2. networks - electrical power grids;
3. spatio-temporal interactions - Newton's cradle;
4. nonlinear systems - operational amplifier with saturation;
5. self-adaptivity - self-leveling suspensions of the vehicles.

1.2.11 Exercise 1.11

Exercise: Analyze the expression "define your terms, gentleman, define your terms. It saves argument!" Who pronounced it?

Solution: This is a sentence by Dr. Samuel Johnson. He was an important social and literary celebrity of the eighteenth century. He was suggesting to fix well structured definitions as the first step to develop the arguments of a discussion. The sentence seems to suggest that, once fixed basic pillars, the construction may be easier to do. If a definition is clear, indeed, it is more simple to follow a scientific reasoning.

1.2.12 Exercise 1.12

Exercise: Propose simple experiments that show nonlinear phenomena in your everyday experience.

Solution: Some examples of nonlinear phenomena we can experience in our everyday life are propagation of waves (for instance sea tides), the diffusion of smoke of cigarettes in the air, the flow of lava, or the wind.

1.3 Solutions of exercises of Chapter 2

1.3.1 Exercise 2.1

Exercise: Consider the tent map

$$x_{k+1} = \begin{cases} 2x_k & \texttt{if } 0 \leq x_k \leq \frac{1}{2} \\ 2(1-x_k) & \texttt{if } \frac{1}{2} < x_k \leq 1 \end{cases} \tag{1.5}$$

1. Draw the map nonlinearity.
2. Build a simple MATLAB® program to calculate the time series generated by the map.
3. Compare it with the logistic map.

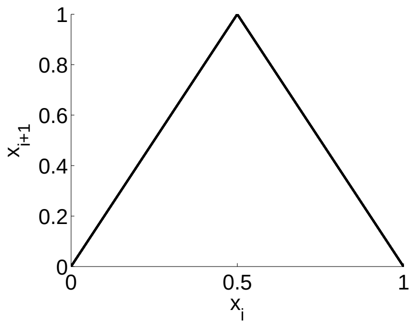

FIGURE 1.16
Nonlinearity of the tent map.

4. Generalize the map by substituting the factor 2 appearing in Equation (1.5) with a parameter a. Then build the bifurcation diagram of the map with respect to this parameter and compare with that of the logistic map.
5. Calculate and plot the Lyapunov exponents with respect to a.

Note: the solution of the initial value problem associated to the tent map is $x_k = \frac{1}{\pi}\cos^{-1}(\cos(2^k \pi x(0)))$.

Solution:

1. The following commands may be used to plot the nonlinearity of the tent map:

```
a=2;
hold on;
xx=linspace(0,1,10001);
for i=1:10001
        if xx(i)<0.5
                x(i) = a*xx(i);
        else
                x(i) = a*(1-xx(i));
        end
        plot(xx(i),x(i),'k','linewidth',2)
end
xlabel('x_i','FontSize',20)
ylabel('x_{i+1}','FontSize',20)
```

The result is shown in Figure 1.16.

For $a = 2$ the times series of the tent map goes to zero after a number of iterations related to the number of bits in the initial condition. This is due to the limited precision of MATLAB®. In order to overcome this problem, we can set $a = 1.9999$. The following commands may be used to plot the time series:

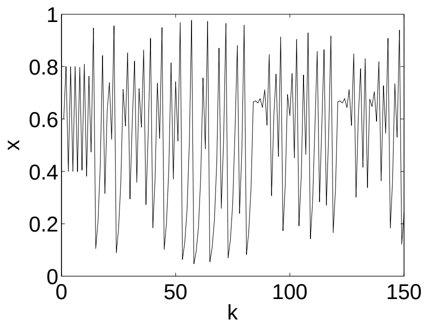

FIGURE 1.17
A time series generated by the tent map.

```
a=1.9999;
x=0.7;
for i=1:150
        if x < 0.5
                x = a*x;
        else
                x = a*(1-x);
        end
        y(i)=x;
end
plot(y,'k')
ylim([0 1]);
xlabel('k','FontSize',20)
ylabel('x','FontSize',20)
```

The time series is shown in Figure 1.17; it exhibits a chaotic behavior.

2. The tent map can be seen as a piece-wise linear approximation of the logistic map. For the logistic map the maximum of the nonlinearity is equal to $a/4$, for the tent map it is equal to $a/2$. The tent map is chaotic for $a = 2$, while the logistic map for $a = 4$.

3. The generalized tent map can be expressed as:

$$x_{k+1} = \begin{cases} a\, x_k & \texttt{if } 0 \leq x_k \leq \frac{1}{2} \\ a\,(1 - x_k) & \texttt{if } \frac{1}{2} < x_k \leq 1 \end{cases} \tag{1.6}$$

4. The following commands may be used to plot the bifurcation diagram of the tent map:

```
n = 1000;
x = zeros(n+1,1);
x(1)=0.51;
for a=0:0.003:2
        for i=1:n
                if x(i)<0.5
                        x(i+1) = a*x(i);
```

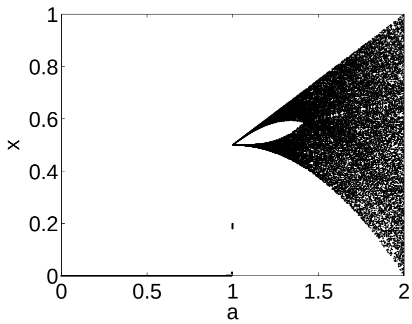

FIGURE 1.18
Bifurcation diagram of the tent map.

```
                else
                        x(i+1) = a*(1-x(i));
                end
        end
end
plot(a,x (900:end ),'k.','MarkerSize',4)
hold on
end
xlabel('a','FontSize',20)
ylabel('x','FontSize',20)
```

The bifurcation diagram is shown in Figure 1.18.

5. The Lyapunov exponent of the tent map is given by $\lambda = \log a$. In fact, the Lyapunov exponent $\lambda = \lambda(x_0)$ can be computed considering the following expression:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln |f'(x_{k-1})| \tag{1.7}$$

For the tent map $f'(x_{k-1}$ is equal to:

$$f'(x_{k-1}) = \begin{cases} a & \texttt{if } 0 \le x_{k-1} \le \frac{1}{2} \\ -a & \texttt{if } \frac{1}{2} < x_{k-1} \le 1 \end{cases} \tag{1.8}$$

Therefore, the Lyapunov exponent is equal to:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln a = \ln a \tag{1.9}$$

In accordance with the bifurcation diagram, the Lyapunov exponent assumes a positive value for $a > 1$. For these values of the parameter,

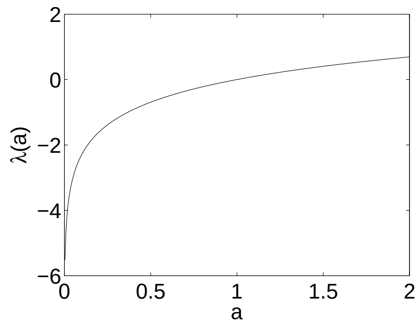

FIGURE 1.19
Lyapunov exponent of the tent map for $a \in [0\ \ 2]$. The initial condition is $x(0) = 0.51$.

in fact, the tent map is chaotic. The following MATLAB® code may be used to plot the Lyapunov exponent:

```
n =1000;
a_vector=0:0.004:2;
lamba_a=zeros(length(a_vector),1);
for j=1:length(a_vector)
        x =.51;
        lyaptmp =0;
        a= a_vector (j)
        lyaptmp=log(a);
        lambda_a(j)=lyaptmp ;
end
plot(a_vector,lambda_a,'k')
xlabel('a','FontSize',20)
ylabel('\lambda(a)','FontSize',20)
```

The Lyapunov exponent vs. a is shown in Figure 1.19.

1.3.2 Exercise 2.2

Exercise: Consider the logistic map with $a = 4$ and initial condition $x(0) = \frac{1}{3}$. Derive without using the computer the time series $x(1), x(2), \ldots, x(n), \ldots$ and comment on the result obtained.

Solution: The first five iterations of the logistic map with $a = 4$ and $x(0) = \frac{1}{3}$ are:

$$\begin{array}{l} x(1) = \frac{2^3}{3^2} \\ x(2) = \frac{2^5}{3^4} \\ x(3) = \frac{2^7}{3^8}\, 7^2 \\ x(4) = \frac{2^9}{3^{16}}\, 7^2\, 17^2 \\ x(5) = \frac{2^{11}}{3^{32}}\, 7^2\, 17^2\, 5983^2 \end{array} \tag{1.10}$$

While from the first two terms a simple iteration rule seems to appear, this is no more evident in the next samples. In fact, for $r = 4$ a general expression, not easy to derive, exists:

$$x(i) = \sin^2(2^i \theta \pi) \tag{1.11}$$

with $\theta = \frac{1}{\pi} \sin^{-1}(x_0^{1/2})$.

1.3.3 Exercise 2.3

Exercise: The asymmetric tent map is defined as

$$x_{k+1} = \begin{cases} a x_k & \texttt{if } 0 \le x_k \le \frac{1}{a} \\ \frac{a}{a-1}(1 - x_k) & \texttt{if } \frac{1}{a} < x_k \le 1 \end{cases} \tag{1.12}$$

where a, b, and c are parameters with $a > 0$, $b > 1$ and $a + b > ab$.

1. Implement it.
2. Derive a time series for given values of a, b, and c.
3. Draw the bifurcation diagrams with respect to the parameters of the map.
4. Derive the Lyapunov exponents as function of each parameter.

Solution: The equation of the asymmetric tent map is a more general model of the tent map. In fact, for $a = 2$ the symmetric tent map is retrieved.

1. The map can be implemented with the following MATLAB® commands:

```
a= 3;
hold on;
xx=linspace(0,1,10001);
for i=1:10001
    if ((xx(i)<=1/a) &&(xx(i)>=0))
        x(i) = a*xx(i);
    elseif ((xx(i)>1/a)&&(xx(i)<=1))
        x(i) = a*(1-xx(i))/(a-1);
    end
    plot(xx(i),x(i),'k.','linewidth',2)
end
xlabel('x_k','FontSize',20)
ylabel('x_{k+1}','FontSize',20)
```

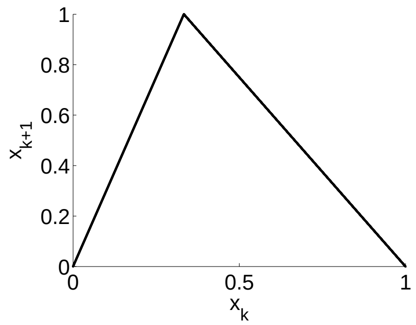

FIGURE 1.20
x_k vs. x_{k-1} for the asymmetric tent map with $a = 3$.

An example of the plot illustrating x_k vs. x_{k-1} obtained with $a = 3$ is shown in Figure 1.20.

2. The time series is calculated and plotted with the following commands:

```
a=3;
hold on;
x=0.1;
for i=1:150
    if ((x<=1/a)&&(x>=0))
        x = a*x;
    elseif ((x>1/a)&&(x<=1))
        x = a*(1-x)/(a-1);
    end
    y(i)=x;
end
plot(y,'k')
xlabel('t','FontSize',20)
ylabel('x','FontSize',20)
```

The result is shown in Figure 1.21.

3. The bifurcation diagram is built by varying the parameter a in the interval [0, 4]. The following commands may be used for the purpose:

```
n = 1000;
x = zeros(n+1,1);
x(1)=0.51;
for a=0:0.01:4
    for i=1:n
        if x(i)<=1/a
            x(i+1)=a*x(i);
        else
            x(i+1)=a*(1-x(i))/(a-1);
        end
    end
    plot(a,x(900:end),'k.','MarkerSize',4)
    hold on
end
```

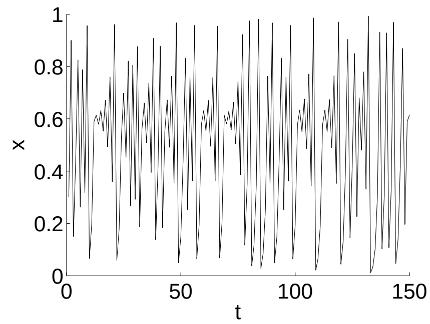

FIGURE 1.21
Time series of asymmetric tent map with $a = 3$.

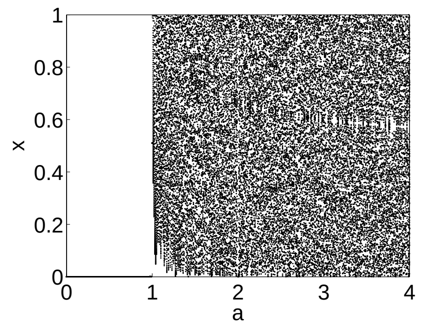

FIGURE 1.22
Bifurcation diagram of asymmetric tent map with respect to a.

```
xlabel('a','FontSize',20)
ylabel('x','FontSize',20)
```

The bifurcation diagram shown in Figure 1.22 reveals that the behavior of the map is chaotic for $a > 1$.

4. The following commands may be used to plot the Lyapunov exponent:

```
n =1000;
a_vector =0:0.002:4;
lamba_a = zeros (length(a_vector),1);
for j=1: length (a_vector)
    x =.1;
    lyaptmp =0;
    a= a_vector(j)
    for i=1:n
        if ((x<=1/a)&&(x>=0))
            y(i) = a*x;
```

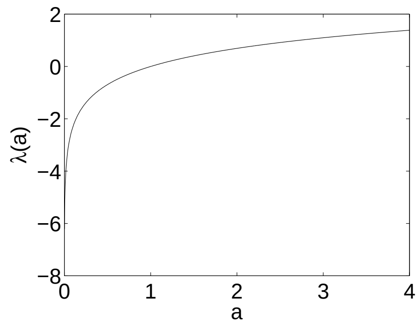

FIGURE 1.23
Lyapunov exponent of the asymmetric tent map for $a \in [0,\ 4]$ with initial condition $x(0) = 0.1$.

```
        elseif ((x>1/a)&&(x<=1))
            y(i) = a*(1-x)/(a-1);
        end
        x=y(i);
        if ((x<=a)&&(x>=0))
            lyap =(1/ n)* log (abs(a));
        elseif ((x>a)&&(x<=1))
            lyap =(1/ n)* log (abs(-a/(a-1)));
        end
        lyaptmp = lyap + lyaptmp;
    end
    lambda_a(j)= lyaptmp;
end
plot(a_vector,lambda_a,'k')
xlabel('a','FontSize',20)
ylabel('\lambda(a)','FontSize',20)
```

The Lyapunov exponent is shown in Figure 1.23. It assumes a positive value for $a > 1$ according with what observed in the bifurcation diagram.

1.3.4 Exercise 2.4

Exercise: Consider the delayed logistic map

$$x_{k+1} = rx_k(1 - x_{k-1}) \tag{1.13}$$

It represents a population at the $k + 1$ generation that depends not only on the population at the k generation but also on that at the $k-1$ generation.

1. Draw the bifurcation diagram with respect to r.
2. Plot x_k vs. x_{k-1}.
3. Derive a surface plot reporting x_{k+1} as a function of x_k and x_{k-1}.

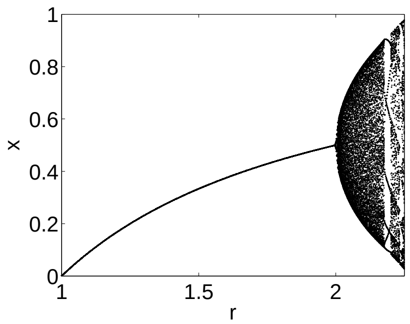

FIGURE 1.24
Bifurcation diagram of delayed logistic map.

4. Discuss the results obtained.

Solution:

1. The following commands may be used to calculate the bifurcation diagram of the map:

```
n = 1000;
x = zeros(n+1,1);
x(1)=0.51;
x(2)=0.51;
for r=1:0.0015:2.25
        for i=2:n
                x(i+1)=r*x(i)*(1-x(i-1));
        end
        plot(r,x(900:end),'k.','MarkerSize',4)
        hold on
end
xlabel('r','FontSize',20)
ylabel('x','FontSize',20)
xlim([1 2.25])
```

The result is shown in Figure 1.24.

2. The following commands may be used to generate the $x_k - x_{k-1}$ graph:

```
hold on
x= linspace (0,1,101);
for i=2:101
    y(i) =  2.15*x(i)*(1-x(i-1));
end
for i=2:101
   plot(y(i-1),y(i),'k.-','linewidth' ,2)
end
xlabel('x_{k-1} ','FontSize',20)
ylabel('x_{k} ','FontSize',20)
```

The graph is shown in Figure 1.25.

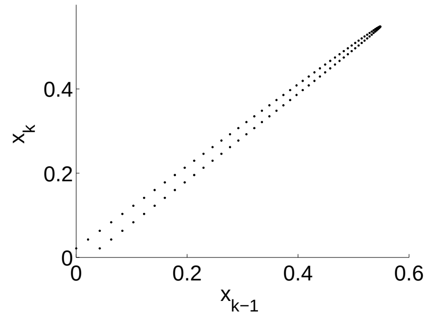

FIGURE 1.25
x_k vs. x_{k-1} of the delayed logistic map for $a = 2.15$.

3. The surface plot reporting x_{k+1} as a function of x_k and x_{k-1}, along with a system trajectory, is obtained with the following MATLAB® commands:

```
x=linspace(0,1,101);
[X,Y]=meshgrid(x,x);
Z=2.15.*X.*(1-Y);
figure,surface(X,Y,Z)
figure,mesh(X,Y,Z)
hold on
x=linspace(0,1,1001);
for i=3:1000
    y(i)=2.15*x(i-1)*(1-x(i-2));
    x(i)=y(i);
end
y=y(1,200:end);
plot3(y(2:end-1),y(1:end-2),y(3:end),'k.','linewidth' ,2)
xlabel('x_k ','FontSize',20)
ylabel('x_{k-1} ','FontSize',20)
zlabel('x_{k+1} ','FontSize',20)
```

   The surface plot is shown in Figure 1.26.

4. We note that the logistic map is a second-order discrete-time map. In fact, the actual sample depends on two past samples. For this reason, reporting x_k as function of x_{k-1} does not provide any insight on the system nonlinearity. On the contrary, the surface plot is meaningful. Finally, we note that the map is able to generate chaotic motion and that a period doubling cascade is observed also for this map.

1.3.5 Exercise 2.5

Exercise: The following map is said to be the Bernoulli map:

$$x_{k+1} = f(x_k) \tag{1.14}$$

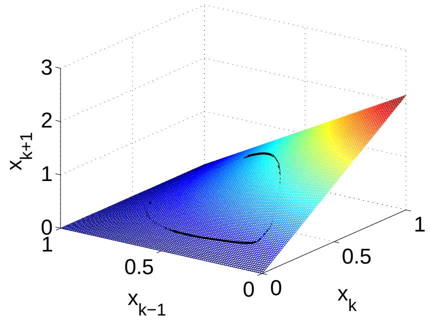

FIGURE 1.26
Surface plot reporting x_{k+1} as function of x_k and x_{k-1}, along with a system trajectory, in the delayed logistic map for $a = 2.15$.

where $f(x_k) = 2x_k \mod 1$. Draw the time series generated by this map.

Solution: As for the tent map with $a = 2$, also in this case the limited precision of the numerical calculation yields to uncorrect results. To overcome this problem, we set the parameter equal to 1.9999 instead than equal to 2. The following commands may be used to calculate and plot the time series generated by the Bernoulli map:

```
x(1) = 0.6;
for i=1:150
    x(i+1) = mod(1.9999*x(i),1);
end
plot(x,'k')
xlabel('k','FontSize',20)
ylabel('x','FontSize',20)
ylim([0 1]); xlim([0 150]);
```

The time series is shown in Figure 1.27.

1.3.6 Exercise 2.6

Exercise: Make a qualitative comparison of the results derived analyzing the behavior of the maps at the points 1)-5).

Solution: The following first-order discrete-time maps have been analyzed:

1. the tent map;
2. the logistic map;
3. the asymmetric tent map;

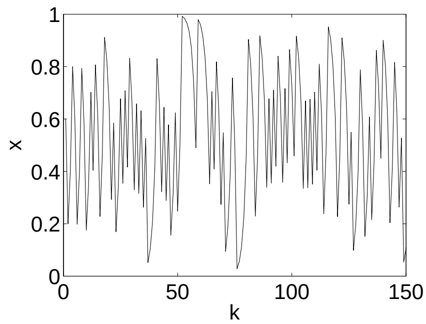

FIGURE 1.27
Time series of the Bernoulli map with $a = 1.9999$.

4. the delayed logistic map;
5. the Bernoulli map.

All the maps are characterized by a single parameter and the interval where it can change is different for each map. All the maps can produce a chaotic behavior for a specific range of the parameter. It is important to note that the logistic map with $a = 4$ is topologically conjugate to the tent map and Bernoulli map with $a = 2$. The parameter of the tent map varies between $0 \leq a \leq 2$; a bifurcation point occurs at $a = 1$ after which the map becomes chaotic. On the contrary, for the logistic map a bifurcation point occurs at $a = 3$, where a cascade of period doubling starts. The Bernoulli map has also a bifurcation point for $a = 1$ and it becomes chaotic for increasing values of this parameter.

Making a comparison between the delayed logistic map and the non-delayed one, we note that they have different behavior since the bifurcation diagram has a different shape and bifurcations occur for different values of the parameters. For example, in the delayed logistic map the first bifurcation occurs for $a = 2$ instead than $a = 3$ of the classical one.

1.3.7 Exercise 2.7

Exercise: Consider the cubic map $x_{k+1} = ax_k - x_k^3$.

1. Find the equilibrium points and study their stability.
2. Draw the bifurcation diagram with respect to a.

Solution:

1. The equilibrium points of the map are calculated from the equation $x_{k+1} = x_k$. For the cubic map the solutions are:

$$x_k = 0 \tag{1.15}$$

and

$$x_k = \pm\sqrt{a-1} \tag{1.16}$$

The Jacobian associated to the system is equal to:

$$J = \frac{\partial x_{k+1}}{\partial x_k} = a - 3x_k^2 \tag{1.17}$$

We study the stability of the first equilibrium point by substituting $x_k = 0$ to the Jacobian:

$$J = \left.\frac{\partial x_{k+1}}{\partial x_k}\right|_{x_k=0} = a \tag{1.18}$$

Therefore, for $|a| < 1$ the equilibrium point is asymptotically stable, while $|a| > 1$ it is unstable.

Let us now study the other two equilibrium points by substituting $x_k = \pm\sqrt{a-1}$ to the Jacobian. The result is:

$$J = \left.\frac{\partial x_{k+1}}{\partial x_k}\right|_{x_k=\pm\sqrt{a-1}} = 3 - 2a \tag{1.19}$$

The condition $|3 - 2a| < 1$ can be rewritten as $1 < a < 2$ and in this case the equilibrium points $x_k = \pm\sqrt{a-1}$ are asymptotically stable. The condition $|3-2a| > 1$ can be rewritten as $a < 1 \cup a > 2$ and in this case equilibrium points $x_k = \pm\sqrt{a-1}$ are unstable.

2. The following code may be used to obtain the bifurcation diagram:

```
n = 1000;
x = zeros(n+1,1);
x(1)=0.51;
for a=1:0.002:3
        for i=1:n
                x(i+1)= a*x(i)-x(i)^3;
        end
        plot(a,x (900:end ),'k.','MarkerSize',4)
        hold on
end
```

The bifurcation diagram is shown in Figure 1.28.

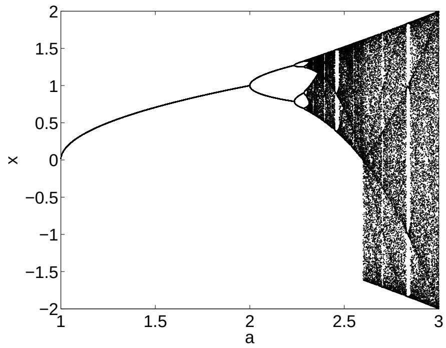

FIGURE 1.28
Bifurcation diagram of the cubic map $x_{k+1} = ax_k - x_k^3$.

1.3.8 Exercise 2.8

Exercise: Consider two coupled logistic maps

$$\begin{aligned} x_{k+1} &= rx_k(1 - x_k) + \sigma(y_k - x_k) \\ y_{k+1} &= ry_k(1 - y_k) + \sigma(x_k - y_k) \end{aligned} \tag{1.20}$$

Derive the bifurcation diagrams considering fixed the parameter σ and varying r and vice versa.

Solution: The following commands are used to obtain the bifurcation diagram of the system of two coupled logistic maps while varying the parameter r and keeping constant σ ($\sigma = 0.1$):

```
n =1000;
x= zeros(n+1,1);
y= zeros(n+1,1);
x(1)=0.51;
y(1)=0.72;
sigma=0.1;
figure
for a =1:0.004:3.7
    for i=1:n
        x(i+1)= a*x(i)*(1-x(i))+sigma*(y(i)-x(i));
        y(i+1) = a*y(i)*(1-y(i))+sigma*(x(i)-y(i));
    end
    plot(a,x(900:end),'k.','MarkerSize',4)
    hold on
end
xlabel('a','FontSize',20)
ylabel('x','FontSize',20)
xlim([1 3.7])
set(gca,'FontSize',20)
```

The bifurcation diagram is shown in Figure 1.29 (in the diagram the vari-

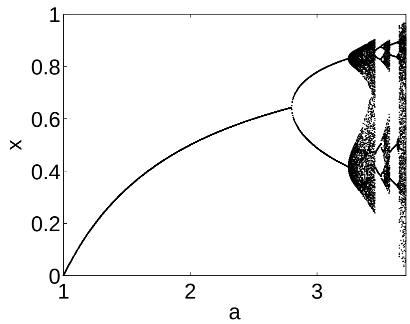

FIGURE 1.29
Bifurcation diagram of the map (1.20) while varying r with $\sigma = 0.1$.

able x_k is considered; similar results are obtained when y_k is taken into account).

The bifurcation diagram with respect to σ with fixed r ($r = 3.8$) is obtained with the following commands:

```
n =1000;
x= zeros(n+1,1);
y=zeros(n+1,1);
x(1)=0.51;
y(1)=0.72;
r=3.8;
for sigma=-0.2:0.0001:0.05
    for i=1:n
        x(i+1)=r*x(i)*(1-x(i))+sigma*(y(i)-x(i));
        y(i+1)=r*y(i)*(1-y(i))+sigma*(x(i)-y(i));
    end

    plot(sigma,x(900:end),'k.','MarkerSize',4)
    hold on
end
xlabel('\sigma','FontSize',20)
ylabel('x','FontSize',20)
xlim([-0.2 0.05])
ylim([-0.2 1])
set(gca,'FontSize',20)
```

It is shown in Figure 1.30.

1.3.9 Exercise 2.9

Exercise: Consider a natural number n and the following map

$$\begin{aligned} x_{k+2} &= \left[\frac{y_k+n}{y_{k+1}}\right] x_{k+1} - x_k \\ y_{k+2} &= \left[\frac{y_k+n}{y_{k+1}}\right] y_{k+1} - y_k \end{aligned} \tag{1.21}$$

with initial conditions $x_0 = 0$, $y_0 = 1$, $x_1 = 1$, $y_1 = n$, where $[x]$ indicates the

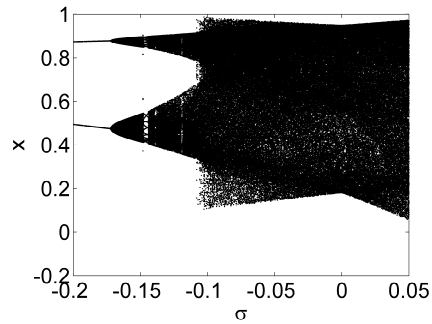

FIGURE 1.30
Bifurcation diagram of the map (1.20) while varying σ with $r = 3.8$.

largest integer not greater than x. x_i/y_i with $i = 2, 3, \ldots$ represents the Farey sequence. Analyze the map.

Solution: The map is studied with the following commands:

```
T=5000;
n=1;
x(1)=0;
x(2)=1;
y(1)=1;
y(2)=n;
f(1)=x(1)/y(1);
f(2)=x(2)/y(2);
hold on;
for n=1:1:1000
    for k=1:T
        x(k+2) = floor((x(k)+n)/x(k+1))*x(k+1)-x(k);
        y(k+2) = floor((y(k)+n)/y(k+1))*y(k+1)-y(k);
        f(k+2) = x(k+2)/y(k+2);
    end
    plot(n,x(4000:end),'k');
    %plot(n,f(4000:end),'k');
end
xlabel('n','FontSize',20);
ylabel('x','FontSize',20);
```

used to generate the bifurcation diagrams of Figure 1.31. The diagram in Figure 1.31(a) is obtained by plotting the last 1000 samples of the variable x_k, while that in Figure 1.31(b) the samples of the ratio between the two variables x_k and y_k. For a given value of n, from the map definition it is clear that each variable can take only integer values in the interval $[1,\ n]$. For large enough n, the sequence of these n symbols is not short-term periodic and not trivial. The same applies for the behavior of the ratio as it appears clear in the bifurcation diagram.

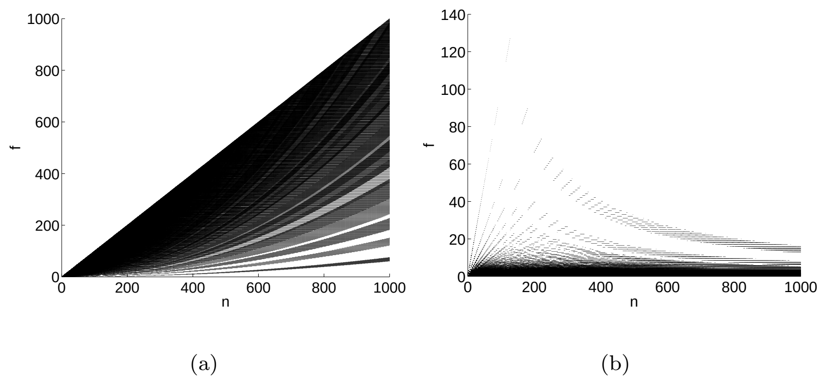

FIGURE 1.31
Bifurcation diagram of the map (1.21) with respect to n: (a) x_k; (b) $f_k = x_k/y_k$.