Taking the lower branch once and then always the upper branch we get

$$x_0 = 5,\ x_1 = 2,\ x_2 = 5, x_3 = 4 + \sqrt{5} \approx 6.236,\ x_4 = 4 + \sqrt{x_2 - 1} \approx 6.288.$$

Another possible trajectory is the periodic one given by

$$x_0 = 5,\ x_1 = 2,\ x_2 = 5, x_3 = 2,\ x_4 = 5.$$

Exercise 1.8 In the same way as in example 1.7 one can obtain the first-order condition

$$v'(m_t/p_t)/p_t = u'(m_t/p^e_{t+1})/p^e_{t+1}.$$

Multiplying by m_t and using the equilibrium condition $m_t = M$ for all t this implies

$$V(M/p_t) = U(M/p^e_{t+1}).$$

From the definition of adaptive expectations we obtain $p_t = [p^e_{t+1} - (1-\gamma)p^e_t]/\gamma$. Substituting this into the above condition yields

$$V(\gamma M/[p^e_{t+1} - (1-\gamma)p^e_t]) = U(M/p^e_{t+1}).$$

The price forecast is pre-determined because of the interpretation of adaptive expectations.

Chapter 2

Exercise 2.1 We have $z(1)_t = \lambda(1)^t w(1)$ and $z(2)_t = \lambda(2)^t w(2)$, where $w(1)$ and $w(2)$ are eigenvectors corresponding to $\lambda(1)$ and $\lambda(2)$, respectively. Because the eigenvalues $\lambda(1)$ and $\lambda(2)$ are assumed to be different from each other, it follows that $w(1)$ and $w(2)$ are linearly independent vectors of $\mathbb{R}^n$.

Now suppose that there exist numbers α and β, not both equal to 0, such that

$$\alpha z(1)_t + \beta z(2)_t = 0$$

holds for all $t \in \mathbb{N}_0$. For $t = 0$ this implies that $\alpha w(1) + \beta w(2) = 0$. Since $w(1)$ and $w(2)$ are linearly independent, it follows that $\alpha = \beta = 0$. This contradiction proves the claim.

Exercise 2.2 (a) Define $y_t = x_{t-1}$. Then the equation can be written as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}.$$

(b) The system matrix has a double eigenvalue 2 with an eigenvector $w = (2,1)^\top$. Since there does not exists another eigenvector that is linearly independent of w, one has to find a generalized eigenvector satisfying $Av = 2v + w$. This holds for example for $v = (1,0)^\top$. The general solution is therefore given by

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \alpha 2^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta \left[2^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2^{t-1} t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 2^t[2\alpha + \beta(1+t)] \\ 2^{t-1}(2\alpha + \beta t) \end{pmatrix}.$$

(c) The initial condition is $x_1 = 0$ and $y_1 = 1$. Substituting these values and $t = 1$ into the result of part (b) it follows that $\alpha = 1$ and $\beta = -1$. The particular solution is therefore given by

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 2^t(1-t) \\ 2^{t-1}(2-t) \end{pmatrix}.$$

Exercise 2.3 (a) The equations are

$$\begin{aligned} \pi_t &= \pi_{t-1} + \alpha y_t, \\ y_t &= \beta y_{t-1} + \gamma \pi_{t-1}, \end{aligned}$$

which can also be written as

$$\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 + \alpha\gamma & \alpha\beta \\ \gamma & \beta \end{pmatrix} \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix}.$$

(b) For $\alpha = 1$ and $\beta = 4/9$, $\gamma = 2/9$ we obtain the system matrix

$$A = \begin{pmatrix} 11/9 & 4/9 \\ 2/9 & 4/9 \end{pmatrix}.$$

This matrix has the eigenvalues $4/3$ and $1/3$ with corresponding eigenvectors $(4,1)^\top$ and $(-1,2)^\top$, respectively. The general solution is therefore given by

$$\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} = c_1 (4/3)^t \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 (1/3)^t \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 3^{-t} \begin{pmatrix} 4^{t+1} c_1 - c_2 \\ 4^t c_1 + 2c_2 \end{pmatrix}.$$

(c) For the initial condition $\pi_0 = 0$ and $y_0 = 1$ it follows that $c_1 = 1/9$ and $c_2 = 4/9$ and therefore $\pi_t = 4(4^t - 1)3^{-(t+2)}$ and $y_t = (4^t + 8)3^{-(t+2)}$.

(d)

$$\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 + \alpha\gamma(1-b) & \alpha\beta \\ \gamma(1-b) & \beta \end{pmatrix} \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix}.$$

$$A = \begin{pmatrix} -4/9 & 4/9 \\ -13/9 & 4/9 \end{pmatrix}.$$

The eigenvalues of A are $\{2i/3, -2i/3\}$ with corresponding eigenvectors $(4-6i, 13)^\top$ and $(4+6i, 13)^\top$, respectively. The first eigenvalue can also be written as $r[\cos(\theta) + i\sin(\theta)]$ with $r =$

$2/3 \in (0,1)$ and $\theta = \pi/2$. The general solution is therefore

$$\begin{aligned}\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} &= (2/3)^t \Big\{ c_1 \Big[\cos(\pi t/2) \begin{pmatrix} 4 \\ 13 \end{pmatrix} - \sin(\pi t/2) \begin{pmatrix} -6 \\ 0 \end{pmatrix} \Big] \\ &\quad + c_2 \Big[\sin(\pi t/2) \begin{pmatrix} 4 \\ 13 \end{pmatrix} + \cos(\pi t/2) \begin{pmatrix} -6 \\ 0 \end{pmatrix} \Big] \Big\} \\ &= (2/3)^t \begin{pmatrix} (4c_1 - 6c_2)\cos(\pi t/2) + (6c_1 + 4c_2)\sin(\pi t/2) \\ 13[c_1 \cos(\pi t/2) + c_2 \sin(\pi t/2)] \end{pmatrix}.\end{aligned}$$

For $t = 0$, $\pi_0 = 0$, and $y_0 = 1$ it follows that $c_1 = 1/13$ and $c_2 = 2/39$ such that

$$\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} = (2/3)^t \begin{pmatrix} (26/39)\sin(\pi t/2) \\ \cos(\pi t/2) + (2/3)\sin(\pi t/2) \end{pmatrix}.$$

Exercise 2.4 As in exercise 2.3(d) we have

$$A = \begin{pmatrix} 1 + \alpha\gamma(1-b) & \alpha\beta \\ \gamma(1-b) & \beta \end{pmatrix}.$$

This implies $T = 1 + \beta + \alpha\gamma(1-b)$ and $D = \beta$. As b moves from 0 to $+\infty$ the point (T, D) travels along a ray starting at $(1+\beta+\alpha\gamma, \beta)$ in area A_8 and protruding horizontally to the left through areas A_4, A_1, A_3, and A_7.

$b \in (0,1)$: A_8 saddle point dynamics;
$b \in (1, 1 + 2(1+\beta)/(\alpha\gamma))$: A_4, A_1, A_3 stable;
$b \in (1 + 2(1+\beta)/(\alpha\gamma), +\infty)$: A_7 saddle point dynamics.

Exercise 2.5 (a) Since the non-homogeneity is a linear function of time, we try the guess $\bar{x}_t = A + Bt$. Substituting the guess into the equation yields

$$Bt + (A + B) = (\lambda B + \alpha)t + \lambda A,$$

which holds for all t if and only if $B = \alpha/(1-\lambda)$ and $A = -\alpha/(1-\lambda)^2$. Hence, the particular solution is

$$\bar{x}_t = \frac{\alpha[(1-\lambda)t - 1]}{(1-\lambda)^2}.$$

The general solution of the homogeneous equation is $x_t = C\lambda^t$. It follows that the general solution of the non-homogeneous equation is

$$x_t = C\lambda^t + \frac{\alpha[(1-\lambda)t - 1]}{(1-\lambda)^2}.$$

(b) In this case there is resonance and we try the guess $\bar{x}_t = A + Bt + Ct^2$. Substitution into the equation yields

$$Ct^2 + (B + 2C)t + (A + B + C) = Ct^2 + (B + \alpha)t + A,$$

which can hold for all t if and only if $B = -\alpha/2$ and $C = \alpha/2$. The value of A is irrelevant and we set it therefore equal to 0. This yields the particular solution

$$\bar{x}_t = \alpha(t^2 - t)/2.$$

The general solution of the homogeneous equation is $x_t = D$. It follows that the general solution of the non-homogeneous equation is

$$x_t = D + \alpha(t^2 - t)/2.$$

Exercise 2.6 For $\lambda = \mu \neq 1$ the equation reads as $x_{t+1} = \lambda x_t + \nu t^2 + \lambda^t$. We try

$$\bar{x}_t = A\lambda^t + Bt\lambda^{t-1} + Ct^2 + Dt + E.$$

This works for

$$B = 1\,,\ C = \frac{\nu}{1-\lambda}\,,\ D = -\frac{2\nu}{(1-\lambda)^2}\,,\ E = \frac{\nu(1+\lambda)}{(1-\lambda)^3}.$$

The value for A is irrelevant and we set it arbitrarily to 0. A particular solution is therefore given by

$$\bar{x}_t = t\lambda^{t-1} + \frac{\nu t^2}{1-\lambda} - \frac{2\nu t}{(1-\lambda)^2} + \frac{\nu(1+\lambda)}{(1-\lambda)^3}.$$

When $\lambda = \mu = 1$ the equation boils down to $x_{t+1} = x_t + \nu t^2 + 1$. We try the guess

$$\bar{x}_t = At^3 + Bt^2 + Ct + D,$$

which works for

$$A = \frac{\nu}{3}\,,\ B = -\frac{\nu}{2}\,,\ C = 1 + \frac{\nu}{6}.$$

The value for D is irrelevant and we set it to 0. A particular solution is therefore given by

$$\bar{x}_t = \frac{\nu t^3}{3} - \frac{\nu t^2}{2} + \left(1 + \frac{\nu}{6}\right) t.$$

2.7 (a) The system matrix is

$$A = \begin{pmatrix} -3/2 & 8 & 9/2 \\ -9/4 & 3 & 1/4 \\ 1/2 & 8 & 5/2 \end{pmatrix}.$$

The eigenvalues are $3+4i$, $3-4i$, and -2 with corresponding eigenvectors $(2, i, 2)^\top$, $(2, -i, 2)^\top$, and $(2, 1, -2)^\top$, respectively. The general solution is therefore given by

$$\alpha 5^t \left[\cos(\theta t)\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \sin(\theta t)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] + \beta 5^t \left[\sin(\theta t)\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \cos(\theta t)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] + \gamma(-2)^t \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

where $\theta = \arctan(4/3)$. This can also be written as

$$\begin{pmatrix} 2 \times 5^t[\alpha\cos(\theta t) + \beta\sin(\theta t)] - (-2)^{t+1}\gamma \\ 5^t[\beta\cos(\theta t) - \alpha\sin(\theta t)] + (-2)^t\gamma \\ 2 \times 5^t[\alpha\cos(\theta t) + \beta\sin(\theta t)] + (-2)^{t+1}\gamma \end{pmatrix}.$$

(b) Since the non-homogeneity is constant, we try a constant particular solution $(\bar{x}, \bar{y}, \bar{z})^\top$. Substituting into the equation it follows that

$$\begin{aligned} \bar{x} &= -(3/2)\bar{x} + 8\bar{y} + (9/2)\bar{z}, \\ \bar{y} &= -(9/4)\bar{x} + 3\bar{y} + (1/4)\bar{z} + 10, \\ \bar{z} &= (1/2)\bar{x} + 8\bar{y} + (5/2)\bar{z}. \end{aligned}$$

These equations hold for $\bar{x} = 4$, $\bar{y} = -1$, and $\bar{z} = 4$. The general solution of the non-homogeneous equation is therefore

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} 2 \times 5^t[\alpha\cos(\theta t) + \beta\sin(\theta t)] - (-2)^{t+1}\gamma + 4 \\ 5^t[\beta\cos(\theta t) - \alpha\sin(\theta t)] + (-2)^t\gamma - 1 \\ 2 \times 5^t[\alpha\cos(\theta t) + \beta\sin(\theta t)] + (-2)^{t+1}\gamma + 4 \end{pmatrix}.$$

Setting $t = x_0 = y_0 = z_0 = 0$ yields $\alpha = -2$, $\beta = 1$, and $\gamma = 0$. The particular solution starting at the given initial point is therefore

$$x_t = z_t = 2 \times 5^t[\sin(\theta t) - 2\cos(\theta t)] + 4\,,\ y_t = 5^t[\cos(\theta t) + 2\sin(\theta t)] - 1.$$

Exercise 2.8 Let x_t denote the balance on birthday t. Then we have $x_0 = 0$ and $x_1 = 100$. The balance evolves according to

$$x_{t+1} = (1+r)x_t + 100(t+1).$$

The general solution of the homogeneous equation is

$$x_t = A(1+r)^t.$$

For a particular solution we try

$$\bar{x}_t = B + Ct.$$

Substituting this conjecture into the equation we obtain

$$B + C(t+1) = (1+r)(B+Ct) + 100(t+1),$$

which holds for all t if and only if

$$C = (1+r)C + 100$$

and

$$B + C = (1+r)B + 100.$$

This is the case if and only if

$$C = -100/r \text{ and } B = -100(1+r)/r^2.$$

The general solution of the non-homogeneous equation is therefore

$$x_t = A(1+r)^t - \frac{100(1+r+rt)}{r^2}.$$

To satisfy the initial condition $x_1 = 100$ we have to choose

$$A = 100(1+r)/r^2.$$

The final result is therefore

$$x_t = \frac{100}{r^2}[(1+r)^{t+1} - 1 - r - rt].$$

Chapter 3

Exercise 3.1 (a) The unique fixed point is $x = 1/2$. Every point $x \neq 1/2$ is a periodic point of period 2.

(b) There are two fixed points, namely $x = (-\sqrt{5}-1)/2$ and $x = (\sqrt{5}-1)/2$. Periodic points of period 2 must satisfy $x = f(f(x)) = f(1-x^2) = 1-(1-x^2)^2 = 2x^2 - x^4$. This equation can also be written as

$$x^4 - 2x^2 + x = x(x-1)(x^2+x-1) = 0.$$

This holds for the two fixed points and for $x \in \{0,1\}$. Hence $x = 0$ and $x = 1$ are the only periodic points of period 2.

(c) The fixed point equation $A/(1+x) = x$ has the solutions $x = (-\sqrt{1+4A}-1)/2$ and $x = (\sqrt{1+4A}-1)/2$. Only the solution with the plus sign is an element of the system domain. The unique fixed point is therefore $x = (\sqrt{1+4A}-1)/2$. Periodic points of period 2 must satisfy $f = f(f(x)) = A(1+x)/(1+x+A)$. This equation can be written as

$$x^2 + x - A = 0,$$

which is only satisfied by the fixed point. Hence, there do not exist any periodic points of period 2.

(d) Fixed points must simultaneously satisfy $x = 1-y$ and $y = 1-x^2$. This implies that $(x,y) \in \{(0,1),(1,0)\}$. These are the fixed points. It holds that $f(f(x,y)) = f(1-y, 1-x^2) = (x^2, y(2-y))$. A periodic point of period 2 must therefore satisfy $x = x^2$ and $y = y(2-y)$. This gives the four solutions $\{(0,0),(0,1),(1,0),(1,1)\}$. Hence, the periodic points of period 2 are given by $\{(0,0),(1,1)\}$.