Chapter 2

Solving linear systems

2.1 Vectors and linear equations

Exercise 2.1.1.

1. Redo Example 2.1.1 with the first elementary row operation $R_2 - R_1$.

Solution:

System	Matrix representation	Elementary matrix
$\begin{cases} x - y = 1 \\ x + y = 2 \end{cases}$	$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$	
$\Downarrow R_2 - R_1$		$E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$
$\begin{cases} x - y = 1 \\ 2y = 1 \end{cases}$	$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$	
$\Downarrow \frac{1}{2}R_2$	$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
$\begin{cases} x - y = 1 \\ y = \dfrac{1}{2} \end{cases}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	
$\Downarrow R_1 + R_2$	$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$	$E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
$\begin{cases} x = \dfrac{3}{2} \\ y = \dfrac{1}{2} \end{cases}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$	

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}.$$

2. Determine whether the following matrices are in reduced row echelon form and row echelon form, respectively:

$$a) \begin{bmatrix} 1 & 0 & 0 & 9 & -2 \\ 0 & 1 & 0 & -2 & \frac{1}{2} \\ 0 & 0 & 1 & -5 & \frac{1}{2} \end{bmatrix}, \quad b) \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix}, \quad c) \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

Solution: a) and b) are in reduced row echelon form (hence also in row echelon form). c) is in row echelon form. □

3. Solve the following systems using Gauss–Jordan eliminations:

$$a) \begin{cases} x + 3z = 1 \\ 2x + 3y = 3 \\ 4y + 5z = 5 \end{cases} \quad b) \begin{cases} x + 2y + 3z = 1 \\ 2x + 3y + 4z = 3 \\ 3x + 4y + 5z = 5 \end{cases} \quad c) \begin{cases} x + 2y + 3z = 1 \\ 2x + 3y + 4z = 3 \\ 5x + 9y + 13z = 7 \end{cases}$$

Solution: a) We re-write the system of linear equations in the matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

Then the corresponding augmented matrix is

$$[A : \mathbf{b}] = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 \\ 0 & 4 & 5 & 5 \end{bmatrix}.$$

By the elementary row operations on $[A : \mathbf{b}]$ we have

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 \\ 0 & 4 & 5 & 5 \end{bmatrix} &\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 3 & -6 & 1 \\ 0 & 4 & 5 & 5 \end{bmatrix} \\ &\xrightarrow{R_2/3} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & \frac{1}{3} \\ 0 & 4 & 5 & 5 \end{bmatrix} \\ &\xrightarrow{R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 13 & \frac{11}{3} \end{bmatrix} \end{aligned}$$

$$\xrightarrow{R_3/13} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{11}{39} \end{bmatrix}$$

$$\xrightarrow[R_1-3R_3]{R_2+2R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{39} \\ 0 & 1 & 0 & \frac{35}{39} \\ 0 & 0 & 1 & \frac{11}{39} \end{bmatrix}.$$

Then we have an equivalent system with augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & \frac{6}{39} \\ 0 & 1 & 0 & \frac{35}{39} \\ 0 & 0 & 1 & \frac{11}{39} \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{39} \\ \frac{35}{39} \\ \frac{11}{39} \end{bmatrix}.$$

b) We re-write the system of linear equations in the matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

Then the corresponding augmented matrix is

$$[A : \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 5 & 5 \end{bmatrix}.$$

By the elementary row operations on $[A : \mathbf{b}]$ we have

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 5 & 5 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow{(-1)\cdot R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow[R_1-2R_2]{R_3+2R_2} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then we have an equivalent system with augmented matrix

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

c) We re-write the system of linear equations in the matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 9 & 13 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ \text{and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}.$$

Then the corresponding augmented matrix is

$$[A : \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 3 \\ 5 & 9 & 13 & 7 \end{bmatrix}.$$

By the elementary row operations on $[A : \mathbf{b}]$ we have

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 3 \\ 5 & 9 & 13 & 7 \end{bmatrix} &\xrightarrow[R_3-5R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & -1 & -2 & 2 \end{bmatrix} \\ &\xrightarrow{(-1)\cdot R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -2 & 2 \end{bmatrix} \\ &\xrightarrow[R_1-2R_2]{R_3+R_2} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Then we have an equivalent system with augmented matrix

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the last equation is contradictory. The system has no solution. □

□

4. Consider a linear system $Ax = b$ with A an $m \times n$ matrix and b an $m \times 1$ matrix. Is it true if there is more than one solution for x in $\mathbb{R}^n$, there must be infinitely many? You may use the fact that

$$\begin{aligned} A(x+y) &= Ax + Ay \\ A(tx) &= tAx, \end{aligned}$$

for every $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, which is called the **linearity** of matrix multiplication.

Solution: Let x_1 and x_2 be distinct solutions. Then $A(x_1-x_2) = Ax_1-Ax_2 = 0$ and for every $t \in \mathbb{R}$, we have

$$A(x_1 + t(x_1 - x_2)) = Ax_1 + tA(x_1 - x_2) = b.$$

Hence $x = x_1 + t(x_1 - x_2)$, $t \in \mathbb{R}$ are all solutions of $Ax = b$. That is, $Ax = b$ has infinitely many solutions. □

5. Let k be a real number. Consider the following linear system of equations:

$$\begin{cases} x_2 + 2x_3 + x_4 = 1 \\ 2x_1 + x_2 + 3x_3 = 2 \\ x_1 + 4x_3 + 2x_4 = 3 \\ kx_2 + x_4 = 1. \end{cases} \tag{2.1}$$

Find all possible values of k such that system (2.1) i) has a unique solution; ii) has no solutions and iii) has infinitely many solutions.

Solution: We re-write the system of linear equations in the matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 4 & 2 \\ 0 & k & 0 & 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}.$$

Then the corresponding augmented matrix is

$$[A : \mathbf{b}] = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 0 & 2 \\ 1 & 0 & 4 & 2 & 3 \\ 0 & k & 0 & 1 & 1 \end{bmatrix}.$$

By the elementary row operations on $[A : \mathbf{b}]$ we have

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 0 & 2 \\ 1 & 0 & 4 & 2 & 3 \\ 0 & k & 0 & 1 & 1 \end{bmatrix} &\xRightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 4 & 2 & 3 \\ 2 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & k & 0 & 1 & 1 \end{bmatrix} \\ &\xRightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 4 & 2 & 3 \\ 0 & 1 & -5 & -4 & -4 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & k & 0 & 1 & 1 \end{bmatrix} \\ &\xRightarrow[R_3 - R_2]{R_4 - kR_2} \begin{bmatrix} 1 & 0 & 4 & 2 & 3 \\ 0 & 1 & -5 & -4 & -4 \\ 0 & 0 & 7 & 5 & 5 \\ 0 & 0 & 5k & 1+4k & 1+4k \end{bmatrix} \end{aligned}$$

$$\xrightarrow{R_3/7} \begin{bmatrix} 1 & 0 & 4 & 2 & 3 \\ 0 & 1 & -5 & -4 & -4 \\ 0 & 0 & 1 & 5/7 & 5/7 \\ 0 & 0 & 5k & 1+4k & 1+4k \end{bmatrix}$$

$$\xrightarrow{R_4-(5k)\cdot R_3} \begin{bmatrix} 1 & 0 & 4 & 2 & 3 \\ 0 & 1 & -5 & -4 & -4 \\ 0 & 0 & 1 & 5/7 & 5/7 \\ 0 & 0 & 0 & 1+3k/7 & 1+3k/7 \end{bmatrix}.$$

Then we have an equivalent system with augmented matrix

$$\begin{bmatrix} 1 & 0 & 4 & 2 & 3 \\ 0 & 1 & -5 & -4 & -4 \\ 0 & 0 & 1 & 5/7 & 5/7 \\ 0 & 0 & 0 & 1+3k/7 & 1+3k/7 \end{bmatrix},$$

i) If $1+3k/7 \neq 0$, that is, $k \neq -7/3$ the original system has a unique solutiuon;
ii) If $1+3k/7 = 0$, that is, $k = -7/3$ the original system has infinitely many solutions.
Therefore, for every $k \in \mathbb{R}$, the system has at least one solution. □

2.2 Matrix operations

Exercise 2.2.1.

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. i) Compute AB. ii) Does BA exist?

Solution: i)

$$AB = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{bmatrix}.$$

ii) BA does not exist since their sizes do not match as B is 2×4 and A is 2×2. □

2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. i) Compute AB. ii) If B is block partitioned into $B = [B_1 : B_2]$, is it true $AB = [AB_1 : AB_2]$?

Solution: i)

$$AB = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 7 & 1 & 4 & 3 \end{bmatrix}$$

ii) According to the definition of matrix multiplication, the statement is true. □

3. Show Lemma 2.2.6.

Solution: Follow the approach for the proof of Theorem 2.2.5. $\square$

4. Let $A = [a_1 : a_2 \cdots : a_n]$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be $m \times n$ and $n \times r$ matrices. Show that $AB = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

Solution: By the definition of matrix multiplication we have

$$\begin{aligned}(AB)_{i,j} &= a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j} \\ &= (a_1 b_1)_{i,j} + (a_2 b_2)_{i,j} + \cdots + (a_n b_n)_{i,j} \\ &= (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)_{i,j}.\end{aligned}$$

Therefore, we have $AB = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$. $\square$

5. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. Use Question 4 to compute AB.

Solution: Let $A = [a_1 : a_2]$, $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{aligned}AB &= a_1 b_1 + a_2 b_2 \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & -3 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & 2 & 1 \\ 7 & 1 & 4 & 3 \end{bmatrix}.\end{aligned}$$

$\square$

6. Let A and B be $m \times n$ and $n \times r$ matrices. Show that i) every column of AB is a linear combination of the columns of A; ii) every row of AB is a linear combination of the rows of B.

Solution: Let $A = [a_1 : a_2 \cdots : a_n]$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be $m \times n$ and $n \times r$ matrices.

$$(AB)_{i,j} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j}.$$

The j-th column of AB is

$$(AB)_j = a_1 b_{1,j} + a_2 b_{2,j} + \cdots + a_n b_{n,j},$$

which is a linear combination of the columns of A. $\square$

7. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find all matrices B such that $AB = BA$.

Solution: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $BA = AB$ leads to a linear system of $x = (a, b, c, d) \in \mathbb{R}^4$:

$$\begin{cases} 3b - 2c = 0 \\ 2a + 3b - 2d = 0 \\ 3a + 3c - 3d = 0 \\ 3b - 2c = 0. \end{cases}$$

Using Guass-Jordan elimination, we obtain the solution

$$x = s \begin{bmatrix} -1 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; s, t \in \mathbb{R}.$$

That is,

$$B = \begin{bmatrix} t - s & 2s/3 \\ s & t \end{bmatrix}.$$

8. Let A and B be $n \times n$ matrices. Explain that in general we have $(A + B)(A - B) \neq A^2 - B^2$ and $(A + B)^2 \neq A^2 + 2AB + B^2$.

Solution: We note that in general, $AB \neq BA$. So $(A + B)(A - B) = A^2 - AB + BA - B^2 \neq A^2 - B^2$ and $(A + B)^2 \neq A^2 + 2AB + B^2$ if $AB \neq BA$.

9. Let A be an $n \times n$ matrix. Define $V = \{B : AB = BA\}$. Show that i) $V \neq \emptyset$; ii) if $B_1 \in V$ and $B_2 \in V$, then every linear combination of B_1 and B_2 is in V.

Solution: i) Since $AI = IB$ and $I \in V$, we have $V \neq \emptyset$.

ii) If $B_1 \in V$ and $B_2 \in V$, then for every $s \in \mathbb{R}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} (sB_1 + tB_2)A =& sB_1A + tB_2A \\ =& sAB_1 + tAB_2 \\ =& A(sB_1 + tB_2). \end{aligned}$$

Therefore, we have $sB_1 + tB_2 \in V$. We have shown that V is a vector space. $\square$

10. Give an example that $A^2 = 0$ but $A \neq 0$.

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We have $A^2 = 0$ but $A \neq 0$.

11. Give an example that $A^2 = I$ but $A \neq \pm I$.

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have $A^2 = I$ but $A \neq \pm I$.

12. Let A be an $n \times n$ matrix. If we want to define a limit $\lim_{m\to\infty} A^m$, how would you define the closeness (distance) between matrices?

Solution: If we identify A as a vector in $\mathbb{R}^{n^2}$. Then we can borrow norms on $\mathbb{R}^{n^2}$ for defining the distance between matrices. For example, we can define

$$\text{distance}(A, B) = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - b_{ij})^2 \right)^{\frac{1}{2}},$$

where a_{ij} and b_{ij} are entries at (i, j)-position of A and B, respectively. □

2.3 Inverse matrices

Exercise 2.3.1.

1. Determine whether or not the following matrices are invertible. Find the inverse of each matrix if it exists.

$$a) \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad b) \quad \begin{bmatrix} -1 & 2 \\ 3 & 6 \end{bmatrix}, \quad c) \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Solution: a) $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$, b) $\begin{bmatrix} -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{12} \end{bmatrix}$, c) not invertible.

□

2. Determine whether or not the following matrices are invertible. Find the inverse of each matrix if it exists.

$$a) \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad b) \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \quad c) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

□

3. Determine whether or not the following matrices are invertible. Find the inverse of each matrix if it exists.

$$a) \quad \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 3 & 6 \end{bmatrix}, \quad c) \quad \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: a) $\begin{bmatrix} -2 & 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{4} & \frac{1}{12} \end{bmatrix}$, c) not invertible.

□

4. For the given matrix A, use elimination to find A^{-1} and record each elementary row operation and the corresponding elementary matrix at the same time.

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & 5 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 1 & -1 & 5 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Row operation} \qquad \text{Elementary Matrix}$$

$$\Downarrow \qquad R_1 \leftrightarrow R_3 \qquad E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 5 & 0 & 0 & 1 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad \text{Row operation} \qquad \text{Elementary Matrix}$$

$$\Downarrow \qquad R_2 - 2R_1 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 6 & -8 & 0 & 1 & -2 \\ 3 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad \text{Row operation} \qquad \text{Elementary Matrix}$$

$$\Downarrow \qquad R_3 - 3R_1 \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 6 & -8 & 0 & 1 & -2 \\ 0 & 3 & -14 & 1 & 0 & -3 \end{bmatrix} \qquad \text{Row operation} \qquad \text{Elementary Matrix}$$

$$\Downarrow \qquad R_3 - R_2/2 \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 6 & -8 & 0 & 1 & -2 \\ 0 & 0 & -10 & 1 & -\frac{1}{2} & -2 \end{bmatrix}$$

Row operation — Elementary Matrix

$$\Downarrow \qquad R_3/(-10) \qquad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 6 & -8 & 0 & 1 & -2 \\ 0 & 0 & 1 & -\frac{1}{10} & \frac{1}{20} & \frac{1}{5} \end{bmatrix}$$

Row operations — Elementary Matrices

$$\Downarrow \qquad R_1 - 5R_3 \qquad E_6 = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_2 + 8R_3 \qquad E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 6 & 0 & -\frac{4}{5} & \frac{14}{10} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{1}{10} & \frac{1}{20} & \frac{1}{5} \end{bmatrix}$$

Row operation — Elementary Matrix

$$\Downarrow \qquad R_2/6 \qquad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{4}{30} & \frac{14}{60} & -\frac{2}{30} \\ 0 & 0 & 1 & -\frac{1}{10} & \frac{1}{20} & \frac{1}{5} \end{bmatrix}$$

Row operation — Elementary Matrix

$$\Downarrow \qquad R_1 + R_2 \qquad E_9 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{11}{30} & \frac{1}{60} & -\frac{2}{30} \\ 0 & 1 & 0 & -\frac{4}{30} & \frac{14}{60} & -\frac{2}{30} \\ 0 & 0 & 1 & -\frac{1}{10} & \frac{1}{20} & \frac{1}{5} \end{bmatrix}$$

Then we have

$$A^{-1} = \frac{1}{60} \begin{bmatrix} 22 & 1 & -4 \\ -8 & 14 & -4 \\ -6 & 3 & 12 \end{bmatrix},$$

which can be written as the product of elementary matrices $E_9E_8E_7E_6E_5E_4E_3E_2E_1$.

□

5. For what values of $\lambda \in \mathbb{R}$ is the following matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & \lambda \end{bmatrix}$$

invertible?

Solution: We use elimination to determine when the reduced row echelon form is an identity matrix. We have

$$\begin{aligned} A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & \lambda \end{bmatrix} &\xRightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 4 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & \lambda \end{bmatrix} \\ &\xRightarrow{R_1/2} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & -1 & \lambda \end{bmatrix} \\ &\xRightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -2 \\ 0 & -3 & \lambda - 1 \end{bmatrix} \\ &\xRightarrow{R_2/(-6)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & -3 & \lambda - 1 \end{bmatrix} \\ &\xRightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \lambda \end{bmatrix}. \end{aligned}$$

The reduced row echelon form of A is I if and only if $\lambda \neq 0$. That is, A is invertible if and only if $\lambda \neq 0$.

□

6. Let A be an $n \times n$ matrix. If $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ satisfies that $r_2 = r_3 + r_1$, is A invertible?

Solution: A is not invertible since using elementary eliminations to substract r_1 and r_3 from r_2 will result in a matrix with a zero row which is not invertible. □

7. Let A be an $n \times n$ matrix. If $A = \begin{bmatrix} c_1 : c_2 : \cdots : c_n \end{bmatrix}$ satisfies that $c_2 = c_3 + c_1$, is A invertible?

Solution: By the previous question, A^T is not invertible and hence A is not invertible. □

8. Let $v,\, w \in \mathbb{R}^n$ be vectors. Is the matrix $A = \begin{bmatrix} \|v\| & 1 \\ |v \cdot w| & \|w\| \end{bmatrix}$ invertible?

Solution: A is not invertible if and only if $\|v\| \cdot \|w\| = |v \cdot w|$. □

9. Give an example of a 3×3 dominant matrix and find its inverse.

10. Find a sufficient condition on a, b, c and $d \in \mathbb{R}$ such that the matrix

$$A = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2cd & c^2 + d^2 \end{bmatrix}$$

is invertible.

11. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. i) Compute A^2; ii) Show that for every $k \geq 3$, $k \in \mathbb{N}$, $A^k = 0$.

Solution: $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore for every $k \geq 3$, $k \in \mathbb{N}$, $A^k = A^3 A^{k-3} = 0$. . □

12. Let A be an $n \times n$ matrix. Show that if $A^k = 0$, then $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}.$$

Solution: Hint: We have $(I - A)(I + A + A^2 + \cdots + A^{k-1}) = I$. □

13. Let A be an $n \times n$ matrix and $A = tI + N$, $t \in \mathbb{R}$ with $N^4 = 0$ for some $k \in \mathbb{N}$. Compute A^4 in terms of t and N.

Solution: Note that $tIN = N(tI)$. We have $A^4 = N^4 + (tI)N^3 + (tI)^2N^2 + (tI)^3N + (tI)^4 = tN^3 + t^2N^2 + t^3N + t^4I$. □

14. Let $D = \text{diag}\{\lambda_1,\, \lambda_2,\, \cdots,\lambda_n\}$ be a diagonal matrix with the main diagonal entries $\lambda_1,\, \lambda_2,\, \cdots,\lambda_n$. Show that D is invertible if and only if $\lambda_i \neq 0$, for every $i = 1,\, 2,\, \cdots, n$.

Solution: Hint: if $\lambda_i \neq 0$, for every $i = 1,\, 2,\, \cdots, n$, the reduced row echelon form of D is the identity matrix. □

15. Let A be an $n \times n$ matrix. i) If $A^3 = I$, find A^{-1}; ii) If $A^k = I$ for some $k \in \mathbb{N}$, find A^{-1}; iii) If $A^k = 0$ for some $k \in \mathbb{N}$, is it possible that A is invertible?

Solution: i) $A^{-1} = A^2$, ii) $A^{-1} = A^{k-1}$, iii). No. Notice that AB is invertible if and only if both A and B are invertible, where A and B are square matrices of the same size. □

16. Show that A is invertible if and only if A^k is invertible for every $k \in \mathbb{N}, k \geq 1$.

Solution: Notice that AB is invertible if and only if both A and B are invertible. The conclusion follows. $\square$

17. Let A and B be $n \times n$ invertible matrices. i) Give an example to show that $A+B$ may not be invertible; ii) Show that $A+B$ is invertible if and only if $A^{-1}+B^{-1}$ is invertible.

Solution: i) Let $A = I$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be 2×2 matrixes. Then A and B are both invertible while $A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

ii). Note that $A + B = A(A^{-1} + B^{-1})B$. $A + B$ is invertible if and only if $A^{-1} + B^{-1}$ is invertible. $\square$

2.4 *LU* decomposition

Exercise 2.4.1.

1. Let $-l_{ij}$ be the entry of the 4×4 E_{ij}^{-1} matrix below the main diagonal. Which one of the following products can be obtained by directly writing $-l_{ij}$ into the (i, j) position of the products? i) $E_{31}^{-1}E_{32}^{-1}E_{41}^{-1}E_{42}^{-1}E_{43}^{-1}$; ii) $E_{32}^{-1}E_{21}^{-1}E_{31}^{-1}E_{42}^{-1}E_{43}^{-1}$.

Solution: The point is that the row of the identity matrix to be added to another row should not the changed. i) Row 1, 2, 3 were not changed when they are used to change row 4 and row Row 1 and 2 were not changed when they are used to change row 3. So the matrix $E_{31}^{-1}E_{32}^{-1}E_{41}^{-1}E_{42}^{-1}E_{43}^{-1}$ can be obtained by directly writing $-l_{ij}$ into the (i, j) position of the products.

ii) Since row 2 were changed with E_{21}^{-1} before applying to E_{32}^{-1}, $E_{32}^{-1}E_{21}^{-1}E_{31}^{-1}E_{42}^{-1}E_{43}^{-1}$ can not be obtained by directly writing $-l_{ij}$ into the (i, j) position of the products.

$\square$

2. Find the *LU* decomposition of

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & 5 \end{bmatrix}.$$

Solution: Using elementary eliminations we have

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & \frac{4}{3} \\ 0 & 0 & 5 \end{bmatrix} := U,$$

where

$$E_{21} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{4} & 1 \end{bmatrix}.$$

Then we have $A = LU$, where

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{4} & 1 \end{bmatrix}.$$

□

3. Let $b = (1, 2, 3)$ and $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$. Use the LU decomposition of A to solve system $Ax = b$.

Solution: Since $A = LU$, we have $LUx = b$. Let $Ux = y$. Then $Ly = b$ with solution $y = (1, \frac{4}{3}, 3)$. Solving $Ux = y$ we have

$$x = \left(\frac{2}{15}, \frac{2}{15}, \frac{3}{5}\right).$$

□

4. Is it true that a matrix A does not have an LU decomposition? Justify your answer.

Solution: Yes, for example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

□

2.5 Transpose and permutation

Exercise 2.5.1.

1. Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Find A^{-1} and A^T.

Solution: A is a permutation matrix and is orthogonal.

$$A^{-1} = A^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

□

2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. i) Find AA^T and A^TA. ii) Determine which one of AA^T and A^TA is invertible. iii) If one of AA^T and A^TA is invertible, does it contradict Theorem 2.3.6?

Solution: i)

$$AA^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ii) A^TA is invertible.

iii) No. Theorem 2.3.6 is about square matrices. □

3. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{bmatrix}.$$

i) Find a permutation matrix P_1 such that $B = P_1A$; ii) Find a permutation matrix P_2 such that $A = P_2B$. iii) Compute P_1P_2 and P_2P_1.

Solution: Examining how the rows of A were rearranged to obtain B, we have

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_2 = P_1^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_1P_2 = P_2P_1 = I.$$

□

4. Let

$$A = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 7 \end{bmatrix}.$$

Find a permutation matrix P, a lower triangular matrix L and a diagonal matrix D such that $PA = LU$.

Solution:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/7 & 13/25 & 1 & 0 \\ 2/7 & 19/25 & 1/2 & 1 \\ 3/7 & 1 & 0 & 0 \end{bmatrix}, U = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 0 & 25/7 & 29/7 & 40/7 \\ 0 & 0 & 14/25 & 3/5 \\ 0 & 0 & 0 & -1/2 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

□

5. Let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Show that R_θ is an orthogonal matrix.

Solution: R_θ is an orthogonal matrix since we have

$$R_\theta {R_\theta}^T = I.$$

□

6. Let $x \in \mathbb{R}^n$ with $x^T x = 1$. Define the **Householder matrix** by

$$H = I - 2xx^T.$$

i) Show that H is an orthogonal matrix; ii) Show that H is symmetric.

Solution: i) Verify by the definition that $H^T H = I$. Indeed, we have

$$\begin{aligned} H^T H &= (I - 2xx^T)^T (I - 2xx^T) \\ &= (I - 2xx^T)(I - 2xx^T) \\ &= (I - 2xx^T) - 2xx^T(I - 2xx^T) \\ &= (I - 2xx^T) - 2xx^T + 4xx^T xx^T \\ &= I - 4xx^T + 4xx^T \\ &= I. \end{aligned}$$

ii) $H^T = (I - 2xx^T)^T = H$. H is symmetric. □

7. Let $S = \begin{bmatrix} I & A \\ A^T & O \end{bmatrix}$, where I is $m \times m$ and A is $m \times n$, O the zero matrix. Find a block diagonal matrix D and block lower triangular matrix L such that

$$S = LDL^T.$$

Solution: Let $L = \begin{bmatrix} I & O \\ -A^T & I \end{bmatrix}$. We have

$$LS = \begin{bmatrix} I & O \\ -A^T & I \end{bmatrix} \begin{bmatrix} I & A \\ A^T & O \end{bmatrix} = \begin{bmatrix} I & A \\ O & -A^T A \end{bmatrix}.$$

Then

$$LSL^T = \begin{bmatrix} I & O \\ -A^T & I \end{bmatrix} \begin{bmatrix} I & A \\ A^T & O \end{bmatrix} \begin{bmatrix} I & -A \\ O & I \end{bmatrix} = \begin{bmatrix} I & O \\ O & -A^T A \end{bmatrix}.$$

□

8. Show that AA^T is invertible if and only if the rows of A are linearly independent.

Solution: We know that the columns of A are linearly independent if and only if $A^T A$ is invertible. If follows that $(A^T)^T A^T$ is invertible if and only if the columns of A^T are linearly independent. That is, AA^T is invertible if and only if the rows of A are linearly independent. □

9. We say A is **skew-symmetric** if $A^T = -A$. i) Show that if A is a skew-symmetric $n \times n$ matrix then $a_{ii} = 0$ for every $i = 1, 2, \cdots, n$. ii) If A is both symmetric and skew-symmetric, then $A = 0$.

Solution: i) We have $A^T = -A$. Then it follows that for every $i = 1, 2, \cdots, n$,

$$(A^T)_{ii} = (A)_{ii} = (-A)_{ii} \Rightarrow a_{ii} = -a_{ii}.$$

We have $a_{ii} = 0$ for every $i = 1, 2, \cdots, n$.

ii) If A is both symmetric and skew-symmetric, then $A^T = -A = A$. That is, $A = 0$. □

10. Let A be an $n \times n$ matrix. Show that i) $A + A^T$ is symmetric; ii) $A - A^T$ is skew-symmetric; iii) For every square matrix B, there exist a unique symmetric matrix B_1 and a unique skew-symmetric matrix B_2 such that $B = B_1 + B_2$.

Solution: i) Since $(A + A^T)^T = A^T + A = A + A^T$, $A + A^T$ is symmetric.

ii) Since $(A - A^T)^T = A^T - A = -(A - A^T)$, $A - A^T$ is skew-symmetric. □

11. A matrix is called **lower triangular** if every entry above the main diagonal is zero and is called **upper triangular** if every entry below the main diagonal is zero. Let A be an $n \times n$ invertible matrix. i) Show that if A is lower triangular, A^{-1} is also lower triangular; ii) Show that if A is upper triangular, then A^{-1} is also upper triangular.

Solution: i) We use matrix partition. Suppose A and $B = A^{-1}$ are partitioned as following:

$$A = \begin{bmatrix} A_{11} & O \\ A_{21} & a_{nn} \end{bmatrix}, \ B = A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & b_{nn} \end{bmatrix}.$$

Then we have

$$BA = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & b_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & O \\ A_{21} & a_{nn} \end{bmatrix} = \begin{bmatrix} B_{11}A_{11} + B_{12}A_{21} & B_{12}a_{nn} \\ B_{21}A_{11} + b_{22}A_{21} & b_{nn}a_{nn} \end{bmatrix} = I_n,$$

where I_n is the $n \times n$ identity matrix. Since $a_{nn} \neq 0$, we have $B_{12} = O$. It follows that $B_{11}A_{11} = I_{n-1}$. By the same token, we can show that right upper block of B_{11} is zero. Repeat the same argument on the sub-martrices of B_{11}, we obtain that B is lower triangular.

ii) Notice that if A is invertible we have $(A^{-1})^T = (A^T)^{-1}$. The statement follows from i). $\square$