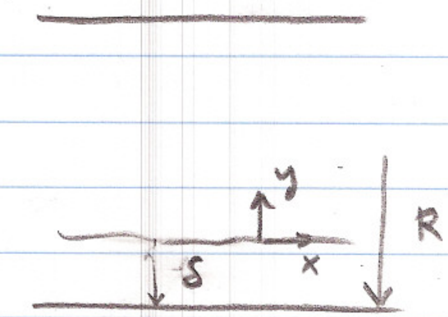
Problem 2.14



Assume

1. $\delta / R << 1$; therefore liquid can be infinitely thick
2. Both phases are incompressible & inviscid
3. No gravitational effect
4. Axi-symmetric conditions

In view of assumption 1, the waves can be treated as 2-D, in (x, y) coordinates. The solution proceeds as:

$$\phi_L = U_L x + \phi_L' \qquad (1)$$

$$\phi_G = U_G x + \phi_G' \qquad (2)$$

Integration of Euler's equation for the two phases (similar to Eq. (2.119)) gives:

$$\rho_L \left(\frac{\partial \phi_L'}{\partial t} + U_L \frac{\partial \phi_L'}{\partial x} \right) = - P_L + P_{0L} \qquad (3)$$

$$\rho_G \left(\frac{\partial \phi_G'}{\partial t} + U_G \frac{\partial \phi_G'}{\partial x} \right) = - P_G + P_{0G} \qquad (4)$$

At equilibrium we would have

$$P_{0L} - P_{0G} = \frac{\sigma}{R-h} \qquad (5)$$

Also

$$P_L - P_G = \frac{\sigma}{R-h-\zeta} + \sigma \frac{\partial^2 \zeta}{\partial x^2} \qquad (6)$$

Now assume

$$\phi_L' = A_L e^{-ky} e^{i(\omega t - kx)} \qquad (7)$$

$$\phi_G' = A_G \cosh\left[k(y+\delta) \right] e^{i(\omega t - kx)} \qquad (8)$$

These equations satisfy the following

$$\nabla^2 \phi_L' = \nabla^2 \phi_G' = 0 \qquad (9)$$

$$\nabla \phi_L' \Big|_{y \to \infty} = 0 \qquad (10)$$

$$\frac{d\phi_G'}{dy} \Big|_{y=-\delta} = 0 \qquad (11)$$

Assume a disturbance of the form

$$\zeta = \zeta_0 \, e^{i(\omega t - kx)} \qquad (12)$$

We now must satisfy

$$v_L\Big|_{y=\zeta} \approx \frac{\partial \phi_L}{\partial y}\Big|_{y=0} = \frac{\partial \zeta}{\partial t} + U_L \frac{\partial \zeta}{\partial x} \qquad (13)$$

$$v_G\Big|_{y=\zeta} \approx \frac{\partial \phi_G}{\partial y}\Big|_{y=0} = \frac{\partial \zeta}{\partial t} + U_G \frac{\partial \zeta}{\partial x} \qquad (14)$$

$$v_L\Big|_{y=\zeta} = v_G\Big|_{y=\delta} \qquad (15)$$

We also note that

$$-(P_L - P_G) + (P_{0L} + P_{0G}) = \frac{\sigma}{R-\delta} - \frac{\sigma}{(R-\delta)-\zeta} - \sigma \frac{\partial^2 \zeta}{\partial x^2} \qquad (16\text{-}a)$$

$$\frac{\sigma}{(R-\delta)-\zeta} \approx \frac{1}{R-\delta}\,\sigma \left[1 - \frac{\zeta}{R-\delta}\right] \qquad (16\text{-}b)$$

Eqs. (13), (14), (15), then lead to, respectively

$$k A_L = -i \zeta_0 (\omega - U_L k) \qquad (17)$$

$$k A_G \sinh(k\delta) = i \zeta_0 (\omega - U_G k) \qquad (18)$$

$$\rho_L \left[i\omega A_L + U_L A_L (-ik) \right] - \rho_G \left[i\omega A_G \cosh(k\delta) + U_G A_G \cosh(k\delta)(-ik) \right] = \sigma \zeta_0 k^2 - \frac{\sigma \zeta_0^2}{(R-\delta)^2} \qquad (19)$$

$$(17) \Rightarrow \quad A_L = -i \, \frac{J_0(\omega - U_L k)}{k} \qquad (20)$$

$$(18) \Rightarrow \quad A_G = i \, \frac{J_0(\omega - U_G k)}{k \sinh(k\delta)} \qquad (21)$$

substitution from Eqs. (20) and (21) in (19) gives the desired result.