega_n t)\, \omega_n^2 + 2B (\cos \omega_n t)\, \omega_n - Bt (\sin \omega_n t)\, \omega_n^2\right] \end{aligned}$$

The differential equation becomes

$$\begin{aligned} & \left[-2C (\sin \omega_n t)\, \omega_n - Ct (\cos \omega_n t)\, \omega_n^2 + 2B (\cos \omega_n t)\, \omega_n - Bt (\sin \omega_n t)\, \omega_n^2\right] \\ & + \omega_n^2 Ct \cos \omega_n t + Bt \sin \omega_n t \\ = \; & (A/m) \sin \omega_n t. \end{aligned}$$

Next we combine like terms:

$$\begin{aligned} & (-2C\omega_n + (-\omega_n^2 + 1)tB) \sin \omega_n t \\ & + 2B\omega_n \cos \omega_n t = (A/m) \sin \omega_n t. \end{aligned}$$

This equation is an identity of two equalities: $-2C\omega_n + (-\omega_n^2 + 1)tB = A/m$, and $2B\omega_n = 0$, which must now be solved for C and B simultaneously:

$$-2C\omega_n - Bt\omega_n^2 + Bt = A/m$$

$$2B\omega_n = 0.$$

The solution is

$$B = 0 \text{ and } C = -\frac{1}{2\omega_n} \frac{A}{m}.$$

Note that above simultaneous equations must be satisfied for all time t. The total solution is

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= C_1 \cos \omega_n t + C_2 \sin \omega_n t - \frac{A}{m} \frac{1}{2\omega_n} t \cos \omega_n t. \end{aligned}$$

The constants of integration, C_1 and C_2, are determined by satisfying the initial conditions. Let $x(0) = x_0$ and $\dot{x}(0) = v_0$, and we find the total solution to be given by

$$x(t) = x_o \cos \omega_n t + \left(\frac{2v_o w_n + A/m}{2\omega_n^2} \right) \sin \omega_n t - \frac{A}{m} \frac{1}{2\omega_n} t \cos \omega_n t.$$

51. Derive Equation 2.37,

$$x(t) = \frac{x_{st}\omega_n^2}{2\varepsilon(\varepsilon+\omega)}\left[\sin(\varepsilon+\omega)t\sin\varepsilon t\right],$$

from the equation of motion.

Solution: Begin with the equation

$$x(t) = \frac{x_{st}}{1-(\omega/\omega_n)^2}[\cos\omega t - \cos\omega_n t].$$

Use the trig identities for sums and differences of angles to obtain Equation 2.36:

$$x(t) = \frac{x_{st}}{1-(\omega/\omega_n)^2}\left[2\sin\frac{(\omega_n+\omega)t}{2}\sin\frac{(\omega_n-\omega)t}{2}\right],$$

and make the substitution $\omega_n = 2\varepsilon + \omega$. Then,

$$\begin{aligned}
x(t) &= \frac{2x_{st}}{1-(\omega/\omega_n)^2}[\sin(\varepsilon+\omega)t\sin\varepsilon t] \\
&= \frac{2x_{st}}{\left(\frac{{\omega_n}^2-\omega^2}{{\omega_n}^2}\right)}[\sin(\varepsilon+\omega)t\sin\varepsilon t] \\
&= \frac{2x_{st}{\omega_n}^2}{(\omega_n-\omega)(\omega_n+\omega)}[\sin(\varepsilon+\omega)t\sin\varepsilon t] \\
&= \frac{2x_{st}{\omega_n}^2}{2\varepsilon(2\varepsilon+2\omega)}[\sin(\varepsilon+\omega)t\sin\varepsilon t].
\end{aligned}$$

Finally, we obtain Equation 2.37 given by

$$x\,(t) = \frac{x_{st}{\omega_n}^2}{2\varepsilon(\varepsilon+\omega)}[\sin(\varepsilon+\omega)t\sin\varepsilon t].$$

52. A cylinder of mass m is mounted as shown in Figure 2.67 in a water tunnel. The configuration is end-on with the cylinder axis shown to be transverse to the flow direction. When there is no flow, a vertical force of F_{static} on the cylinder results in a deflection of y. With flow in the tunnel, shedding vortices impart alternating forces on the cylinder. The velocity of the water is v in the y direction, the distance between vortices is d, and the magnitude of the lateral forces is F_v. The lateral forces can be modeled by the harmonic function $F(t) = F_v \sin \omega t$. Find the amplitude of lateral steady-state response of the cylinder.

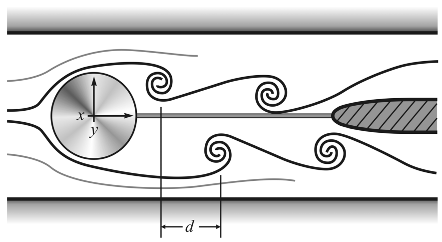
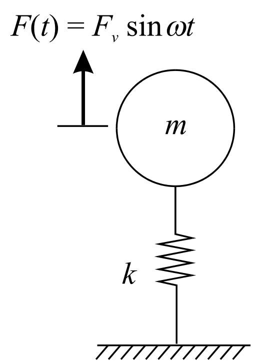


Figure 2.67: Cross-section of an elastically mounted cylinder excited by shedding vortices.

Solution: This problem can be modeled as a mass supported by a spring as shown below.



It is given that the force F_{static} is required to deflect the system a distance y. This gives us the stiffness of the leaf spring as $k = F_{static}/y$. It is said that the vortices are shed every d/v seconds. This gives us the forcing frequency of $\omega = 2\pi/(d/v)$. The equation of motion of the simple model is then

$$m\ddot{x} + kx = F_v \sin \omega t,$$

where $x(t)$ is the vertical deflection. The steady-state response (assuming very small damping so that the transient response dies out) amplitude is

$$\frac{F_v}{\sqrt{(k - m\omega^2)}},$$

where $k = F_{static}/y$ and $\omega = 2\pi/(d/v)$. It is assumed that F_v and m are known.