Chapter Two

1. The equation of motion for an operator in the Heisenberg picture is given by (2.2.19), so

$$\dot{S}_x = \frac{1}{i\hbar}[S_x, H] = -\frac{1}{i\hbar}\frac{eB}{mc}[S_x, S_z] = \frac{eB}{mc}S_y \qquad \dot{S}_y = -\frac{eB}{mc}S_x \quad \dot{S}_z = 0$$

and $\ddot{S}_{x,y} = -\omega^2 S_{x,y}$ for $\omega \equiv eB/mc$. Thus S_x and S_y are sinusoidal with frequency ω and S_z is a constant.

2. The Hamiltonian is not Hermitian, so the time evolution operator will not be unitary, and probability will not be conserved as a state evolves in time. As suggested, set $H_{11} = H_{22} = 0$. Then $H = a|1\rangle\langle 2|$ in which case $H^2 = a^2|1\rangle\langle 2|1\rangle\langle 1| = 0$. Since H is time-independent,

$$\mathcal{U}(t) = \exp\left(-\frac{i}{\hbar}Ht\right) = 1 - \frac{i}{\hbar}Ht = 1 - \frac{i}{\hbar}at|1\rangle\langle 2|$$

even for finite times t. Thus a state $|\alpha, t\rangle \equiv \mathcal{U}(t)|2\rangle = |2\rangle - (iat/\hbar)|1\rangle$ has a time-dependent norm. Indeed $\langle\alpha|\alpha\rangle = 1 + a^2t^2/\hbar^2$ which is nonsense. In words, it says that if you start out in the state $|2\rangle$, then the probability of finding the system in this state is unity at $t = 0$ and then grows with time. You can be more formal, and talk about an initial state $c_1|1\rangle + c_2|2\rangle$, but the bottom line is the same; probability is no longer conserved in time.

3. We have $\hat{\mathbf{n}} = \sin\beta\hat{\mathbf{x}} + \cos\beta\hat{\mathbf{z}}$ and $\mathbf{S} \doteq (\hbar/2)\boldsymbol{\sigma}$, so $\mathbf{S}\cdot\mathbf{n} \doteq (\hbar/2)(\sin\beta\sigma_x + \cos\beta\sigma_z)$ and we want to solve the matrix equation $\mathbf{S}\cdot\mathbf{n}\psi = (\hbar/2)\psi$ in order to find the initial state column vector ψ. This is, once again, a problem whose solution best makes use of the Pauli matrices, which are not introduced until Section 3.2. On the other hand, we can also make use of Problem 1.9 to write down the initial state. Either way, we find

$$\begin{aligned}
|\alpha, t=0\rangle &= \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle \qquad \text{so,} \\
|\alpha, t\rangle &= \exp\left[-\frac{i}{\hbar}\frac{eB}{mc}tS_z\right]|\alpha, t=0\rangle = e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)|-\rangle
\end{aligned}$$

for $\omega \equiv eB/mc$. From (1.4.17a), the state $|S_x;+\rangle = (1/\sqrt{2})|+\rangle + (1/\sqrt{2})|-\rangle$, so

$$\begin{aligned}
|\langle S_x;+|\alpha,t\rangle|^2 &= \left|\frac{1}{\sqrt{2}}e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right) + \frac{1}{\sqrt{2}}e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)\right|^2 \\
&= \frac{1}{2}\cos^2\left(\frac{\beta}{2}\right) + \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\beta}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2}\cos(\omega t)\sin\beta = \frac{1}{2}(1 + \sin\beta\cos\omega t)
\end{aligned}$$

which makes sense. For $\beta = 0$, the initial state is a z-eigenket, and there is no precession, so you just get 1/2 for the probability of measuring S_x in the positive direction. The same works out for $\beta = \pi$. For $\beta = \pi/2$, the initial state is $|S_x;+\rangle$ so the probabilty is $+1$ at $t = 0$ and 0 at $t = \pi/\omega = T/2$. Now from (1.4.18a), $S_x = (\hbar/2)[|+\rangle\langle-| + |-\rangle\langle+|]$, so

$$\begin{aligned}
\langle\alpha, t|S_x|\alpha, t\rangle &= \left[e^{i\omega t/2}\cos\left(\frac{\beta}{2}\right)\langle+| + e^{-i\omega t/2}\sin\left(\frac{\beta}{2}\right)\langle-|\right] \\
&\quad \frac{\hbar}{2}\left[e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)|+\rangle + e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right)|-\rangle\right] \\
&= \frac{\hbar}{2}\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\left[e^{i\omega t} + e^{-\omega t}\right] = \frac{\hbar}{2}\sin\beta\cos\omega t
\end{aligned}$$

Again, this makes perfect sense. The expectation value is zero for $\beta = 0$ and $\beta = \pi$, but for $\beta = \pi/2$, you get the classical precession of a vector that lies in the xy-plane.

4. First, restating equations from the textbook,

$$\begin{aligned}
|\nu_e\rangle &= \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle \\
|\nu_\mu\rangle &= \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \\
\text{and} \qquad E &= pc\left(1 + \frac{m^2c^2}{2p^2}\right)
\end{aligned}$$

Now, let the initial state $|\nu_e\rangle$ evolve in time to become a state $|\alpha, t\rangle$ in the usual fashion

$$\begin{aligned}
|\alpha, t\rangle &= e^{-iHt/\hbar}|\nu_e\rangle \\
&= \cos\theta e^{-iE_1t/\hbar}|\nu_1\rangle - \sin\theta e^{-iE_2t/\hbar}|\nu_2\rangle \\
&= e^{-ipct/\hbar}\left[e^{-im_1^2c^3t/2p\hbar}\cos\theta|\nu_1\rangle - e^{-im_2^2c^3t/2p\hbar}\sin\theta|\nu_2\rangle\right]
\end{aligned}$$

The probability that this state is observed to be a $|\nu_e\rangle$ is

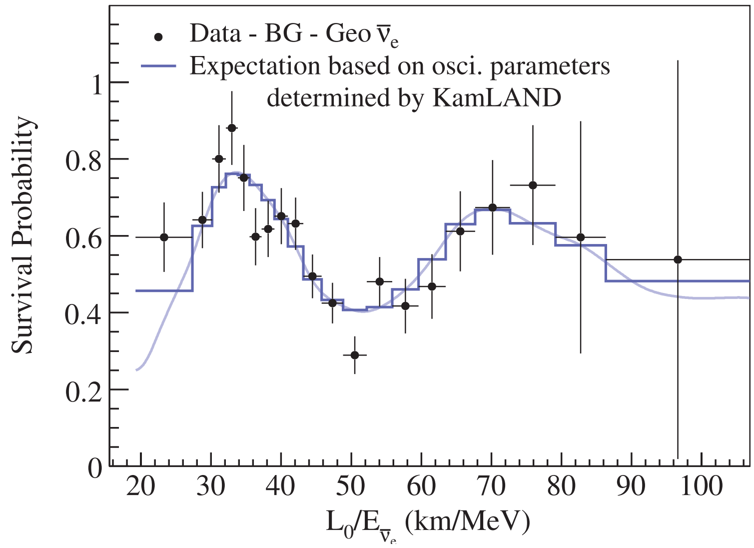
$$\begin{aligned}
P(\nu_e \to \nu_e) = |\langle\nu_e|\alpha, t\rangle|^2 &= \left|e^{-im_1^2c^3t/2p\hbar}\cos^2\theta + e^{-im_2^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\
&= \left|\cos^2\theta + e^{i\Delta m^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\
&= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos\left[\frac{\Delta m^2c^3t}{2p\hbar}\right] \\
&= 1 - \sin^2 2\theta\sin^2\left[\frac{\Delta m^2c^3t}{4p\hbar}\right]
\end{aligned}$$

Writing the nominal neutrino energy as $E = pc$ and the flight distance $L = ct$ we have

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta\sin^2\left[\Delta m^2c^4\frac{L}{4E\hbar c}\right]$$

It is quite customary to ignore the factor of c^4 and agree to measure mass in units of energy, typically eV.

The neutrino oscillation probability from KamLAND is plotted here:



The minimum in the oscillation probability directly gives us $\sin^2 2\theta$, that is

$$1 - \sin^2 2\theta \approx 0.4 \qquad \text{so} \qquad \theta \approx 25^\circ$$

The wavelength gives the mass difference parameter. We have

$$40\frac{\text{km}}{\text{MeV}} = 2\pi\frac{4\hbar c}{\Delta m^2} = \frac{8\pi \times 200 \text{ MeV fm}}{\Delta m^2}$$

where we explicitly agree to measure Δm^2 in eV^2. Therefore

$$\Delta m^2 = 40\pi \times 10^{12}\text{eV}^2 \times 10^{-15}/10^3 = 1.2 \times 10^{-4} \text{ eV}^2$$

The results from a detailed analysis by the collaboration, in Phys.Rev.Lett.100(2008)221803, are $\tan^2\theta = 0.56$ ($\theta = 37^\circ$) and $\Delta m^2 = 7.6 \times 10^{-5}$ eV^2. The full analysis not only includes the fact that the source reactors are at varying distances (although clustered at a nominal distance), but also that neutrino oscillations are over three generations.

5. *Note: This problem is worked through rather thoroughly in the text. See page 85.* First, $\dot{x} = (1/i\hbar)[x, H] = (1/i\hbar)[x, p^2/2m] = p/m$ (using Problem 1.29). However $\dot{p} = (1/i\hbar)[p, p^2/2m] = 0$ so $p(t) = p(0)$, a constant. Therefore $x(t) = x(0) + p(0)t/m$, and $[x(t), x(0)] = [x(0) + p(0)t/m, x(0)] = [p(0), x(0)]t/m = -i\hbar t/m$. By the generalized uncertainty principle(1.4.53), this means that the uncertainty in position grows with time. This conclusion is also a consequence of a study of "wave packets."

6. This is the proof of the so-called "dipole sum rule." Using Problem 1.29,

$$[H,x] = \left[\frac{p^2}{2m} + V(x), x\right] = -i\hbar\frac{p}{m} \qquad \text{so} \qquad [[H,x],x] = -\frac{\hbar^2}{m}$$

Now $[[H,x],x] = [H,x]x - x[H,x] = Hx^2 - xHx - xHx + x^2H = Hx^2 + x^2H - 2xHx$, and so $\langle a''|[[H,x],x]|a''\rangle = 2E''\langle a''|x^2|a''\rangle - 2\langle a''|xHx|a''\rangle = -\hbar^2/m$ from above. Inserting a complete set of states $|a'\rangle$ into each of the two terms on the left, we come up with

$$\begin{aligned}
\frac{\hbar^2}{2m} &= \langle a''|xHx|a''\rangle - E''\langle a''|x^2|a''\rangle \\
&= \sum_{a'}\left[\langle a''|xH|a'\rangle\langle a'|x|a''\rangle - E''\langle a''|x|a'\rangle\langle a'|x|a''\rangle\right] = \sum_{a'}(E'-E'')|\langle a''|x|a'\rangle|^2
\end{aligned}$$

7. We solve this in the Heisenberg picture, letting the operators be time dependent. Then

$$\begin{aligned}
\frac{d}{dt}\mathbf{x}\cdot\mathbf{p} &= \frac{1}{i\hbar}[\mathbf{x}\cdot\mathbf{p}, H] = \frac{1}{i\hbar}\left[xp_x + yp_y + zp_z, \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + V(\mathbf{x})\right] \\
&= \frac{1}{2i\hbar m}\left\{[x,p_x^2]p_x + [y,p_y^2]p_y + [z,p_z^2]p_z\right\} + \frac{1}{i\hbar}\mathbf{x}\cdot[\mathbf{p}, V(\mathbf{x})] \\
&= \frac{1}{m}(p_x^2+p_y^2+p_z^2) - x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} - z\frac{\partial V}{\partial z} = \frac{\mathbf{p}^2}{m} - \mathbf{x}\cdot\boldsymbol{\nabla}V
\end{aligned}$$

using (2.2.23). What does this mean if $d\mathbf{x}\cdot\mathbf{p}/dt = 0$? The original solution manual is elusive, so I'm not sure what Sakurai was getting at. In Chapter Three, we show that for the orbital angular momentum operator $\mathbf{L}$, one has $\mathbf{L}^2 = \mathbf{x}^2\mathbf{p}^2 - (\mathbf{x}\cdot\mathbf{p})^2 + i\hbar\mathbf{x}\cdot\mathbf{p}$, so it appears that there is a link between this quantity and conservation of angular momentum. So,...?

8. Firstly, $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ and (from Problem 5 above) $x(t) = x(0) + (p(0)/m)t$, so $\langle x(t)\rangle = \langle x(0)\rangle + (\langle p(0)\rangle/m)t = 0$ and $\langle(\Delta x)^2\rangle = \langle x^2\rangle$ at all times. Therefore we want

$$\langle(\Delta x)^2\rangle = \langle x^2(t)\rangle = \langle x^2(0)\rangle + \frac{t}{m}\langle x(0)p(0) + p(0)x(0)\rangle + \frac{t^2}{m^2}\langle p^2(0)\rangle$$

where the expectation value can be calculated for the state at $t=0$. For this (minimum uncertainty) state, we have $\Delta x = x(0) - \langle x(0)\rangle = x(0)$ and $\Delta p = p(0) - \langle p(0)\rangle = p(0)$, so from Problem 1.18(b) we have $\Delta p(0)|\rangle = ia\Delta x(0)|\rangle$ where a is real. Therefore

$$\langle(\Delta x)^2\rangle = \langle x^2(0)\rangle + \frac{t}{m}\left[ia\langle x^2(0)\rangle - ia\langle x^2(0)\rangle\right] + \frac{t^2}{m^2}(-ia)(ia)\langle x^2(0)\rangle = \langle x^2(0)\rangle\left[1 + \frac{a^2t^2}{m^2}\right]$$

where $\hbar^2/4 = \langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = a^2\langle x^2(0)\rangle$ sets $a^2 = \hbar^2/4\langle(\Delta x)^2\rangle|_{t=0}$. San Fu Tuan's original solution manual states that this agrees with the expansion of wave packets calculated using wave mechanics. This point should probably be investigated further.

9. The matrix representation of H in the $|a'\rangle$, $|a''\rangle$ basis is $H = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}$, so the characteristic equation for the eigenvalues is $(-E)^2 - \delta^2 = 0$ and $E = \pm\delta \equiv E_\pm$ with eigenstates $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$. This gives $|a'\rangle = (|E_+\rangle + |E_-\rangle)/\sqrt{2}$ and $|a''\rangle = (|E_+\rangle - |E_-\rangle)/\sqrt{2}$. Since the Hamiltonian is time-independent, the time evolved state is $\exp(-iHt/\hbar)|a'\rangle = (e^{-i\delta t/\hbar}|E_+\rangle + e^{i\delta t/\hbar}|E_-\rangle)/\sqrt{2}$. The probability to find this state at time t in the state $|a''\rangle$ is $|\langle a''|\exp(-iHt/\hbar)|a'\rangle|^2$, or

$$\frac{1}{4}\left|\left(\langle E_+| - \langle E_-\right)|\left(e^{-i\delta t/\hbar}|E_+\rangle + e^{i\delta t/\hbar}|E_-\rangle\right)\right|^2 = \frac{1}{4}\left|e^{-i\delta t/\hbar} - e^{i\delta t/\hbar}\right|^2 = \sin^2\left(\frac{\delta t}{\hbar}\right)$$

This is the classic two-state problem. Spin-1/2 is one example. Another is ammonia.

10. *This problem is nearly identical to Problem 9, only instead speciying two ways to determine the time-evolved state, plus Problem 2 tossed in at the end. Perhaps it should be removed from the next edition.*

(a) The energy eigenvalues are $E_\pm \equiv \pm\Delta$ with normalized eigenstates $|E_\pm\rangle = (|R\rangle \pm |L\rangle)/\sqrt{2}$.

(b) We have $|R\rangle = (|E_+\rangle + |E_-\rangle)/\sqrt{2}$ and $|L\rangle = (|E_+\rangle - |E_-\rangle)/\sqrt{2}$, so, with $\omega \equiv \Delta/\hbar$,

$$\begin{aligned} |\alpha, t\rangle &= e^{-iHt/\hbar}|\alpha, t=0\rangle = e^{-iHt/\hbar}|R\rangle\langle R|\alpha\rangle + e^{-iHt/\hbar}|L\rangle\langle L|\alpha\rangle \\ &= \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|E_+\rangle + e^{i\omega t}|E_-\rangle\right]\langle R|\alpha\rangle + \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|E_+\rangle - e^{i\omega t}|E_-\rangle\right]\langle L|\alpha\rangle \end{aligned}$$

(c) The initial condition means that $\langle R|\alpha\rangle = 1$ and $\langle L|\alpha\rangle = 0$, so we calculate

$$|\langle L|\alpha, t\rangle|^2 = \frac{1}{4}\left|\left(\langle E_+| - \langle E_-\right)|\left(e^{-i\omega t}|E_+\rangle + e^{i\omega t}|E_-\rangle\right)\right|^2 = \frac{1}{4}\left|e^{-i\omega t} - e^{i\omega t}\right|^2 = \sin^2\omega t$$

(d) *This is the only part of the problem that is "new." Indeed, Problem 9 could have been done this way, instead of using the time propagation operator.* Using (2.1.27) we write

$$i\hbar\frac{\partial}{\partial t}\langle R|\alpha, t\rangle = \langle R|H|\alpha, t\rangle \qquad \text{and} \qquad i\hbar\frac{\partial}{\partial t}\langle L|\alpha, t\rangle = \langle L|H|\alpha, t\rangle$$

Let $\psi_R(t) \equiv \langle R|\alpha, t\rangle$ and $\psi_L(t) \equiv \langle L|\alpha, t\rangle$. These coupled equations become

$$i\hbar\dot{\psi}_R = \frac{1}{\sqrt{2}}\left(\Delta\langle E_+| - \Delta\langle E_-\right)|\alpha, t\rangle = \Delta\psi_L \qquad \text{and} \qquad i\hbar\dot{\psi}_L = \Delta\psi_R$$

or $\dot{\psi}_R = -i\omega\psi_L$ and $\dot{\psi}_L = -i\omega\psi_R$, so $\psi_R(t) = Ae^{i\omega t} + B^{-i\omega t}$ and $\psi_L(t) = Ce^{i\omega t} + D^{-i\omega t}$. These are just (b) where $A = \langle R|E_+\rangle$, $B = \langle R|E_-\rangle$, $C = \langle L|E_+\rangle$, and $D = \langle L|E_-\rangle$.

(e) See Problem 2. It can be embellished by in fact solving the most general time-evolution problem, but in the end, the point will still be that probability is not conserved.

11. Restating this problem: *Using the one-dimensional simple harmonic oscillator as an example, illustrate the difference between the Heisenberg picture and the Schrödinger picture. Discuss in particular how (a) the dynamic variables x and p and (b) the most general state vector evolve with time in each of the two pictures.*

This problem, namely 2.10 in the previous edition, is rather open ended, atypical for most of the problems in the book. Perhaps it should be revised. Most of the problem is in fact covered on pages 94 to 96. Anyway, we start from the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2 = \left(N + \frac{1}{2}\right)\hbar\omega$$

(a) In the Schrödinger picture, x and p do not evolve in time. In the Heisenberg picture

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{i\hbar}[x,H] = \frac{1}{2im\hbar}[x,p^2] = \frac{1}{2im\hbar}i\hbar(2p) = \frac{p}{m} \\ \frac{dp}{dt} &= \frac{1}{i\hbar}[p,H] = \frac{m\omega^2}{2i\hbar}[p,x^2] = \frac{m\omega^2}{2i\hbar}(-i\hbar)(2x) = -m\omega^2x \end{aligned}$$

using Problem 1.29. These are just the classical Hamilton's equations, with a force $-\omega^2 x$. Solving these coupled equations are simple, yielding sinusoidal motion at frequency ω for x and p. One can also recognize that the two pictures coincide at $t = 0$, and then get Heisenberg from Schrödinger using $x_H(t) = \exp(iHt/\hbar)x(0)\exp(-iHt/\hbar)$ and expanding the exponentials. Similarly for momentum.

(b) In the Heisenberg picture, state vectors are stationary. For the Schrödinger picture, it is easiest to expand in terms of eigenstates of N, that is $|\alpha,t\rangle = \sum c_n(t)|n\rangle$, so (2.1.27) gives

$$i\hbar\sum_n \dot{c}_n(t)|n\rangle = H|\alpha,t\rangle = \sum_n\left(n+\frac{1}{2}\right)\hbar\omega c_n(t)|n\rangle$$

in which case $c_n(t) = \exp[-i(n+1/2)\omega t]$, using orthonormality of the $|n\rangle$.

12. *Not enough information is given in the problem statement. The state $|0\rangle$ is one for which $\langle x\rangle = 0 = \langle p\rangle$.* As described in the solution to Problem 11, in the Heisenberg picture, the position operator is $x(t) = x(0)\cos(\omega t) + (p(0)/m)\sin(\omega t)$, and $\langle x\rangle = \langle t=0|x(t)|t=0\rangle$. Since $e^{ip/\hbar}xe^{-ipa/\hbar} = e^{ip/\hbar}\{[x, e^{-ipa/\hbar}] + e^{-ipa/\hbar}x\} = e^{ip/\hbar}i\hbar(-ia/\hbar)e^{-ipa/\hbar} + x = x + a$, using Problem 1.29, the expectation value of position is

$$\begin{aligned} \langle x\rangle &= \langle 0|e^{ip/\hbar}x(0)e^{-ipa/\hbar}|0\rangle\cos(\omega t) + \langle 0|e^{ip/\hbar}p(0)e^{-ipa/\hbar}|0\rangle\sin(\omega t) \\ &= \langle 0|[x(0)+a]|0\rangle\cos(\omega t) + \langle 0|p(0)|0\rangle\sin(\omega t) = a\cos(\omega t) \end{aligned}$$

Since the state $e^{-ipa/\hbar}|0\rangle$ represents a position displaced by a distance a (See Problem 1.28), we have the classical motion of a harmonic oscillator starting from rest with amplitude a.

13. Making use of (1.6.36), we recognize $\mathcal{T}(a) = \exp(-ipa/\hbar)$ as the operator that translates in x by a distance a. Therefore $\langle x'|\mathcal{T}(a) = \langle x' - a|$ and

$$\langle x'|e^{-ipa/\hbar}|0\rangle = \langle x' - a|0\rangle = \frac{1}{\pi^{1/4}}\frac{1}{x_0^{1/2}}\exp\left[-\frac{1}{2}\left(\frac{x'-a}{x_0}\right)^2\right]$$

The probability to find the state $e^{-ipa/\hbar}|0\rangle$ in the ground state $|0\rangle$ is the square of

$$\langle 0|e^{-ipa/\hbar}|0\rangle = \int dx'\langle 0|x'\rangle\langle x'|e^{-ipa/\hbar}|0\rangle = \frac{1}{\pi^{1/2}}\frac{1}{x_0}\int_{-\infty}^{\infty} dx' e^{-[(x'-a)^2+x'^2]/2x_0^2}$$

The integral is simple to do by completing the square. Write

$$(x'-a)^2 + x'^2 = 2\left[x'^2 - ax' + \frac{a^2}{2}\right] = 2\left[\left(x' - \frac{a}{2}\right)^2\right] + \frac{a^2}{2}$$

and shift the integration variable by $a/2$. You end up with

$$\langle 0|e^{-ipa/\hbar}|0\rangle = \frac{1}{\pi^{1/2}}\frac{1}{x_0}e^{-a^2/4x_0^2}\int_{-\infty}^{\infty} dy e^{-y^2/x_0^2} = e^{-a^2/4x_0^2}$$

so the probability is just $e^{-a^2/2x_0^2}$. This is indeed time-independent.

14. Rearranging, we have $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$ and $p = i\sqrt{\hbar m\omega/2}(a^\dagger - a)$, therefore

$$\begin{aligned}
x|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle\right] \\
p|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}|n+1\rangle - \sqrt{n}|n-1\rangle\right] \\
\langle m|x|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle m|(a + a^\dagger)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m.n+1}\right] \\
\langle m|p|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\langle m|(a^\dagger - a)|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\delta_{m.n+1} - \sqrt{n}\delta_{m,n-1}\right] \\
\langle m|\{x,p\}|n\rangle &= \langle m|(xp + px)|n\rangle = \langle m|xp|n\rangle + \langle n|xp|m\rangle^* \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\langle m|x|n+1\rangle - \sqrt{n}\langle m|x|n-1\rangle\right. \\
&\quad \left.-\sqrt{m+1}\langle n|x|m+1\rangle + \sqrt{m}\langle n|x|m-1\rangle\right] \\
&= i\frac{\hbar}{2}\left[(n+1)\delta_{nm} + \sqrt{(n+1)(n+2)}\delta_{n+2,m} - \sqrt{n(n-1)}\delta_{n-2,m} - n\delta_{nm}\right. \\
&\quad \left.-(m+1)\delta_{nm} - \sqrt{(m+1)(m+2)}\delta_{n,m+2} + \sqrt{m(m-1)}\delta_{n,m-2} + m\delta_{nm}\right] \\
&= i\hbar\left[\sqrt{(n+1)(n+2)}\delta_{n+2,m} - \sqrt{n(n-1)}\delta_{n-2,m}\right]
\end{aligned}$$

$$\begin{aligned}
\langle m|x^2|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\langle m|x|n-1\rangle + \sqrt{n+1}\langle m|x|n+1\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\sqrt{n(n-1)}\delta_{n-2,m} + (2n+1)\delta_{nm} + \sqrt{(n+1)(n+2)}\delta_{n+2,m}\right] \\
\langle m|p^2|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\langle m|p|n+1\rangle - \sqrt{n}\langle m|p|n-1\rangle\right] \\
&= -\frac{\hbar m\omega}{2}\left[\sqrt{(n+1)(n+2)}\delta_{n+2,m} - (2n+1)\delta_{nm} + \sqrt{n(n-1)}\delta_{n-2,m}\right]
\end{aligned}$$

Now, the virial theorem in three dimensions is quoted as

$$\left\langle \frac{\mathbf{p}^2}{m} \right\rangle = \langle \mathbf{x}\cdot\boldsymbol{\nabla}V\rangle \qquad \text{or} \qquad \left\langle \frac{p^2}{m} \right\rangle = \left\langle x\frac{dV}{dx} \right\rangle$$

in one dimension. For the harmonic oscillator, $xdV/dx = m\omega^2x^2$. So, evaluating the expectation value in the state $|n\rangle$ using the calculations above, we have

$$\left\langle \frac{p^2}{m} \right\rangle = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega\left(n+\frac{1}{2}\right) \qquad \text{and} \qquad \left\langle x\frac{dV}{dx} \right\rangle = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega\left(n+\frac{1}{2}\right)$$

and the virial theorem is indeed satisfied.

15. Turning around what is given, $\langle p'|x'\rangle = (2\pi\hbar)^{-1/2}e^{-ip'x'/\hbar}$. Then

$$\begin{aligned}
\langle p'|x|\alpha\rangle &= \int dx'\langle p'|x'\rangle\langle x'|x|\alpha\rangle = \int dx' x'\langle p'|x'\rangle\langle x'|\alpha\rangle \\
&= i\hbar\int dx'\frac{\partial}{\partial p'}\langle p'|x'\rangle\langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\int dx'\langle p'|x'\rangle\langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle
\end{aligned}$$

For the Hamiltonian $H = p^2/2m + m\omega^2x^2/2$ with eigenvalues E, the wave equation in momentum space is $\langle p'|H|\alpha\rangle = E\langle p'|\alpha\rangle \equiv Eu_\alpha(p')$, and the second term in $\langle p'|H|\alpha\rangle$ is

$$\frac{m\omega^2}{2}\langle p'|x^2|\alpha\rangle = \frac{m\omega^2}{2}i\hbar\frac{\partial}{\partial p'}\langle p'|x|\alpha\rangle = -\frac{m\hbar^2\omega^2}{2}\frac{\partial^2}{\partial p'^2}\langle p'|\alpha\rangle = -\frac{m\hbar^2\omega^2}{2}\frac{d^2u_\alpha}{dp'^2}$$

With a little rearranging, the wave equation becomes

$$-\frac{m\hbar^2\omega^2}{2}\frac{d^2u_\alpha}{dp'^2} + \frac{1}{2m}p'^2u_\alpha(p') = Eu_\alpha(p')$$

which is the same as (2.5.13) but with $m\omega^2$ replaced with $1/m$. Inserting this same substitution into (2.5.28) therefore gives the wave functions in momentum space.

16. From (2.3.45a), $x(t) = x(0)\cos\omega t + [p(0)/m\omega]\sin\omega t$, so

$$C(t) \equiv \langle 0|x(t)x(0)|0\rangle = \langle 0|x(0)x(0)|0\rangle\cos\omega t + (1/m\omega)\langle 0|p(0)x(0)|0\rangle\sin\omega t$$

The matrix elements can be calculated by the techniques in Problem 14. You find that $\langle 0|x(0)x(0)|0\rangle = \hbar/2m\omega$ and $\langle 0|p(0)x(0)|0\rangle = -i\hbar/2$. *(Note: Error in old solutions manual.)* Therefore $C(t) = (\hbar/2m\omega)\cos\omega t - i(\hbar/2m\omega)\sin\omega t$.

17. Write $|\alpha\rangle = a|0\rangle + b|1\rangle$, with a, b real and $a^2 + b^2 = 1$. Using Problem 14,

$$\langle\alpha|x|\alpha\rangle = a^2\langle 0|x|0\rangle + ab\langle 0|x|1\rangle + ab\langle 1|x|0\rangle + b^2\langle 1|x|1\rangle = 2ab\sqrt{\frac{\hbar}{2m\omega}}$$

The maximum is obtained when $a = b = 1/\sqrt{2}$ so $\langle x\rangle = \sqrt{\hbar/2m\omega}$.

The state vector in the Schrödinger picture is $|\alpha, t\rangle = e^{-iHt/\hbar}|\alpha\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega t/2}|0\rangle + e^{-3\omega t/2}|1\rangle\right]$ and the expectation value $\langle\alpha, t|x|\alpha, t\rangle$, again using Problem 14, is

$$\langle x\rangle = \frac{1}{2}e^{-i\omega t}\langle 0|x|1\rangle + \frac{1}{2}e^{i\omega t}\langle 1|x|0\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(e^{-i\omega t} + e^{i\omega t}) = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$

In the Heisenberg picture, use $x(t)$ from (2.3.45a), and again Problem 14. In this case, we note that $\langle 0|p|1\rangle = \langle 1|p|0\rangle = 0$, so we read off $\langle x\rangle = \sqrt{\hbar/2m\omega}\cos\omega t$.

To evaluate $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, we just need to calculate $\langle x^2\rangle$. Use the state vector in the Schrödinger picture, and read off matrix elements of x^2 from Problem 14, to get

$$\langle x^2\rangle = \frac{1}{2}\langle 0|x^2|0\rangle + \frac{1}{2}e^{-i\omega t}\langle 0|x^2|1\rangle + \frac{1}{2}e^{i\omega t}\langle 1|x^2|0\rangle + \frac{1}{2}\langle 1|x^2|1\rangle = \frac{1}{2}\frac{\hbar}{2m\omega}[1+3] = \frac{\hbar}{m\omega}$$

so $\langle(\Delta x)^2\rangle = (\hbar/m\omega)(1 - \frac{1}{2}\cos^2\omega t)$.

18. Somehow, it seems this problem should be worked by considering $\langle 0|x^{2n}|0\rangle$, but I don't see it. So, instead, work the left and right sides separately. For the right side, from Problem 14, $\exp[-k^2\langle 0|x^2|0\rangle/2] = \exp[-k^2\hbar/4m\omega]$. For the left side, use position space to write

$$\langle 0|e^{ikx}|0\rangle = \int dx'\langle 0|e^{ikx}|x'\rangle\langle x'|0\rangle = \int dx' e^{ikx'}|\langle x|0\rangle|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\int dx' e^{ikx'}e^{-m\omega x'^2/\hbar}$$

Put $x' = u\sqrt{\hbar/m\omega}$ and write $-u^2 + iku\sqrt{\hbar/m\omega} = -(u - ik\sqrt{\hbar/m\omega}/2)^2 - \hbar k^2/4m\omega$. Then, putting $w = u - ik\sqrt{\hbar/m\omega}/2$, we have

$$\langle 0|e^{ikx}|0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}e^{-\hbar k^2/4m\omega}\int dw e^{-w^2} = \frac{1}{\sqrt{\pi}}e^{-\hbar k^2/4m\omega}\sqrt{\pi} = e^{-\hbar k^2/4m\omega}$$

and the two sides are indeed equal.

19. It will be useful to note that, from (2.3.21), $(a^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle$. So

$$\begin{aligned} a\left[e^{\lambda a^\dagger}|0\rangle\right] &= a\left[\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(a^\dagger)^n|0\rangle\right] = a\left[\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right] = \sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{n!}}a|n\rangle \\ &= \sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}|n-1\rangle = \lambda\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}|m\rangle = \lambda\left[e^{\lambda a^\dagger}|0\rangle\right] \end{aligned}$$

so $e^{\lambda a^\dagger}|0\rangle$ is an eigenvector of a with eigenvalue λ. For the normalization, we need the inner produce of $e^{\lambda a^\dagger}|0\rangle$ with itself. However, $\langle 0|e^{\lambda^* a}e^{\lambda a^\dagger}|0\rangle = \langle 0|e^{\lambda^*\lambda}|0\rangle = e^{|\lambda|^2}$ since $e^{\lambda a^\dagger}|0\rangle$ is an eigenvector of a with eigenvalue λ. Thus $|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$ is the normalized eigenvector.

Now we have $a|\lambda\rangle = \lambda|\lambda\rangle$ and $\langle\lambda|a^\dagger = \langle\lambda|\lambda^*$, so $\langle\lambda|(a^\dagger \pm a)|\lambda\rangle = \lambda^* \pm \lambda$; $\langle\lambda|(a)^2|\lambda\rangle = \lambda^2$; $\langle\lambda|(a^\dagger)^2|\lambda\rangle = (\lambda^*)^2$; $\langle\lambda|a^\dagger a|\lambda\rangle = \lambda^*\lambda$; and $\langle\lambda|aa^\dagger|\lambda\rangle = \langle\lambda|(1+a^\dagger a)|\lambda\rangle = 1+\lambda^*\lambda$. Therefore

$$\begin{aligned} \langle x\rangle &= \langle\lambda|x|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\lambda^*+\lambda) \\ \langle x^2\rangle &= \frac{\hbar}{2m\omega}\left[\lambda^2+(\lambda^*)^2+\lambda^*\lambda+(1+\lambda^*\lambda)\right] \\ (\Delta x)^2 &= \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega} \\ \langle p\rangle &= \langle\lambda|p|\lambda\rangle = i\sqrt{\frac{m\hbar\omega}{2}}(\lambda^*-\lambda) \\ \langle p^2\rangle &= -\frac{m\hbar\omega}{2}\left[\lambda^2+(\lambda^*)^2-\lambda^*\lambda-(1+\lambda^*\lambda)\right] \\ (\Delta p)^2 &= \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2} \end{aligned}$$

so $\Delta x\Delta p = \hbar/2$ and the minimum uncertainty relation is indeed satisfied. Now, from above,

$$|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty}f(n)|n\rangle \qquad \text{so} \qquad |f(n)|^2 = e^{-|\lambda|^2}\frac{|\lambda|^{2n}}{n!}$$

which is a Poisson distribution $P_n(\mu) = e^{-\mu}\mu^n/n!$ with mean $\mu \equiv |\lambda|^2$. Note that the mean value of n is not the same as the most probable value, which is an integer, although they approach the same value for large μ, when the Poisson distribution approaches a Gaussian. However, $P_n(\mu)/P_{n-1}(\mu) = \mu/n > 1$ only if $n < \mu$, so the most probable value of n is the largest integer n_m less than $|\lambda^2|$, and the energy is $(n_m+1)\hbar\omega$. To evaluate $e^{-ip\ell/\hbar}|0\rangle = e^{\ell\sqrt{m\omega/2\hbar}(a^\dagger-a)}|0\rangle$, use $e^{A+B} = e^Ae^Be^{-[A,B]/2}$ where A and B each commute with $[A,B]$. (See Gottfried, 1966, page 262; Gottfried, 2003, problem 2.13; or R. J. Glauber, Phys. Rev. 84(1951)399, equation 39.) With $\lambda \equiv \ell\sqrt{m\omega/2\hbar}$, we then easily prove the last part, as

$$e^{-ip\ell/\hbar}|0\rangle = e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}e^{-\ell\sqrt{m\omega/2\hbar}a}e^{-\ell^2m\omega/4\hbar}|0\rangle = e^{-m\ell^2\omega/4\hbar}e^{\ell\sqrt{m\omega/2\hbar}a^\dagger}|0\rangle = e^{-\lambda^2/2}e^{\lambda a^\dagger}|0\rangle$$

20. *Note the entry in the errata;* $\mathbf{J}^2$ *is not yet defined at this point in the text.* The solution is straightforward. We have $[a_\pm, a_\pm^\dagger] = 1$ and $[a_\pm, a_\mp^\dagger] = 0 = [a_\pm^\dagger, a_\mp^\dagger] = [a_\pm, a_\mp]$. Then

$$\begin{aligned}
[J_z, J_+] &= \frac{\hbar^2}{2}\left(a_+^\dagger a_+ a_+^\dagger a_- - a_+^\dagger a_- a_+^\dagger a_+ - a_-^\dagger a_- a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{2}\left(a_+^\dagger a_+ a_+^\dagger a_- - a_+^\dagger a_-(a_+ a_+^\dagger - 1) - a_-^\dagger a_- a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{2}\left(a_+^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_- + a_+^\dagger a_- a_-^\dagger a_-\right) = \frac{\hbar^2}{2} a_+^\dagger \left(a_- - a_-^\dagger a_- a_- + a_- a_-^\dagger a_-\right) \\
&= \frac{\hbar^2}{2} a_+^\dagger \left(a_- - a_-^\dagger a_- a_- + (1 + a_-^\dagger a_-) a_-\right) = \hbar^2 a_+^\dagger a_- = +\hbar J_+
\end{aligned}$$

and similarly for $[J_z, J_-]$. Put $N_\pm = a_\pm^\dagger a_\pm$ so $J_z = (\hbar/2)(N_+ - N_-)$ with $[N_+, N_-] = 0$. From (3.5.24), $\mathbf{J}^2 = J_+ J_- + J_z^2 - \hbar J_z$, so $J_+ J_- = \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ = \hbar^2 N_+(1 + a_-^\dagger a_-) = \hbar^2 N_+(1 + N_-)$, so $\mathbf{J}^2 = \frac{\hbar^2}{4}(N_+^2 + 2N_+N_- + N_-^2 + 2N_+ + 2N_-) = \frac{\hbar^2}{4}(N^2 + 2N) = \frac{\hbar^2}{2} N\left(\frac{N}{2} + 1\right)$. Finally, noting that we can write both $\mathbf{J}^2$ and J_z in terms of $N_\pm$, which commute, we clearly have $[\mathbf{J}^2, J_z] = 0$.

21. Starting with (2.5.17a), namely $g(x,t) = \exp(-t^2 + 2tx)$, carry out the suggested integral

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x,t) g(x,s) e^{-x^2} dx &= = \int_{-\infty}^{\infty} e^{2st-(t+s)^2+2x(t+s)-x^2} dx \\
&= e^{2st}\int_{-\infty}^{\infty} e^{-[x-(t+s)]^2} dx = \pi^{1/2} e^{2st}
\end{aligned}$$

$$\text{i.e.}\qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx\right]\frac{1}{(n!)^2} t^n s^m = \pi^{1/2}\sum_{n=0}^{\infty}\frac{2^n}{n!} t^n s^n$$

The sum on the right only includes terms where t and s have the same power, so the normalization integral on the left must be zero if $n \neq m$. When $n = m$ this gives

$$\begin{aligned}
\left[\int_{-\infty}^{\infty} H_n(x) H_n(x) e^{-x^2} dx\right]\frac{1}{(n!)^2} &= \pi^{1/2}\frac{2^n}{n!} \\
\text{or}\qquad \int_{-\infty}^{\infty} H_n^2(x) e^{-x^2} dx &= \pi^{1/2} 2^n n!
\end{aligned}$$

which is (2.5.29). In order to normalize the wave function (2.5.28), we compute

$$\int_{-\infty}^{\infty} u_n^*(x) u_n(x) dx = |c_n|^2 \int_{-\infty}^{\infty} H_n^2\left(x\sqrt{\frac{m\omega}{\hbar}}\right) e^{-m\omega x^2/\hbar} dx = |c_n|^2 \sqrt{\frac{\hbar}{m\omega}}\pi^{1/2} 2^n n! = 1$$

so that $c_n = (m\omega/\pi\hbar)^{1/4}(2^n n!)^{-1/2}$, taking c_n to be real. Compare to (B.4.3).

22. This is a harmonic oscillator with $\omega = \sqrt{k/m}$ for $x > 0$, with $\langle x|n\rangle = 0$ at $x = 0$, that is, solutions with odd n. So, the ground state has energy $3\hbar\omega/2$. The wave function is given by (B.4.3), times $\sqrt{2}$ for normalization, that is $u(x) = 2(m\omega/\pi\hbar)^{1/4}e^{-m\omega x^2/2\hbar}x\sqrt{m\omega/\hbar}$, for $x > 0$, and $u(x) = 0$ for $x < 0$. We than calculate the expectation value

$$\langle x^2\rangle = \frac{4m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\int_0^\infty x^4 e^{-m\omega x^2/\hbar}dx = \frac{4m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\frac{3}{8}\left(\frac{\hbar}{m\omega}\right)^2\sqrt{\frac{\pi\hbar}{m\omega}} = \frac{3}{2}\frac{\hbar}{m\omega}$$

23. From (B.2.4), $u_n(x) = \langle x|n\rangle = \sqrt{2/L}\sin(n\pi x/L)$ and $E_n = n^2\pi^2\hbar^2/2mL^2$, so

$$\psi(x,t) = \langle x|\alpha,t\rangle = \langle x|e^{-iHt/\hbar}|\alpha,0\rangle = \sum_n \langle x|e^{-iHt/\hbar}|n\rangle\langle n|\alpha,0\rangle = \sum_n c_n e^{-iE_nt/\hbar}u_n(x)$$

where $c_n \equiv \langle n|\alpha,0\rangle$. Now, I take a hint from the previous solutions manual, that "known to be exactly at $x = L/2$ with certainty" and "You need not worry about normalizations" mean that $\langle x|\alpha,0\rangle \equiv \psi(x,0) = \delta(x - L/2)$, so $c_n = \int_0^L \psi(x,0)u_n(x)dx = \sqrt{2/L}\sin(n\pi/2)$. I don't like this; it seems that $\psi(x,0) = \sqrt{\delta(x - L/2)}$ is a better choice, but how well defined is "known with certainty"? Anyway, $c_n = 0$ if n is even, and $c_n = \sqrt{2/L}(-1)^{(n-1)/2}$ if n is odd, and $|c_n|^2 = 0$ or $|c_n|^2 = 2/L$, i.e. independent of n, for n odd. Then, insert in above.

24. Write the energy eigenvalue as $-E < 0$ for a bound state, so the Schrödinger Equation is $(-\hbar^2/2m)d^2u/dx^2 - \nu_0\delta(x)u(x) = -Eu(x)$. Thus $u(x) = A\exp(-x\sqrt{2mE}/\hbar)$ for $x > 0$, and $u(x) = A\exp(+x\sqrt{2mE}/\hbar)$ for $x < 0$, and $du/dx = \mp(\sqrt{2mE}/\hbar)u(x)$. Now integrate the Schrödiner Equation from $-\varepsilon$ to $+\varepsilon$, and then take $\varepsilon \to 0$. You end up with

$$\lim_{\varepsilon\to 0}\left\{-\frac{\hbar^2}{2m}\frac{\sqrt{2mE}}{\hbar}[-u(\varepsilon) - u(-\varepsilon)]\right\} - \nu_0 u(0) = \frac{\hbar^2}{m}\frac{\sqrt{2mE}}{\hbar}u(0) - \nu_0 u(0) = 0$$

which gives $E = m\nu_0^2/2\hbar^2$. This is unique, so there is only the ground state.

25. For this problem, I just reproduce the solution from the manual for the revised edition. (Note that "problem 22" means "problem 24" here.) See the errata for some comments.

Using the result of problem 22, where $2mE/\hbar^2 = \lambda^2 m^2/\hbar^4$ in our notation, we have $\psi(x,t=0) = A\exp[-m\lambda|x|/\hbar^2]$. The normalization is then $2A^2\int_0^\infty \exp[-2m\lambda x/\hbar^2]dx = 1$ or $2A^2[\hbar^2/2m\lambda] = 1$ and hence $A = (m\lambda/\hbar^2)^{1/2}$. From (2.5.7) and (2.5.16), we have

$$\psi(x,t>0) = \int_{-\infty}^{\infty} dx'\psi(x',0)K(x,x';t)$$

$$= (m\lambda/\hbar^2)^{1/2}(m/2\pi i\hbar t)^{1/2}\int_{-\infty}^{\infty} \exp[-m\lambda|x'|/\hbar^2]\exp[i(x-x')^2 m/2\hbar t]dx'$$

where we have used $\psi(x',0) = (m\lambda/\hbar^2)^{1/2}\exp[-m\lambda|x'|/\hbar^2]$.

26. With $V(x) = \lambda x$, $\lambda > 0$ and $-\infty < x < \infty$, the eigenvalues E are continuous. The wave function is oscillatory for $x < a$ and decaying for $x > a$, where $a \equiv E/\lambda$ is the classical turning point. Indeed, the wave function is proportional to the Airy function $Ai(z)$ where $z \propto (x - a)$. See Figure 2.3. On the other hand, for $V(x) = \lambda|x|$, there are now quantized bound states. This parity-symmetric potential has even and odd wave functions. The even wave functions have $Ai'(z) = 0$ at $x = 0$, and the odd wave functions have $Ai(z) = 0$ at $x = 0$. These conditions lead to quantized energies through (2.5.34) and (2.5.35). As shown in Figure 2.4, the odd energy levels have been confirmed by "bouncing neutrons."

27. *Note: This was Problem 36 in Chapter Five in the Revised Edition. It was moved to this chapter because "density of states" is explicitly worked out now in this chapter. It seems, though, that I should have reworded the problem a bit. See the errata.*

Refer back to the discussion in Section 2.5. The wave function is

$$u_E(\mathbf{x}) = \frac{1}{L}e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \text{where} \qquad k_x = \frac{2\pi}{L}n_x \qquad \text{and} \qquad k_y = \frac{2\pi}{L}n_y$$

and n_x and n_y are integers, with $\mathbf{p} = \hbar\mathbf{k}$. The energy is

$$\begin{aligned} E &= \frac{\mathbf{p}^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2) = \frac{2\pi^2\hbar^2}{mL^2}(n_x^2 + n_y^2) = \frac{2\pi^2\hbar^2}{mL^2}\mathbf{n}^2 \\ \text{so} \qquad dE &= \frac{4\pi^2\hbar^2}{mL^2}ndn \end{aligned}$$

The number of states with $|\mathbf{n}|$ between n and $n + dn$, and ϕ and $\phi + d\phi$, is

$$dN = ndnd\phi = m\left(\frac{L}{2\pi\hbar}\right)^2 dEd\phi$$

so the density of states is just $m(L/2\pi\hbar)^2$. Remarkably, this result is independent of energy.

28. We want to solve (2.5.1) in cylindrical coordinates, that is find $u(\rho, \phi, z)$ where

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{2m_e E}{\hbar^2}u \equiv -k^2 u$$

subject to $u(\rho_a, \phi, z) = u(\rho_b, \phi, z) = u(\rho, \phi, 0) = u(\rho, \phi, L) = 0$. For $u(\rho, \phi, z) = w(\rho, z)\Phi(\phi)$,

$$\frac{1}{w}\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right) + \rho^2\frac{\partial^2 w}{\partial z^2}\right] + \rho^2 k^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

The first two terms are independent of ϕ, and the third term is independent of ρ and z, so they both must equal some constant but with opposite sign. Write $(1/\Phi)\partial^2\Phi/\partial\phi^2 = -m^2$, giving $\Phi(\phi) = e^{\pm im\phi}$ with m an integer so that $\Phi(\phi + 2\pi) = \Phi(\phi)$. Now with $w(\rho, z) = R(\rho)Z(z)$,

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{\rho^2}{Z}\frac{\partial^2 Z}{\partial z^2} + \rho^2 k^2 = m^2 \qquad \text{so} \qquad \frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \frac{m^2}{\rho^2} + k^2 + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

and similarly put $(1/Z)\partial^2 Z/\partial z^2 = -\alpha^2$ so that $Z(\alpha) = e^{\pm i\alpha z}$. Enforcing $Z(0) = 0 = Z(L)$ leads to $Z(z) = \sin\alpha_\ell z$ where $\alpha_\ell = \ell\pi/L$ and $\ell = 1, 2, 3\ldots$. The ρ equation is therefore

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left(k^2 - \alpha_\ell^2 - \frac{m^2}{\rho^2}\right)R = 0$$

Now define $\kappa^2 \equiv k^2 - \alpha_\ell^2$ and $x \equiv \kappa\rho$. Multiply through by x^2 and this becomes

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - m^2)R = 0$$

i.e., Bessel's equation, with solution $R(\rho) = A_m J_m(\kappa\rho) + B_m N_m(\kappa\rho)$, where $J_m(x)$ and $N_m(x)$ are Bessel functions of the first and second kind, respectively. The cylinder wall boundary conditions tell us that for each m we must have $A_m J_m(\kappa\rho_a) + B_m N_m(\kappa\rho_a) = 0$ and $A_m J_m(\kappa\rho_b) + B_m N_m(\kappa\rho_b) = 0$. Set the determinant to zero, and so we would solve

$$J_m(\kappa\rho_a)N_m(\kappa\rho_b) - J_m(\kappa\rho_b)N_m(\kappa\rho_a)$$

for κ. Denote with k_{mn} the nth solution for κ for a given m. Then

$$E = \frac{\hbar^2}{2m_e}k^2 = \frac{\hbar^2}{2m_e}[\kappa^2 + \alpha_\ell^2] \qquad \text{or} \qquad E_{\ell mn} = \frac{\hbar^2}{2m_e}\left[k_{mn}^2 + \left(\frac{\ell\pi}{L}\right)^2\right]$$

In the presence of a magnetic field, the Hamiltonian becomes (2.7.20), with $\phi = 0$. We recover the problem already solved, essentially, using the gauge transformation (2.7.36), but we need to multiply the wave function by the phase factor $\exp[ie\Lambda(\mathbf{x})/\hbar c]$ as in (2.7.55). In this case, $\mathbf{A} = \boldsymbol{\nabla}\Lambda = \hat{\boldsymbol{\phi}}(1/\rho)\partial\Lambda/\partial\phi$ is given by (2.7.62), so $\Lambda(\mathbf{x}) = B\rho_a^2\phi/2 \equiv \hbar c g\phi/e$, and

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \longrightarrow e^{-ig\phi}\frac{1}{\Phi}\frac{d^2}{d\phi^2}\left(e^{ig\phi}\Phi\right) = \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{2ig}{\Phi}\frac{d\Phi}{d\phi} - g^2 = -m^2 \mp 2gm - g^2 = -(m \pm g)^2$$

for $\Phi(\phi) = e^{\pm im\phi}$. Consequently, the solution is the same, but with (integer) m replaced by $\gamma \equiv m \pm g$. (The solutions to Bessel's equation are perfectly valid for non-integral indices.) The ground state is $\ell = 1$ and $n = 1$, so $E_0 = (\hbar^2/2m_e)(k_{01} + \pi^2/L^2)$ for $B = 0$, and $E_0 = (\hbar^2/2m_e)(k_{\gamma 1} + \pi^2/L^2)$ for $B \neq 0$. For these to be equal, $m \pm g = 0$ for integer m, so

$$g \equiv \frac{e}{\hbar c}\frac{B\rho_a^2}{2} = \pm m \qquad \text{or} \qquad B \times \pi\rho_a^2 = \pm 2\pi\frac{\hbar c}{e}m = \pm\frac{hc}{e}m$$

which is the "flux quantization" condition.

The history of flux quantization is quite fascinating. The original discovery can be found in B. S. Deaver and W. M. Fairbank, "Experimental Evidence for Quantized Flux in Superconducting Cylinders", Phys. Rev. Lett. 7(1961)43. The flux quantum worked out to be $hc/2e$, but it was later appreciated that the charge carriers were Cooper pairs of electrons. See also articles by Deaver and others in "Near Zero: new frontiers of physics", by Fairbank, J. D.; Deaver, B. S., Jr.; Everitt, C. W. F.; Michelson, P. F.. Freeman, 1988.

29. The hardest part of this problem is to identify the Hamilton-Jacobi Equation. See Chapter 10 in Goldstein, Poole, and Safko. With one spacial dimension, this equation is $H(x, \partial S/\partial x, t) + \partial S/\partial t = 0$ to be solved for $S(x,t)$, called Hamilton's Principle Function. So, $H\psi = -(\hbar^2/2m)\partial^2\psi/\partial t^2\psi + V(x)\psi = i\hbar\partial\psi/\partial t$ with $\psi(x,t) = \exp[iS(x,t)/\hbar]$ becomes

$$-\frac{\hbar^2}{2m}\left[\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2} + \left(\frac{i}{\hbar}\frac{\partial S}{\partial x}\right)^2\right]\psi + V(x)\psi = -\frac{\partial S}{\partial t}\psi$$

If $\hbar$ is "small" then the second term in square brackets dominates. Dividing out ψ then leaves us with the Hamilton-Jacobi Equation. Putting $V(x) = 0$ and trying $S(x,t) = X(x) + T(t)$, find $(X'')^2/2m = -T' = \alpha$ (a constant). Thus $T(t) = a - \alpha t$ and $X(x) = \pm\sqrt{2m\alpha}x + b$, where a and b are constants that can be discarded when forming $\psi(x,t) = \exp[i(X+T)/\hbar]$. Hence $\psi(x,t) = \exp[i(\pm\sqrt{2m\alpha}x - \alpha t)/\hbar]$, a plane wave. This exact solution comes about because S is linear in x, so $\partial^2 S/\partial x^2 = 0$ and the first term in the Schrödinger Equation, above, is manifestly zero.

30. You could argue this should be in Chapter 3, but what you need to know about the hydrogen atom is so basic, it would surely be covered in an undergraduate quantum physics class. (See, for example, Appendix B.5.) The wave function for the atom looks like $\psi(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi) = C_{lm}R_{nl}(r)P_l^m(\cos\theta)e^{im\phi}$ where C_{lm}, $R_{nl}(r)$, and $P_l^m(\cos\theta)$ are all real. Since $\boldsymbol{\nabla} = \hat{\mathbf{r}}\partial/\partial r + \hat{\boldsymbol{\theta}}(1/r)\partial/\partial\theta + \hat{\boldsymbol{\phi}}(1/r\sin\theta)\partial/\partial\phi$, we have from (2.4.16)

$$\mathbf{j} = \frac{\hbar}{m_e}\mathrm{Im}\left[\psi^\star\boldsymbol{\nabla}\psi\right] = \hat{\boldsymbol{\phi}}\frac{m\hbar}{m_e r\sin\theta}|\psi|^2$$

so $\mathbf{j} = 0$ if $m = 0$, and is in the positive (negative) ϕ direction if m is positive (negative).

31. Write $ibp' - iap'^2 = -ia(p'^2 - bp'/a + b^2/4a^2) + ib^2/4a = -ia(p' - b/2a)^2 + ib^2/4a$, translate p' in the integral, and use $\int_{-\infty}^{\infty} e^{-cx^2}dx = \sqrt{\pi/c}$. Then

$$\begin{aligned}
K(x'', t; x', t_0) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \exp\left[\frac{ip(x''-x')}{\hbar} - \frac{ip'^2(t-t_0)}{2m\hbar}\right] \\
&= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{i(t-t_0)}}\exp\left[i\frac{m(x''-x')^2}{2\hbar(t-t_0)}\right] = \sqrt{\frac{m}{2\pi\hbar i(t-t_0)}}\exp\left[i\frac{m(x''-x')^2}{2\hbar(t-t_0)}\right]
\end{aligned}$$

To generalize to three dimensions, just realize that the length along the x-axis is invariant under rotations. Therefore, we have

$$K(\mathbf{x}'', t; \mathbf{x}', t_0) = \sqrt{\frac{m}{2\pi\hbar i(t-t_0)}}\exp\left[i\frac{m(\mathbf{x}''-\mathbf{x}')^2}{2\hbar(t-t_0)}\right]$$

32. From (2.6.22), $Z = \sum_{a'} \exp\left[-\beta E_{a'}\right]$, so, defining E_0 to be the ground state energy,

$$\lim_{\beta\to\infty}\left\{-\frac{1}{Z}\frac{\partial Z}{\partial\beta}\right\} = \lim_{\beta\to\infty}\left\{\frac{\sum_{a'} E_{a'}\exp\left[-\beta E_{a'}\right]}{\sum_{a'}\exp\left[-\beta E_{a'}\right]}\right\} = \lim_{\beta\to\infty}\left\{\frac{\sum_{a'} E_{a'}\exp\left[-\beta(E_{a'}-E_0)\right]}{\sum_{a'}\exp\left[-\beta(E_{a'}-E_0)\right]}\right\} = E_0$$

where we multiply top and bottom by $\exp(\beta E_0)$ in the penultimate step. The limit is easy to take because for all terms in which $E_{a'} \neq E_0$, the exponent is negative as $\beta\to\infty$ and the term vanishes. For the term $E_{a'} = E_0$, the numerator is E_0 and the denominator is unity.

To "illustrate this for a particle in a one-dimensional box" is trivial. Just replace $E_{a'}$ with $E_n = \hbar^2\pi^2 n^2/2mL^2$ for $n = 1, 2, 3\ldots$ (B.2.4) and the work above carries through. The old solution manual has a peculiar approach, though, replacing the sum by an integral, presumably valid as $\beta\to\infty$, but I don't really get the point.

33. Recall that, in the treatment (2.6.26) for the propagator, position (or momentum) bras and kets are taken to be in the Heisenberg picture. So, one should recall the discussion on pages 86–88, regarding the time dependence of base kets. In particular, $|a', t\rangle_H = \mathcal{U}^\dagger(t)|a'\rangle$, that is, base kets are time dependent and evolve "backwards" relative to state kets in the Schrödinger picture. So, for a free particle with $H = \mathbf{p}^2/2m$, we have

$$\langle\mathbf{p}'', t|\mathbf{p}', t_0\rangle = \langle\mathbf{p}''|e^{-iHt/\hbar}e^{iHt_0/\hbar}|\mathbf{p}'\rangle = \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}'^2}{2m}(t-t_0)\right]\delta^{(3)}\left(\mathbf{p}''-\mathbf{p}'\right)$$

The solution in the old manual confuses me.

34. The classical action is $S(t_a, t_b) = \int_{t_a}^{t_b} dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2\right)$. Approximating this for the time interval $\Delta t \equiv t_b - t_a$, defining $\Delta x \equiv x_b - x_a$, and writing $x_a + x_b = 2x_b - \Delta x$, we have

$$S(t_a, t_b) \approx \Delta t\left[\frac{1}{2}m\left(\frac{\Delta x}{\Delta t}\right)^2 - \frac{1}{2}m\omega^2\left(x_b - \frac{\Delta x}{2}\right)^2\right] \approx \frac{1}{2}m\left(\frac{\Delta x}{\Delta t}\right)\Delta x - \frac{1}{2}m\omega^2x_b^2\Delta t$$

keeping only lowest order terms. Combine this with (2.6.46) (and sum over all paths) to get the Feynman propagator. Now the problem says to show this is the same as (2.6.26), but (2.6.18) is the solution for the harmonic oscillator. Taking this limit for $\Delta t\to 0$, one gets

$$\begin{aligned}
K(x_b, t_b; x_a, t_a) &= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\left\{\frac{im}{2\hbar\Delta t}\right\}\left\{(x_b^2 + x_a^2)\left(1 - \frac{\omega^2\Delta t^2}{2}\right) - 2x_ax_b\right\}\right] \\
&= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{i}{\hbar}\left\{\frac{1}{2}m\frac{(\Delta x)^2}{\Delta t} - \frac{1}{2}m\omega^2(x_a^2 + x_b^2)\Delta t\right\}\right]
\end{aligned}$$

Taking the limit $\Delta x\to 0$ clearly gives the same expression as inserting our classical action, above, into (2.6.46).

I'm not sure I understand the point of this problem.

35. The "Schwinger action principle" does not seem to be treated in modern references, and also not in (this version of) this textbook. So, I just reprint here San Fu Tuan's old solution.

The Schwinger action principle states that the following condition determines the transformation function $\langle x_2t_2|x_1t_1\rangle$ in terms of a given quantum mechanical Lagrangian L

$$\delta\langle x_2t_2|x_1t_1\rangle = (i/\hbar)\langle x_2t_2|\delta\int_{t_1}^{t_2} L dt|x_1t_1\rangle.$$

To obtain $\langle x_2t_2|x_1t_1\rangle$, let $\delta\langle x_2t_2|x_1t_1\rangle = (i/\hbar)\langle x_2t_2|\delta W_{21}|x_1t_1\rangle$ where W_{21} is action in going from initial state x_1t_1 to final state x_2t_2. Also, let $\delta W_{21} = \delta\omega_{21}$ where $\delta\omega_{21}$ is the well-ordered form (c.f. Finkelstein (1973), p.164) of δW_{21}. Then $\delta\langle x_2t_2|x_1t_1\rangle = \frac{i}{\hbar}\langle x_2t_2|\ \delta\omega_{21}|x_1t_1\rangle = \frac{i}{\hbar}\ \delta\omega'_{21}\langle x_2t_2|x_1t_1\rangle$ and thus $\delta\ln\langle x_2t_2|x_1t_1\rangle = \frac{i}{\hbar}\delta\omega'_{21}$ or

$$\langle x_2t_2|x_1t_1\rangle = \exp[\tfrac{i}{\hbar}\omega'_{21}]. \qquad (1)$$

The corresponding Feynman expression for $\langle x_2t_2|x_1t_1\rangle$ [c.f. Finkelstein (1973), p.144] is

$$\langle x_2t_2|x_1t_1\rangle = \frac{1}{N}\sum_{\text{paths}} \exp[(i/\hbar)S_{21}]. \qquad (2)$$

The classical limit of (2) is such that as $\hbar/S \to$ small, the probability amplitude $\langle x_2t_2|x_1t_1\rangle$ will be important only for those varied paths which lie in a narrow tube between x_1t_1 and x_2t_2 enclosing the classical path. On the other hand, to describe the classical limit for (1) (which has a well-ordered exponent instead of a sum over paths), is to consider first the operator Hamilton-Jacobi equation (c.f. Finkelstein (1973), p.166)

$$H(\frac{\partial\omega}{\partial x}.\dots.x\dots.) + \partial\omega/\partial t = 0. \qquad (3)$$

Since ω'_{21} satisfies (3), which arises from a variation of the final state (and is similar to the Schrödinger picture), it is seen that the correspondence limit of ω'_{21} is S, i.e. the probability amplitude (1) approaches the consideration of all possible paths as in the Feynman path integral case (2). Thus in the classical limit, (1) and (2) become equal provided they both are modulated by the factor 1/N (N = total number of individual steps in going from $x_1t_1 \to x_2t_2$).

36. Wave mechanically, the phase difference comes about because, approximating the neutron by a plane wave, the factor $\exp[-i(\omega t - px/\hbar)]$ (where x is the direction AC or BD in Figure 2.9) is different because p (and $v = p/m_n$) will depend on the height. That is, $p_{BD}^2/2m_n = p_{AC}^2/2m_n - m_n gz$ where $z = l_2 \sin\delta$. The accumulated phase difference is

$$\phi_{BD} - \phi_{AC} = \left[\frac{p_{BD} - p_{AC}}{\hbar} - \omega\left(\frac{1}{v_{BD}} - \frac{1}{v_{AC}}\right)\right] l_1 = \frac{p_{BD} - p_{AC}}{\hbar}\left[1 + \frac{\hbar\omega}{m_n v_{BD} v_{AC}}\right] l_1$$

The experiment in Figure 2.10 was performed with $\lambda = 1.445$Å neutrons. (The book has $\lambda = 1.42$Å?) So $p = h/\lambda = 2\pi\hbar c/c\lambda = 2\pi(200 \times 10^6 \times 10^{-5}\ \text{eV} - \text{Å})/c\lambda = 8.7\ \text{keV}/c$ and $E = \hbar\omega = p^2/2m_n = 4.05\times 10^{-2}$ eV, whereas $m_n gh = (m_n c^2)gh/c^2 \approx 10^{-9}$ eV for $h = 10$ cm. Thus the change in momentum is very small and $\hbar\omega/m_n v_{BD} v_{AC} = m_n E/p^2 = 1/2$. Therefore

$$\phi_{BD} - \phi_{AC} = \frac{p_{BD} - p_{AC}}{\hbar}\frac{3}{2}l_1 \approx \frac{p_{BD}^2 - p_{AC}^2}{2\hbar p}\frac{3}{2}l_1 = -\frac{2m_n^2 gz}{2\hbar p}\frac{3}{2}l_1 = -\frac{3}{2}\frac{m_n^2 g(\lambda/2\pi)l_1 z}{\hbar^2}$$

This differs from (2.7.17) by the factor 3/2, which comes from the ωt contribution to the phase. San Fu Tuan's solution starts with the same expression as I do, but ignores the ωt term when calculating the phase. My thought is that this is in fact a more complicated problem than meets the eye, and I need to think about it more.

37. Since $\mathbf{A} = \mathbf{A}(\mathbf{x})$, write $p_i = (\hbar/i)\partial/\partial x_i$ and work in position space. Then

$$\begin{aligned}
[\Pi_i, \Pi_j]\psi(\mathbf{x}) &= \left[\frac{\hbar}{i}\frac{\partial}{\partial x_i} - \frac{eA_i}{c}, \frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{eA_j}{c}\right]\psi(\mathbf{x}) = -\frac{\hbar}{i}\frac{e}{c}\left\{\left[\frac{\partial}{\partial x_i}, A_j\right] - \left[A_i, \frac{\partial}{\partial x_j}\right]\right\}\psi(\mathbf{x}) \\
&= -\frac{\hbar}{i}\frac{e}{c}\left\{\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right\}\psi(\mathbf{x}) = \frac{i\hbar e}{c}\varepsilon_{ijk}\left(\boldsymbol{\nabla}\times\mathbf{A}\right)_k\psi(\mathbf{x}) = \frac{i\hbar e}{c}\varepsilon_{ijk}B_k\psi(\mathbf{x})
\end{aligned}$$

$$m\frac{d^2x_i}{dt^2} = \frac{d\Pi_i}{dt} = \frac{1}{i\hbar}[\Pi_i, H] = \frac{1}{i\hbar}\left[\Pi_i, \frac{1}{2m}\boldsymbol{\Pi}^2 + e\phi\right] = \frac{1}{2im\hbar}\sum_j\left[\Pi_i, \Pi_j^2\right] + \frac{1}{i\hbar}[p_i, e\phi]$$

Now from Problem 1.29(a), $(1/i\hbar)[p_i, e\phi] = -e\partial\phi/\partial x_i = eE_i$. Also $[\Pi_i, \Pi_j^2] = [\Pi_i, \Pi_j]\Pi_j + \Pi_j[\Pi_i, \Pi_j]$ so $(1/2im\hbar)[\Pi_i, \Pi_j^2] = (e/2mc)(\varepsilon_{ijk}B_k p_j + p_j\varepsilon_{ijk}B_k)$. This amounts to

$$m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E} + \frac{e}{2mc}\left[-\mathbf{B}\times\mathbf{p} + \mathbf{p}\times\mathbf{B}\right] = e\left[\mathbf{E} + \frac{1}{2c}\left(\frac{d\mathbf{x}}{dt}\times\mathbf{B} - \mathbf{B}\times\frac{\mathbf{x}}{dt}\right)\right]$$

As for showing that (2.7.30) follows from (2.7.29) with $\mathbf{j}$ defined as in (2.7.31), just follow the same steps used to prove (2.4.15) with the definition (2.4.16). That is, multiply the Schrödinger equation by ψ^*, and then multiply its complex conjugate by ψ, and subtract the two equations. You just need to use some extra care when writing out (2.7.29) to make sure the $\mathbf{A}(\mathbf{x}')$ is appropriately differentiated. Indeed, the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}'^2\psi + \frac{i\hbar e}{mc}\mathbf{A}\cdot\boldsymbol{\nabla}'\psi + \frac{i\hbar e}{2mc}\left(\boldsymbol{\nabla}'\cdot\mathbf{A}\right)\psi + \frac{e^2}{2mc^2}\mathbf{A}^2\psi + e\phi\psi = i\hbar\frac{\partial\psi}{\partial t}$$

The remainder of the proof is simple from here.

38. The vector potential $\mathbf{A} = -\frac{1}{2}By\hat{\mathbf{x}} + \frac{1}{2}Bx\hat{\mathbf{y}}$ gives $\mathbf{B} = B\hat{\mathbf{z}}$ in a gauge where $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. Reading the Hamiltonian from the previous problem solution, we are led to an interaction

$$\frac{i\hbar e}{mc}\mathbf{A}\cdot\boldsymbol{\nabla}+\frac{e^2}{2mc^2}\mathbf{A}^2 = -\frac{e}{mc}\left(-\frac{1}{2}By\hat{\mathbf{x}}+\frac{1}{2}Bx\hat{\mathbf{y}}\right)\cdot\mathbf{p}+\frac{e^2B^2}{8mc^2}(x^2+y^2) = \frac{eB}{2mc}L_z+\frac{e^2B^2}{8mc^2}(x^2+y^2)$$

where $\mathbf{L} \equiv \mathbf{r}\times\mathbf{p}$. The first term is just $\boldsymbol{\mu}\cdot\mathbf{B}$ for $\boldsymbol{\mu} \equiv (e/2mc)\mathbf{L}$, the magnetic moment of an orbiting electron. The second term gives rise to the quadratic Zeeman effect. See pages 328–330 and Problems 5.18 and 5.19 in the textbook.

39. See the solution to Prob.37. We find $[\Pi_x, \Pi_y] = (i\hbar e/c)B_z = i\hbar eB/c$ or $[Y, \Pi_y] = i\hbar$ for $Y \equiv c\Pi_x/eB$. As in the solution to Prob.38, $A_z = 0$. So, as in Prob.37, the Hamiltonian is

$$H = \frac{\Pi_x^2}{2m}+\frac{\Pi_y^2}{2m}+\frac{p_z^2}{2m} = \frac{p_z^2}{2m}+\frac{\Pi_y^2}{2m}+\frac{1}{2}m\frac{e^2B^2}{m^2c^2}Y^2$$

The second two terms constitute the one dimensional harmonic oscillator Hamiltonian, by virtue of the commutation relation $[Y, \Pi_y] = i\hbar$, with ω replaced by eB/mc.

40. One requires that the phase change $\mu BT/\hbar$ be 2π after traversing a field B of length $l = vT$. The speed $v = p/m = h/\lambda m$. Since $\mu = g_n(e\hbar/2mc)$, we have

$$\frac{\mu BT}{\hbar} = g_n\frac{e\hbar}{2mc}\frac{B}{\hbar}\frac{lm\lambda}{h} = 2\pi \qquad \text{or} \qquad B = \frac{4\pi hc}{eg_nl\lambda}$$

See also (3.2.25). San Fu Tuan's solution is much more complicated. I may be misunderstanding something.