$$\rho_L \left(\frac{\partial \phi_L'}{\partial t} + U_L \frac{\partial \phi_L'}{\partial x} + U_L \frac{\partial \phi_L'}{\partial y} - g\zeta \right)$$

$$\rho_G \left(\frac{\partial \phi_G'}{\partial t} + U_G \frac{\partial \phi_G'}{\partial x} + U_G \frac{\partial \phi_G'}{\partial y} - g\zeta \right) = \sigma \left(- \frac{\partial^2 \zeta}{\partial x^2} - \frac{\partial^2 \zeta}{\partial y^2} \right) \quad \text{— (1)}$$

$$\frac{\partial \zeta}{\partial t} + U_L \frac{\partial \zeta}{\partial x} + U_L \frac{\partial \zeta}{\partial y} \simeq \left. \frac{\partial \phi_L'}{\partial z} \right|_{z=0} \quad \text{— (2)}$$

$$\frac{\partial \zeta}{\partial t} + U_G \frac{\partial \zeta}{\partial x} + U_G \frac{\partial \zeta}{\partial y} \simeq \left. \frac{\partial \phi_G'}{\partial z} \right|_{z=0} \quad \text{— (3)}$$

$$\zeta = \zeta_0 \exp \left[i (\omega t - k_x x - k_y y) \right] \quad \text{— (4)}$$

Employing (4) into (2)

$$-bk = \zeta_0 \left[i\omega + U_L(-ik_x) + U_L(-ik_y) \right] = i\zeta_0 \left[\omega - U_L k_x - U_L k_y \right]$$

Employing (4) into (3)

$$b'k = \zeta_0 \left[i\omega - U_G i k_x - i U_G k_y \right] = i\zeta_0 \left[\omega - U_G k_x - U_G k_y \right]$$

$$\rho_L \left[i\omega b - ik_x U_L b - ik_y U_L b - g\zeta_0 \right]$$

$$- \rho_G \left[i\omega b' - ik_x U_G b' - ik_y U_G b' - g\zeta_0 \right] = \sigma \zeta_0 (k_x^2 + k_y^2)$$

✓ $\nabla^2 \phi_L = 0$

$$(-ik_x)^2 + (-ik_y)^2 + (-k)^2 = -k_x^2 - k_y^2 + k^2 = 0$$

$$\Rightarrow k^2 = k_x^2 + k_y^2 \quad ✓$$

$$\rho_L \left\{ \frac{\zeta_0 w [w - U_L(k_x + k_y)]}{-k^2} - U_L(k_x+k_y) \frac{\zeta_0 [w - U_L(k_x+k_y)]}{k^2} \right.$$

$$\left. - \frac{g\zeta_0}{k} \right\} - \rho_G \left\{ -\frac{\zeta_0 w [w - U_G(k_x+k_y)]}{k^2} + U_G(k_x+k_y) \frac{\zeta_0 [w - U_G(k_x+k_y)]}{k^2} \right.$$

$$\left. - \frac{g\zeta_0}{k} \right\} = \sigma \zeta_0 k$$

$$(\rho_L + \rho_G)\left(\frac{w}{k}\right)^2 - \frac{2U_L(k_x+k_y)}{k}\rho_L\left(\frac{w}{k}\right) - \frac{2\rho_G U_G(k_x+k_y)}{k}\left(\frac{w}{k}\right)$$

$$+ \frac{\rho_L U_L^2 (k_x+k_y)^2}{k^2} + \frac{\rho_G U_G^2 (k_x+k_y)^2}{k^2} - \left(\frac{\rho_L g - \rho_G g}{k} + \sigma k\right) = 0$$

$$\Rightarrow \quad (\rho_L + \rho_G)\left(\frac{w}{k}\right)^2 - \frac{2(U_L\rho_L + U_G\rho_G)(k_x+k_y)}{k}\left(\frac{w}{k}\right) + \frac{(\rho_L U_L^2 + \rho_G U_G^2)(k_x+k_y)^2}{k^2}$$

$$- \frac{g(\rho_L - \rho_G)}{k} - \sigma k = 0$$

$$\frac{w}{k} = \frac{(U_L\rho_L + U_G\rho_G)}{(\rho_L+\rho_G)}\frac{(k_x+k_y)}{k} \pm \frac{1}{2(\rho_L+\rho_G)}\left\{ 4\frac{(U_L\rho_L + U_G\rho_G)^2(k_x+k_y)^2}{k^2} \right.$$

$$\left. - 4\left[\frac{(\rho_L U_L^2 + \rho_G U_G^2)(k_x+k_y)^2}{k^2} - g\frac{(\rho_L - \rho_G)}{k} - \sigma k\right](\rho_L+\rho_G) \right\}^{\frac{1}{2}}$$

$$\frac{w}{k} = \frac{(U_L \rho_L + U_G \rho_G)(k_x + k_y)}{(\rho_L + \rho_G) \quad k} \pm \frac{1}{(\rho_L+\rho_G)} \Big\{ (k_x^2 + k_y^2)(2\rho_L \rho_G U_L U_G - \rho_L \rho_G U_G^2 - \rho_L \rho_G U_L^2) - kg(\rho_L^2 - \rho_G^2) + \sigma k^3 (\rho_L + \rho_G) \Big\}^{1/2}$$

Critical happens when

$$\frac{g}{k}\left(\frac{\rho_L - \rho_G}{\rho_L + \rho_G}\right) + \frac{(k_x + k_y)^2}{k^2} \frac{\rho_L \rho_G}{(\rho_L+\rho_G)^2} (U_L - U_G)^2 - \frac{\sigma k}{\rho_L + \rho_G} = 0$$

Taylor Instability happens when $U_L = U_G = 0$

$$\frac{g}{k}(\rho_L - \rho_G) - \sigma k = 0$$

$$k_c = \sqrt{\frac{g \Delta \rho}{\sigma}} = \frac{1}{L}$$

$$\lambda_c = \frac{2\pi}{k_c} = 2\pi L \qquad \lambda_{c_2} = \sqrt{2}\,\lambda_c = 2\pi\sqrt{2}\,L$$

To find the fastest growing wave length

$$\frac{dw}{dk} = 0 \,, \quad U_L = U_G = 0$$

$$w = \left(g\left(\frac{\rho_L - \rho_G}{\rho_L + \rho_G}\right)k - \frac{\sigma k^3}{\rho_L + \rho_G}\right)^{\frac{1}{2}}$$

$$\frac{dw}{dk} = \frac{g\left(\frac{\rho_L - \rho_G}{\rho_L + \rho_G}\right) - 3\frac{\sigma}{\rho_L + \rho_G}k^2}{\sqrt{gk\left(\frac{\rho_L - \rho_G}{\rho_L + \rho_G}\right) - \frac{\sigma k^3}{\rho_L + \rho_G}}} = 0 \quad \Rightarrow \quad k = \sqrt{\frac{g \Delta \rho}{3\sigma}} = \frac{1}{\sqrt{3}\,L}$$

$$\lambda_{d_2} = \sqrt{2}\,\lambda_d = 2\pi\sqrt{6}\,L$$