Solution

$$\nabla^2 \phi = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) = 0 \qquad (a)$$

$$\frac{\partial \phi}{\partial r} = \dot{R} \quad \text{at} \quad r = R$$

$$\frac{\partial \phi}{\partial r} \to 0 \quad \text{at} \quad r \to \infty$$

The solution for these equations is

$$\phi = R\dot{R} \ln r$$

Therefore

$$\frac{\partial \phi}{\partial t} = \ln r \left[\dot{R}^2 + R\ddot{R}\right]$$

$$U = u_r = \frac{\partial \phi}{\partial r} = \frac{R\dot{R}}{r}$$

Insert in Euler's equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}U^2 + \frac{P}{\rho} = f(t)$$

$$\Rightarrow \left[\dot{R}^2 + R\ddot{R}\right]\ln r + \frac{1}{2}\left(\frac{R\dot{R}}{r}\right)^2 + \frac{P}{\rho} = f(t) \qquad (b)$$

Equation (b) must be applicable to a point far away, ~~were~~ when

$$\left[\dot{R}^2 + R\ddot{R}\right]\ln R_\infty + \frac{P_\infty}{\rho} = f(t) \qquad (c)$$

At the surface of the cylinder:

$$\left[\dot{R}^2 + R\ddot{R}\right]\ln R + \frac{1}{2}\dot{R}^2 + \frac{P_L}{\rho} = f(t) \qquad (d)$$

Subtracting (c) from (d) gives

$$\left[\dot{R}^2 + R\ddot{R}\right]\ln\frac{R}{R_\infty} + \frac{1}{2}\dot{R}^2 + \frac{P_L - P_\infty}{\rho} = 0$$

Since $\lim\limits_{R_\infty \to \infty} \ln(R/R_\infty) \to -\infty$,

the following equations must both apply in order for the above expression to make sense:

$$\dot{R}^2 + R\ddot{R} = 0$$

$$\frac{1}{2}\dot{R}^2 + \frac{P_L - P_\infty}{\rho} = 0$$

These imply:

$$R\ddot{R} - \frac{2(P_L - P_\infty)}{\rho_L} = 0$$