Problem 2.16

Assume;

1. steady - state
2. $\delta / R \ll 1$, so that a planar, 2-D stability analysis at the interphase is justified
3. $\mu_G \ll \mu_L$, so that the gas can be treated as inviscid
4. Axi-symmetric flow
5. Negligible gravitational effect

2.16.2 The conservation equations for the liquid phase can then be represented as:

$$\frac{\partial u_L}{\partial x}+\frac{\partial v_L}{\partial y}=0 \tag{5}$$

$$\rho_L \frac{\partial u_L}{\partial t}=-\frac{\partial P_L}{\partial x}+\mu_L\left(\frac{\partial^2 u_L}{\partial x^2}+\frac{\partial^2 u_L}{\partial y^2}\right) \tag{6}$$

$$\rho_L \frac{\partial v_L}{\partial t}=-\frac{\partial P_L}{\partial y}+\mu_L\left(\frac{\partial^2 v_L}{\partial x^2}+\frac{\partial^2 v_L}{\partial y^2}\right) \tag{7}$$

The gas phase conservation equation are similar, except that they do not include viscosity.

Periodic motion of incompressible fluid surfaces obeying the Navier-Stokes equations can be represented using a potential and a stream function. [6, 17] Each velocity component of the liquid is therefore assumed to consist of an inviscid term and a perturbation representing the effect of viscosity [17]

$$u_L=u_L^0+U_L \tag{8}$$

$$v_L=v_L^0+V_L \tag{9}$$

$$P=P^0 \tag{10}$$

where parameters with superscript *0* represent ideal fluid conditions, and U and V represent the velocities in x and y directions due to viscosity, respectively. The gas phase is assumed to be inviscid; the gas velocity components therefore do not include terms representing viscosity. The above ideal-fluid velocities are represented by a potential function of the form:

$$\Phi_L=A_L\, e^{-ky+\omega t}\cos k\, x \tag{11}$$

$$\Phi_G = A_G \cosh\left[k\left(y+\delta\right)\right]e^{\omega t}\cos k\,x \tag{12}$$

where for both phases:

$$\left(u^o, v^o\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Phi \tag{13}$$

These potential functions fully represent the irrotational, ideal liquid and gas fields. The liquid velocity terms that account for the viscosity-induced rotational motion can be represented by a stream function, ψ_L, that must satisfy [17]:

$$\frac{\partial \psi_L}{\partial t} = \nu_L \nabla^2 \psi_L \tag{14}$$

where:

$$\left(U_L, V_L\right) = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)\psi_L \tag{15}$$

Equation (14) and all the necessary boundary conditions can be satisfied using:

$$\psi_L = B_L e^{-my+\omega t}\sin kx \tag{16}$$

Note that the above equation approximates the liquid core as infinitely-deep. Using Eq. (14), it can be shown that:

$$m^2 = k^2 + \frac{\omega}{\nu_L} \tag{17}$$

The above potential and stream functions are now utilized in a linear stability analysis, using the following boundary conditions at the interphase:

$$v_L = v_G \tag{18}$$

$$-P_L + 2\mu_L\left(\frac{\partial V_L}{\partial y}\right) = -P_G - \sigma\frac{\partial^2 \eta}{\partial x^2} - \Delta P_\sigma \tag{19}$$

$$\mu_L\left(\frac{\partial u_L}{\partial y}+\frac{\partial v_L}{\partial x}\right)=0 \tag{20}$$

where:

$$\eta=\frac{k}{\omega}\left(B_L-A_L\right)e^{\omega t}\cos kx \tag{21}$$

$$P_L-P_G=-\rho_L\left.\frac{\partial \Phi_L}{\partial t}\right|_{y=0}+\rho_G\left.\frac{\partial \Phi_G}{\partial t}\right|_{y=0} \tag{22}$$

The term ΔP_σ in Eqn. (19), which is de-stabilizing, is the surface tension force caused by the curvature of the channel, and can be represented as:

$$\Delta P_\sigma=\frac{\sigma}{R-\delta}-\frac{\sigma}{R-\delta-\eta}\approx\frac{\sigma}{R-\delta}\left(1+\frac{\eta}{R-\delta}\right) \tag{23}$$

Equations (19-21) following substitution for velocities using the aforementioned potential and stream functions, lead to:

$$-A_L+B_L-A_G\sinh(kh)=0 \tag{24}$$

$$2k^2A_L-\left(m^2+k^2\right)B_L=0 \tag{25}$$

$$\begin{gathered}\left[2\mu_L k^2+\frac{k^3\sigma}{\omega}-\frac{\sigma}{(R-\delta)^2}\frac{k}{\omega}+\rho_L\omega\right]A_L+\\ \left[-2\mu_L km-\frac{k^3\sigma}{\omega}+\frac{\sigma}{(R-\delta)^2}\frac{k}{\omega}\right]B_L-\left[\rho_G\alpha\cosh(k\delta)\right]A_G=0\end{gathered} \tag{26}$$

A non-trivial solution is possible if the determinant of the coefficient matrix of Eqns. (24-26) vanishes, and that leads to:

$$\left[\rho^* \coth\left(R^* K H\right)+1\right]\Omega^2 - \frac{K}{R^{*2}(1-H)^2} + K^3 + 4K^4\left[1-\left(1+\frac{\Omega}{K^2}\right)^{1/2}\right] + 4\Omega K^2 = 0 \tag{27}$$

where $\rho^* = \rho_G/\rho_L$; $H = \delta/R$; $R^* = R\dfrac{\sigma}{\rho_L {v_L}^2}$; $K = k\left(\dfrac{\sigma}{\rho_L {v_L}^2}\right)^{-1}$; and $\Omega = \omega\left(\dfrac{{\rho_L}^2 {v_L}^3}{\sigma^2}\right)$. The interface is evidently unstable when $\Omega > 0$, and the neutral condition ($\Omega = 0$) occurs when

$$\left(KR^*\right)_{cr} = \frac{1}{1-H} \tag{28}$$

Disturbances with wavelengths longer than λ_{cr} thus cause instability, where:

$$\lambda_{cr} = 2\pi\left(R-\delta\right) \tag{29}$$

The neutral wavelength thus is similar to the predictions of the Kelvin-Helmholtz stability; and does not depend on liquid viscosity, in agreement with the Taylor stability analysis for film boiling of viscous liquids. The fastest growing wavelength, Ω_d, occurs when $d\Omega/dK = 0$, whereby:

$$\left(\frac{\rho^* R^* H}{\sinh^2\left(R^* K H\right)}\right){\Omega_d}^2 - 4K\left[2+\left(1+\frac{\Omega_d}{{K_d}^2}\right)^{-1/2}\right]\Omega_d - 16{K_d}^3\left[1-\left(1+\frac{\Omega_d}{{K_d}^2}\right)^{1/2}\right] - 3{K_d}^2 = 0 \tag{30}$$