Problem 2.1

We have

$$dP/dx = \mu_L \frac{\partial^2 U}{\partial x^2} \qquad (1)$$

Also

$$dP/dy = -\rho_L g \qquad (2)$$

Assuming incompressible flow and steady-state, mass conservation requires that:

$$\int_0^H U(y)\, dy = 0 \qquad (3)$$

The velocity boundary conditions are:

$$U = 0 \quad \text{at } y = 0 \qquad (4)$$

$$\mu_L \frac{\partial U}{\partial y} = \frac{d\sigma}{dx} \quad \text{at } y = H \qquad (5)$$

Eq. (1) can now be integrated, with boundary conditions (4) & (5), to get:

$$\mu_L U = \left(\frac{d\sigma}{dx} - H \frac{\partial P}{\partial x} \right) y + \frac{1}{2} \frac{\partial P}{\partial x} y^2 \qquad (6)$$

Given that at any location $P = P_\infty + \rho_L g (H - y)$, then

$$\frac{\partial P}{\partial x} = \rho_L g \, dH/dx \qquad (7)$$

We can now substitute for U in Eq. (3) from Eq. (6), and perform the integration to get

$$\frac{\partial \sigma}{\partial x} = \frac{2}{3} H \frac{\partial P}{\partial x} \qquad (8)$$

Elimination of $\frac{\partial P}{\partial x}$ between Eqs. (7) and (8) then gives

$$\frac{d\sigma}{dx} = \frac{2}{3} \rho_L g H \, dH/dx \qquad (9)$$

Substitution in Eq. (6), from (7) and (9) eventually leads to the desired result.

Reference:

R. F. Probstein, Physicochemical Hydrodynamics, Butterworth, 1989.